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Relationship between results on degree sum conditions for cycles, paths and trees

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ for the order of G, that is, $|G| = |V(G)|$. We denote by $\deg_G(v)$ the degree of a vertex v in G. We define $\sigma_2(G) = \min\{\deg_G(u) + \deg_G(v) : uv \notin E(G)\}$ if G is not complete; otherwise $\sigma_2(G) = \infty$.

In 1960, Ore obtained a $\sigma_2(G)$ condition for Hamiltonicity.

Theorem 1 (Ore [10]). *Let G be a connected graph of order at least three. If $\sigma_2(G) \geq |G|$, then G is Hamiltonian.*

After this theorem, many researchers have given $\sigma_2(G)$ conditions for the existence of cycles, paths and trees. The purpose of this paper is to investigate the relationship between such results. In this section, we show well-known results on Hamiltonian cycles. In Section 2 and Section 3, by using the results, we prove the results on a spanning tree with a constraint on leaves, and the results on a partition into vertex-disjoint paths, respectively. The degree sum conditions in this paper are sharp in a sense.

In 1969, Kronk showed that a stronger condition than Ore's condition guarantees the existence of a Hamiltonian cycle passing through specified linear forest. A *linear forest* is a graph in which every component is a path.

Theorem 2 (Kronk [8]). *Let G be a graph and let F be a linear forest in G. If $\sigma_2(G) \geq |G| + |E(F)|$, then G has a Hamiltonian cycle passing through F.*

In 1976, Bermond and Linial, independently, obtained a $\sigma_2(G)$ condition for the circumference.

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Theorem 3 (Bermond [1], Linial [9]). *Let G be a 2-connected graph. Then G has a cycle of order at least* $\min\{\sigma_2(G), |G|\}$.

In 1997, Brandt, Chen, Faudree, Gould and Lesniak showed that the same condition as Ore's condition yields a partition into prescribed number of vertex-disjoint cycles.

Theorem 4 (Brandt et al. [2]). *Let k be a positive integer, and let G be a graph of order at least $4k$. If $\sigma_2(G) \geq |G|$, then G can be partitioned into k vertex-disjoint cycles.*

In 2004, Enomoto and Li showed that if K_1 and K_2 are regarded as cycles, which are called degenerated cycles, then the $\sigma_2(G)$ condition can be weakened.

Theorem 5 (Enomoto and Li [5]). *Let k be a positive integer, and let G be a graph. If $\sigma_2(G) \geq |G| - k + 1$, then G can be partitioned into k vertex-disjoint cycles or degenerated cycles, except $G = C_5$ and $k = 2$.*

In 2000, Egawa, Faudree, Györi, Ishigami, Schelp and Wang gave a $\sigma_2(G)$ condition for each vertex-disjoint cycle of the partition to pass one of specified edges.

Theorem 6 (Egawa et al. [4]). *Let k be a positive integer, and let G be a graph of order at least $4k - 1$. If $\sigma_2(G) \geq |G| + 2k - 2$, then for each k independent edges $e_1, e_2, \ldots, e_k$, G can be partitioned into k vertex-disjoint cycles $C_1, C_2, \ldots, C_k$ such that $e_i \in E(C_i)$.*

Theorem 7 (Egawa et al. [4]). *Let k be a positive integer, and let G be a graph of order at least $3k$. If $\sigma_2(G) \geq |G| + k$, then for each k independent edges $e_1, e_2, \ldots, e_k$, G can be partitioned into k vertex-disjoint cycles or degenerated cycles $D_1, D_2, \ldots, D_k$, such that $e_i \in E(D_i)$.*

Finally, we show a degree sum condition for Hamiltonicity in directed graphs. Let $D = (V, A)$ be a directed graph (digraph) with vertex set $V(D)$ and arc set $A(D)$. We denote by $\deg_D(v)^+$ and $\deg_D(v)^-$ the out-degree and in-degree of a vertex v in D, respectively. In 1972, Woodall gave a in-degree and out-degree sum condition for Hamiltonicity of digraphs.

Theorem 8 (Woodall [12]). *Let D be a digraph of order at least three. If* $\deg_D(u)^+ + \deg_D(v)^- \geq |D|$ *for $uv \notin A(D)$, then D has a directed Hamiltonian cycle.*

2 Spanning trees with a constraint on leaves

In this section, we show that the results in the previous section imply the results on spanning trees with few leaves and spanning trees with specified leaves.

