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Existence of unbounded solutions to the isentropic p-system with a self-gravitational term*

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1 Introduction

A number of authors have studied the viscous p-system of one-dimensional equations:

$$\begin{cases} \partial_t v - \partial_x u = 0, \\ \partial_t u + \partial_x(p(v)) - \nu\partial_x\left(\dfrac{\partial_x u}{v}\right) = \mathcal{G}. \end{cases} \tag{1}$$

Here, $\nu > 0$ is a viscosity constant, $p(v)$ a function of $v > 0$ assumed to take finite positive values, and $\mathcal{G}$ represents a forcing term, later in this section specified as a self-gravitational field. In case the function $p(v)$ takes the form

$$p(v) = av^{-\gamma}$$

with constants $a > 0$ and $\gamma \geq 1$, so-called isothermal or isentropic case, we know, under suitable boundary conditions, that the Cauchy problem of the system (1) with smooth initial data admits a unique global smooth solution for a considerably wide class of forcing terms. For example, for the system on a finite interval with solid boundary or the system with periodic condition, square integrability of $\mathcal{G}$ in time-space variables is enough to solve the system in H^1-class. Neither blowing up nor breaking down of solutions takes place in a finite time. A major problem left for us would be the study of the global behavior of solutions, as experts in this field agree.

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In this context, some global bounds of solutions would be of great use. For the isothermal system with solid boundary condition, the classical result of Matsumura-Nishida [3] (1989) established the global H^1-bound of solutions for any L^∞ forcing term. The global H^1-bound of a solution then ensures the compactness of the orbit, for example, in L^∞. This enables us to take the ω-limit of the orbit to yield a nonempty set:

$$\bigcap_{s\geq 0} \overline{\{(v(t,\,\cdot\,),u(t,\,\cdot\,));t\geq s\}}^{L^\infty\times L^\infty} \neq \emptyset, \tag{2}$$

and some information on the large time behavior of the solution would be deduced from the set. Under the same situation as in Matsumura-Nishida, an effort to derive the global H^1-bound from the isentropic system was done by Matsumura-Yanagi [4] (1996). However, the result requires the isentropic system some data-dependent closeness to the isothermal system. Besides, a result of Novotny-Straskraba [5] (2001) on the two or three dimensional compressible isentropic Navier-Stokes system with a large stationary forcing term shows the existence of a weak solution approaching certain singular stationary solutions. Even in the one dimensional case, boundedness of forcing term (except for the trivial case of $\mathcal{G}\equiv 0$, see Kanel' [2]) seems inadequate to ensure the boundedness of solutions to the isentropic system. Now the problem arises how an unbounded solution, if exists, is derived from the isentropic system.

In this note, we concentrate ourselves to the isentropic system forced by a non-stationary self-gravitational field:

$$\mathcal{G} = -\frac{4\pi G}{\bar{v}}\frac{\partial}{\partial x}\int_0^L K_L(x,y)\,(v(t,y)-\bar{v})\,dy \tag{3}$$

under spatially periodic condition. K_L represents Green's kernel of the minus Laplacian $-\frac{d^2}{dx^2}$ acting on L-periodic functions with average 0:

$$K_L(x,y) = \sum_{n=1}^{\infty}\frac{L}{2\pi^2 n^2}\cos\frac{2\pi n}{L}(x,y),$$

or

$$K_L(x,y) = -\frac{|x-y|}{2}+\frac{(x-y)^2}{2L}+\frac{L}{12}, \quad 0\leq x,y\leq L,$$

$\bar{v}$ the average of v:

$$\bar{v} = \frac{1}{L}\int_0^L v(t,x)dx,$$

and $G>0$ the gravitational constant. The formula (3) is a representation in Lagrangian material coordinates of Newton's gravitation corresponding to periodic mass distribution of a fluid. This field is often adopted in the classical theory of gravitational instability as a model of gravitation admitting force-free infinite homogeneous states of the fluid. See, for example, Weinberg [8], Chapter 15. The field develops along with the variation of v but turns out to be globally bounded with respect to the time-space variables.

No problem arises about the Cauchy problem for (1) with (3) due to the boundedness of the self-gravitational term, however, we first establish the global solvability of the system for the sake of completeness. In what follows, we denote by H^s, $s = 0, 1, 2, \ldots$, the Sobolev spaces of L-periodic real-valued functions on $\mathbf{R}$, and write $L^2 = H^0$, as usual.

Global existence of solutions For any initial data $v_0, u_0 \in H^1$ with $v_0 > 0$, the Cauchy problem with initial condition

$$v(0,x) = v_0(x), \quad u(0,x) = u_0$$

admits a unique global solution belonging to the class:

$$\begin{cases} v \in C^1([0,\infty); L^2) \cap C^0([0,\infty); H^1), \quad v > 0, \\ u \in H^1_{\text{loc}}(0,\infty; L^2) \cap L^2_{\text{loc}}(0,\infty; H^2) (\subset C^0([0,\infty); H^1)). \end{cases}$$

We are concerned with the global behavior of smooth solutions from this class. Specifically, we introduce a condition for the initial data that ensures the unboundedness in the sense that v "grows up" in infinite time:

$$\sup_{t,x} v(t,x) = \infty.$$

2 Main results

In order to describe a situation of unbounded growth of solutions we need to refer to the structure of the whole stationary solutions of the system. For this purpose let us introduce a function on the interval $0 < \theta < (\gamma - 1)^{-1/2}$ as

$$\begin{aligned} I_\gamma(\theta) &= I_{\gamma,+}(\theta) + I_{\gamma,-}(\theta), \\ I_{\gamma,\pm}(\theta) &= \theta \int_0^1 \frac{1}{\sqrt{1-y}} \frac{1}{f_\pm(F_\pm{}^{-1}(\theta^2 y))} dy, \end{aligned}$$

where $f_\pm$ and $F_\pm$ are functions given by

$$\begin{aligned} f_+(r) &= 1 - (1+r)^{-1/\gamma}, & F_+(r) &= \int_0^r f_+(s) ds, \quad r \geq 0, \\ f_-(r) &= -\{1 - (1-r)^{-1/\gamma}\}, & F_-(r) &= \int_0^r f_-(s) ds, \quad 0 \leq r < 1. \end{aligned}$$

We can show that I_γ is monotone increasing for $1 \leq \gamma < 2$, and at either end of the interval, I_γ has the limit:

$$I_\gamma(+0) = (2\gamma)^{1/2}\pi, \quad \begin{cases} I_1(\infty) = \infty, & \gamma = 1, \\ I_\gamma((\gamma-1)^{-1/2} - 0) < \infty, & 1 < \gamma < 2. \end{cases}$$

For a proof see [7].

Noting that the average of a solution is a constant of motion, and that $\bar{u}$ may be considered to vanish by change of unknown functions $(v, u) \mapsto (v, u - \bar{u})$, for every positive parameter V we study the structure of the whole stationary solutions lying in the manifold

$$M_V = \{(v, u) \in H^1 \times H^1; \bar{v} = V, \bar{u} = 0, v > 0\}.$$

Obviously, the trivial solution $(V, 0)$ lies in M_V.

Theorem 1 Assume $1 \leq \gamma < 2$. For $V > 0$ let $k_{\min}$ and $k_{\max}$, respectively, be the smallest and the largest integers j satisfying

$$\left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < \frac{L}{j} < \begin{cases} \infty, & \gamma = 1, \\ \dfrac{I_\gamma((\gamma-1)^{-1/2})}{\sqrt{2\gamma\pi}} \left(\dfrac{a\gamma\pi}{GV^\gamma}\right)^{1/2}, & 1 < \gamma < 2. \end{cases} \tag{4}$$

Then, for $j = k_{\min}, \ldots k_{\max}$ there exists on M_V a stationary solution with least period L/j. The whole stationary solutions lying in M_V except for the trivial one are given by

$$(\tilde{v}^{(j)}(\cdot - \alpha), 0), \qquad 0 \leq \alpha < L/j, \quad j = k_{\min}, \ldots, k_{\max},$$

where $(\tilde{v}^{(j)}, 0)$ is one of the stationary solutions with least period L/j.

How can we read this result ?

- For $V \leq \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$, no integer satisfies the condition (4) and the stationary problem admits on M_V only the trivial solution.
- In case $\gamma = 1$, the condition (4) is satisfied with $j = 1$ for every $V > \frac{a\pi}{GL^2}$.
- In case $1 < \gamma < 2$, while $j = 1$ satisfies (4) for $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$, solutions with least period L are lost for $V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$.

Whether, for every $V > 0$, the stationary problem admits a solution on M_V other than the trivial one depends on the value γ.