A Hamiltonian path can be viewed as a spanning tree with two leaves. As a generalization of it, we can consider a spanning tree with few leaves. For $k \geq 2$, a *k-ended tree* is a tree with at most k leaves. On the other hand, a concept of Hamiltonian-connected can be viewed as a spanning tree whose two specified vertices are the leaves. For $k \geq 2$, a graph G is *k-leaf connected* if for each subset S of $V(G)$ with $|S| = k$, G has a spanning tree such that S is the set of the leaves .

Theorem 9 (Broersma and Tuinstra [3], Ore [10, 11], Gurgel and Wakabayashi [7]). *Let k be an integer with $k \geq 2$, and let G be a connected graph.*

(1) *If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k-ended tree.*

(2) *If $\sigma_2(G) \geq |G| - 1$, then G has a Hamiltonian path.*

(3) *If $\sigma_2(G) \geq |G| + 1$, then G is Hamiltonian-connected.*

(4) *If $\sigma_2(G) \geq |G| + k - 1$, then G is k-leaf-connected.*

We prove Theorem 9 (1) by using Theorem 3 or Theorem 5.

First proof of Theorem 9 (1). Let H be a graph obtained from G by adding a vertex v and by joining v and all vertices of G. Then H is 2-connected and $\sigma_2(H) \geq |G| - k + 1 + 2 = |G| - k + 3$. By Theorem 3, H has a cycle of order at least $|G| - k + 3$. Hence G can be partitioned into a path of order at least $|G| - k + 2$ and at most $k - 2$ isolated vertices (see Fig. 1).

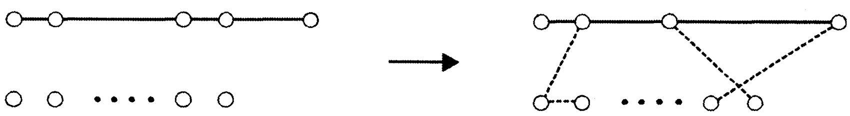

Figure 1: A spanning k-ended tree in the first proof

Since G is connected, we can obtain a spanning k-ended tree from the partition by adding edges connecting the components. □

Second proof of Theorem 9 (1). By Theorem 5, G can be partitioned into k vertex-disjoint cycles or degenerated cycles (see Fig. 2).

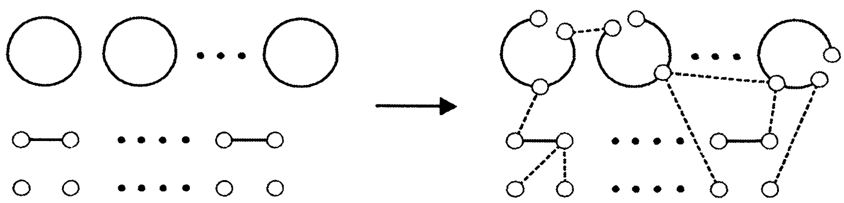

Figure 2: A spanning k-ended tree in the second proof

Since G is connected, we can obtain a spanning k-ended tree from the partition by adding edges connecting components and by deleting one edge of each cycles. □

We prove Theorem 9 (4) by using Theorem 2 or Theorem 7.

First proof of Therorem 9 (4). Let S be a subset of $V(G)$ of order k. We show that G has a spanning tree such that S is the set of the leaves. We construct a graph H obtained from G by adding edges between vertices of S so that $H[S]$ has a Hamiltonian path P, where $H[S]$ is the induced subgraph of H by S. Let u and v be end vertices of P. By the degree sum condition, we have $\sigma_2(H) \geq \sigma_2(G) \geq |G|-k+1 = |H|+|E(P)|$. By Theorem 2, H has a Hamiltonian cycle passing through P (see Fig. 3).