We are now ready to present a condition for initial data leading to unbounded solutions. For $\gamma > 1$ let us introduce a functional, called the energy form:

$$\begin{aligned}
\mathcal{E}(v, u) &= \int_0^L \frac{1}{2} u(x)^2 dx + \mathcal{E}(v), \\
\mathcal{E}(v) &= \int_0^L a\left(\frac{v(x) - \bar{v}}{\bar{v}^\gamma} - \frac{v(x)^{1-\gamma} - \bar{v}^{1-\gamma}}{1-\gamma}\right) dx \\
&\quad - \frac{2\pi G}{\bar{v}} \int_0^L \int_0^L K_L(x, y)(v(x) - \bar{v})(v(y) - \bar{v}) dx dy
\end{aligned}$$

Notice that the value of $\mathcal{E}(v)$ is unchanged by the shift of v in view of the L-periodicity of v.

Theorem 2 Assume $1 < \gamma < 2$. Let $V \geq \left(\frac{aI_\gamma((\gamma-1)^{-1/2}-0)^2}{2\pi GL^2}\right)^{1/\gamma}$ and $\tilde{v}^{(k_{\min})}$ be as in **Theorem 1**.
(i) The subset of $H^1 \times H^1$ given by

$$A_V = \left\{ (v,u) \in M_V \mid \mathcal{E}(v,u) < \begin{cases} \mathcal{E}(\tilde{v}^{(k_{\min})}), & \text{if integers } j \text{ with (4) exist,} \\ 0, & \text{otherwise,} \end{cases} \right\}$$

is nonempty.
(ii) Any solution with initial value from A_V is unbounded, that is,

$$\sup_{t,x} v(t,x) = \infty.$$

The assertion of **Theorem 2** makes a sharp contrast between the isothermal and the isentropic systems. Indeed, in the isothermal case, we can derive the global H^1-bound of any solution from the boundedness of the gravitational field just in the same manner as in Matsumura-Nishida [3]. In particular, growing-up of solutions as in **Theorem 2** never takes place.

3 Sketch of proofs

We sketch the outline of the proof of **Theorem 2**. For details, see [6], [7].

We start by considering the large time behavior of a bounded solution. The following lemma shows that an a priori information on the upper bound of v is enough to derive the global H^1-bound of the solution from the isentropic system with $1 < \gamma \leq 2$. The argument is somewhat similar to that of Matsumura-Nishida [3], deriving the global H^1-bound of solutions from the isothermal system.

Lemma 1 Assume $1 < \gamma \leq 2$. The orbit of a bounded solution, i.e. $\sup_{t,x} v(t,x) < \infty$ is H^1-bounded with positive lower bound $\inf_{t,x} v(t,x) > 0$.

This allows us to study the large time behavior of a bounded solution by taking the ω-limit in L^∞ space of the orbit, as shown by (2). From the next lemma we see that the asymptotics of a bounded solution is under the control of the set of stationary solutions on M_V with $V = \bar{v}$.

Lemma 2 Assume $1 \leq \gamma \leq 2$. For a bounded solution the ω-limit set of the orbit is a subset of stationary solutions with average common with the initial value.

The above lemmas imply that if the orbit of a solution is apart from the set of stationary solutions it is necessarily unbounded. This fact combined with the decrease of energy form along orbits:

$$\frac{d}{dt}E(t) = -\nu \int_0^L \frac{\partial_x u(t,x)^2}{v(t,x)} dx (\leq 0) \quad \text{with} \quad E(t) = \mathcal{E}(v(t,\cdot), u(t,\cdot))$$

means that an initial state with value of energy form less than any values of energy form evaluated at stationary solutions, if exists, leads to a unbounded solution. Thus we come up with the idea of finding a condition for unbounded solutions as presented in **Theorem 2.**

Concerning the condition, there are some questions that need to be answered.

1. Which is minimal amongst the values of energy form evaluated at the stationary solutions on M_V ?

2. What condition ensures that A_V is nonempty ?

3.1 Comparison of the values of energy form

In order to compare the values of energy form evaluated at stationary solutions we consider the value of energy form evaluated at the stationary solution with least period L as a function of L:

$$\varepsilon(L) = \mathcal{E}(\tilde{v}^{(1)}), \qquad \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < L < \frac{I_\gamma((\gamma-1)^{-1/2})}{\sqrt{2\gamma\pi}} \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2},$$

and then express the other values of energy form with this function:

$$\mathcal{E}(\tilde{v}^{(j)}) = j\varepsilon(L/j) = L \times \frac{\varepsilon(L/j)}{L/j}, \quad j = k_{\min}, \ldots, k_{\max}.$$

Thanks to this formula the comparison of the values of energy form is reduced to the study of the behavior of function $\varepsilon(l)/l$. By elementary calculus we can show that

$$\frac{d}{dl}\left(\frac{\varepsilon(l)}{l}\right) < 0, \quad \varepsilon(l) < 0, \qquad \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2} < l < \frac{I_\gamma((\gamma-1)^{-1/2})}{\sqrt{2\gamma\pi}} \left(\frac{a\gamma\pi}{GV^\gamma}\right)^{1/2}.$$

From this we can arrange the values of energy form in order.

Lemma 3 $\mathcal{E}(\tilde{v}^{(k_{\min})}) < \cdots < \mathcal{E}(\tilde{v}^{(k_{\max})}) < \mathcal{E}(V) = 0$. In particular, $\mathcal{E}(\tilde{v}^{(k_{\min})})$ if it is meaningful or else $\mathcal{E}(V) = 0$ is minimal amongst the values of energy form.

3.2 $A_V \neq \emptyset$?

It remains to show that the initial condition expressed by the energy form makes sense. For this end take a stationary solution $(\tilde{v}, 0)$ and consider a disturbance (ϕ, ψ) from the stationary solution with average 0. We then expand the energy form with respect to the disturbance:

$$\mathcal{E}(\tilde{v}+\phi,\psi) = \mathcal{E}(\tilde{v}) + \frac{1}{2}\|\psi\|_{L^2}^2 + \frac{1}{2}Q[\phi] + \mathcal{O}(\|\phi\|_{L^\infty})\|\phi\|_{L^2}^2,$$

where Q is a quadratic form on the Hilbert space $\mathcal{H} = \{\varphi \in L^2; \bar{\varphi} = 0\}$:

$$Q[\varphi] = \int_0^L a\frac{\gamma\varphi(x)^2}{\tilde{v}(x)^{\gamma+1}}dx - \frac{4\pi G}{\tilde{v}}\int_0^L\int_0^L K_L(x,y)\varphi(x)\varphi(y)dxdy.$$

We are interested in a state with value of the energy form less than $\mathcal{E}(\tilde{v})$. Such a state exists in a small neighborhood of the stationary solution if the quadratic form admits a negative value. Notice that corresponds to the quadratic form Q the following self-adjoint operator on $\mathcal{H}$:

$$T_{\lambda,\tilde{w}}\varphi = \frac{\gamma\varphi}{(1+\tilde{w})^{\gamma+1}} - \frac{1}{L}\int_0^L \frac{\gamma\varphi(y)}{(1+\tilde{w}(y))^{\gamma+1}}dy - \lambda\int_0^L K_L(\,\cdot\,,y)\varphi(y)dy$$

with

$$\lambda = \frac{4\pi G V^\gamma}{a}, \quad \tilde{w} = \frac{\tilde{v}-V}{V}$$

in the sense that

$$Q[\varphi] = \frac{a}{V^\gamma}(T_{\lambda,\tilde{w}}\varphi,\varphi)_{L^2}.$$

Thus, we are naturally forced to study study the lower bound of the operator $T_{\lambda,\tilde{w}}$.

In studying the spectrum $\sigma(T_{\lambda,\tilde{w}})$ the following facts are available. The stationary problem for (1) with (3) has another version of the form

$$\Theta(\lambda,\tilde{w}) = 0 \tag{5}$$

with Θ a map from $\mathbf{R} \times \{w \in C^0; \bar{w} = 0, w > -1\}$ to $\{w \in C^0; \bar{w} = 0\}$ given by

$$\Theta(\lambda,w) = -\frac{1}{(1+w)^\gamma} + \frac{1}{L}\int_0^L \frac{dy}{(1+w(y))^\gamma} - \lambda\int_0^L K_L(\,\cdot\,,y)w(y)dy, \tag{6}$$

where C^0 is the space of continuous functions on $\mathbf{R}$ with period L. The operator $T_{\lambda,\tilde{w}}$ is the extension onto $\mathcal{H}$ of the Fréchet derivative Θ_w at $(\lambda,\tilde{w})$. If $\tilde{w} \neq 0$, by the equivariant structure of (6) to the translation of functions, $\tilde{w}'$ is an eigenfunction associated with eigenvalue 0. Furthermore:

- We can study the structure of the null space of $T_{\lambda,\tilde{w}}$ using the well-known structure of solutions of the second order linear ordinary differential equation:

$$\frac{d^2}{dx^2}\left\{\frac{\gamma\varphi}{(1+\tilde{w})^{\gamma+1}}\right\} + \lambda\varphi = 0.$$

  If $\tilde{w} \neq 0$, the null space turns out to be one dimensional.

- In the equation (5) by taking λ as a bifurcation parameter some information on $\sigma(T_{\lambda,\tilde{w}})$ near the origin is derived from the perturbation of a simple eigenvalue due to Crandall and Rabinowitz [1].