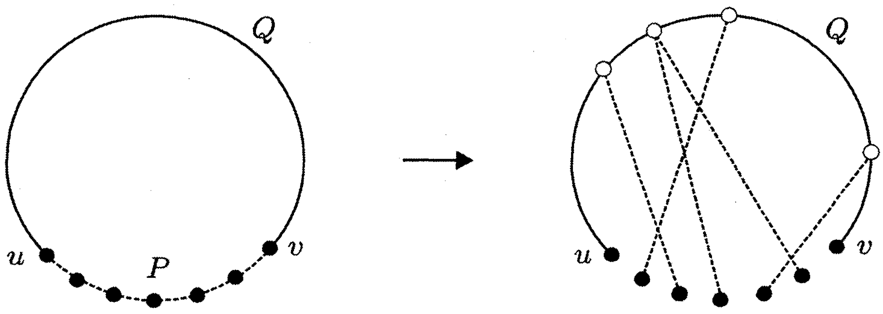

Figure 3: A spanning tree with specified leaves in the first proof

Therefore G has a (u, v)-path Q such that $V(Q) \setminus \{u, v\} = V(G) \setminus S$ and $V(G) \setminus V(Q) = S \setminus \{u, v\}$. Since $\sigma_2(G) \geq |G| + |S| - 1$, G is $(|S| + 1)$-connected. Therefore, each vertex in $S \setminus \{u, v\}$ is adjacent to a vertex of $V(G) \setminus S$, that is, each vertex in $V(G) \setminus V(Q)$ is adjacent to a vertex of $V(Q) \setminus \{u, v\}$. Hence G has a spanning tree such that S is the set of end vertices. □

Second proof of Therorem 9 (4). Let S be a subset of $V(G)$ of order k. We show that G has a spanning tree such that S is the set of the leaves.

Case 1. k is even.

We construct a graph H from G by adding edges between vertices of S so that $H[S]$ has a perfect matching $\{e_1, e_2, \dots, e_{k/2}\}$. Since $k \geq 2$, we have $\sigma_2(H) \geq \sigma_2(G) \geq |G|+k-1 \geq |H|+k/2$. By Theorem 7, H can be partitioned into $k/2$ vertex-disjoint cycles or degenerated cycles $D_1, D_2, \dots, D_{k/2}$ such that $e_i \in E(D_i)$. Therefore G can be partitioned into vertex-disjoint paths such that S is the set of ends.

Since $\sigma_2(G) \geq |G| + |S| - 1$, G is $(|S| + 1)$-connected. Therefore, each vertex of S is adjacent to a vertex of $V(G) \setminus S$. Hence G has a spanning tree such that S is the set of leaves.

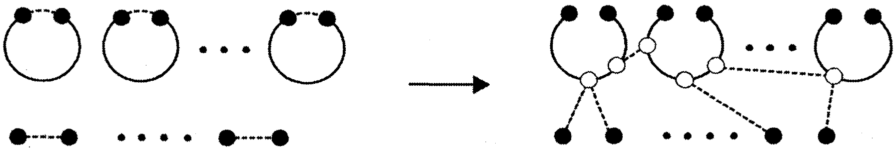

Figure 4: A spanning tree with specified leaves in the second proof

Case 2. k is odd.

Let v be a vertex in S. We construct a graph H from G by deleting a vertex v and by adding edges between vertices of $S \setminus \{v\}$ so that $H[S \setminus \{v\}]$ has a perfect matching $\{e_1, e_2, \ldots, e_{(k-1)/2}\}$. Since $k \geq 3$, we have $\sigma_2(H) \geq \sigma_2(G) - 2 \geq |G| + k - 1 - 2 \geq |H| + (k-1)/2$. By Theorem 7, H can be partitioned into $(k-1)/2$ vertex-disjoint cycles or degenerated cycles $D_1, D_2, \ldots, D_{(k-1)/2}$ such that $e_i \in E(D_i)$. Thus, we can prove in the same way as Case 1. □

3 Partition into vertex-disjoint paths

In this section, we show that the results in Section 1 implies results on a partition into vertex-disjoint paths.

A *k-path system* is a linear forest with at most k components. We can prove the following theorem in the same way as the first proof of Theorem 9 (1).

Theorem 10. *Let k be a positive integer and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a k-path-system.*

Let G be a graph and let $X \subseteq V(G)$. An *X-path* is a path such that the end vertices belong to X and the inner vertices do not belong to X. An *X-path-system* is a graph in which every component is an X-path.