Combining the above observations with the continuous dependence on $(\lambda,\tilde{w})$ of $\inf\sigma(T_{\lambda,\tilde{w}})$, we obtain the sign of the lower bound for every stationary solution.

Lemma 4

- For $\tilde{v} = V$, i.e. $\tilde{w} = 0$, the lower bound of $T_{\lambda,0}$ is explicitly calculated as

$$\inf \sigma(T_{\lambda,0}) = \gamma - \frac{GL^2V^\gamma}{a\pi}.$$

- For $\tilde{v} = \tilde{v}^{(1)}$,
  - $\inf \sigma(T_{\lambda,\tilde{w}})$ is a simple eigenvalue 0 with eigenfunction $\tilde{v}^{(1)\prime}$.
  - Except for the eigenvalue 0 the spectrum of $T_{\lambda,\tilde{w}}$ lies in a positive region:

$$\inf(\sigma(T_{\lambda,\tilde{w}}) \setminus \{0\}) = \kappa_V > 0$$

  with a constant κ_V depending only on V.

- For $\tilde{v} = \tilde{v}^{(k)}$, $k = 2, 3, \dots$, $\inf \sigma(T_{\lambda,\tilde{w}})$ is a negative eigenvalue.

We are now in a position to present a condition for A_V to be nonempty.

Case $V < \left(\dfrac{aI_\gamma((\gamma-1)^{-1/2} - 0)^2}{2\pi GL^2}\right)^{1/\gamma}$:

- When $V \leq \left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma}$, since the stationary problem admits only the trivial solution $(V, 0)$ with $\inf \sigma(T_{\lambda,0}) \geq 0$,

and

- when $\left(\frac{a\gamma\pi}{GL^2}\right)^{1/\gamma} < V < \left(\dfrac{aI_\gamma((\gamma-1)^{-1/2} - 0)^2}{2\pi GL^2}\right)^{1/\gamma}$, since $k_{\min} = 1$ and $\inf \sigma(T_{\lambda,\tilde{w}^{(1)}}) = 0$ with $\tilde{w}^{(j)} = \frac{\tilde{v}^{(j)} - V}{V}$, $j = k_{\min}, \dots, k_{\max}$,

it is hopeless to find an element of A_V in a small neighborhood of the set of stationary solutions. Search for elements of A_V on the whole M_V is an open problem.

Case $V \geq \left(\dfrac{aI_\gamma((\gamma-1)^{-1/2} - 0)^2}{2\pi GL^2}\right)^{1/\gamma}$: Since $\inf \sigma(T_{\lambda,0}) < 0$, and since, if meaningful, $k_{\min} \geq 2$ and hence $\inf \sigma(T_{\lambda,\tilde{w}^{(k_{\min})}}) < 0$, we can find an element of A_V in any small neighborhood of the trivial solution $(V, 0)$ or the stationary solution $(\tilde{v}^{(k_{\min})}, 0)$.

Excluding the first case as unsettled, we now arrive at the statement of **Theorem 2**.

References

[1] M. G. Crandall and P. Rabinowitz, Bifurcation, perturbation of simple eigenvalues, and linearized stability, Arch. Rational Mech. Anal., 52 (1973), 161-180.

[2] Ya. Kanel', On a model system of one-dimensional gas motion (Russian), Differencial'nya Uravnenija 4 (1968), 374-380.

[3] A. Matsumura and T. Nishida, Periodic solutions of a viscous gas equation, in "Recent Topics in Nonlinear PDE IV", Lecture Notes in Numerical and Applied Analysis, Vol. 10 (1989), 49-82.

[4] A. Matsumura and S. Yanagi, Uniform boundedness of the solutions for a one-dimensional isentropic model system of compressible viscous gas, Comm. Math. Phys., 175 (1996), 259-274.

[5] A. Novotny and I. Straskraba, Convergence to equilibria for compressible Navier-Stokes equations with large data, Ann. Mat. Pure Appl., 179 (2001), 263-287.

[6] M. Sawada and Y. Yamamoto, Unboundedness of some solutions to isentropic model equations for the one dimensional periodic motions of a compressible self-gravitating viscous fluid, Springer Proceedings in Mathematics and Statistics, Mathematical Challenge to a New Phase of Materials Science (to appear).

[7] M. Sawada and Y. Yamamoto, Existence of unbounded solutions to isentropic model equations for the one dimensional periodic motions of a compressible viscous fluid with self-gravitation (in preparation).

[8] S. Weinberg, Gravitation and Cosmology, John Wiley & Sons, New York, 1972.