Theorem 11. *Let k be a positive integer, and let G be a graph and let $X \subseteq V(G)$ with $|X| = 2k$. If $\sigma_2(G) \geq |G| + k$, then G has a spanning X-path-system.*

Proof. We construct a graph H from G by adding edges so that $H[X]$ has a perfect matching M. Note that $\sigma_2(H) \geq \sigma_2(G) \geq |G| + k \geq |H| + |M|$. By Theorem 2, H has a Hamiltonian cycle C passing through M (see Fig. 5).

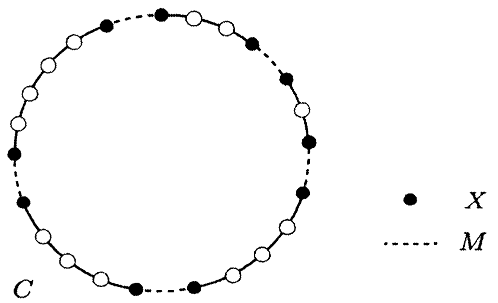

Figure 5: A spanning X-path-system

Then $C - M$ is a spanning X-path-system of G. □

For a graph G and $X, Y \subseteq V(G)$, an *(X,Y)-path* is a path such that one end vertex belongs to X, another end vertex belongs to Y and the inner vertices do not belong to $X \cup Y$. An *(X,Y)-path-system* is a graph in which every component is an (X,Y)-path.

Theorem 12 (Gould and Whalen [6]). *Let k be a positive integer, and let G be a graph of order at least $3k$. Let $X, Y \subseteq V(G)$ with $X \cap Y \neq \emptyset$ and $|X| = |Y| = k$. If $\sigma_2(G) \geq |G| + k$, then G has a spanning (X,Y)-path-system.*

Note that Theorem 11 is a corollary of Theorem 12. We can prove this theorem by using Theorem 8, and moreover we can omit the order condition.

Proof. Let $X = \{x_1, x_2, \ldots, x_k\}$, $Y = \{y_1, y_2, \ldots, y_k\}$ and $Z = V(G) - X - Y$. We construct a digraph D^* from G as follows (see Fig. 6).

(1) Delete edges in $G[X]$ and $G[Y]$.

(2) Replace each edge incident to a vertex x of X with an arc whose tail is x.

(3) Replace each edge incident to a vertex y of Y with an arc whose head is y.

(4) Replace each edge joining vertices z_1 and z_2 of Z with two arcs z_1z_2 and z_2z_1.

(5) Delete an edge x_iy_i, if there exists, for $1 \leq i \leq k$.

We construct a digraph D from D^* by identifying x_i with y_i for $1 \leq i \leq k$.

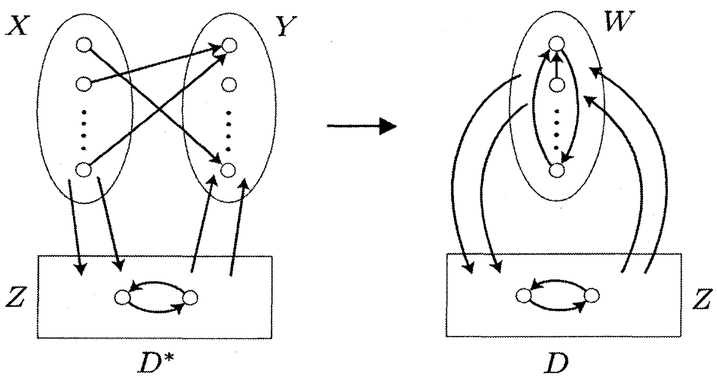

Figure 6: The construction of a digraph D from D^*

Let w_i be the vertex obtained by identifying x_i with y_i for $1 \leq i \leq k$. Let $W = \{w_1, w_2, \ldots, w_k\}$. We now check the out-degree and in-degree of each vertex in D. For $z \in Z$,

$$\deg_D(z)^+ = \deg_G(z) - \deg_X(z) \geq \deg_G(z) - k$$

and

$$\deg_D(z)^- = \deg_G(z) - \deg_Y(z) \geq \deg_G(z) - k.$$

For $w_i \in W$,

$$\deg_D(w_i)^+ = \deg_G(x_i) - \deg_{X \cup \{y_i\}}(x_i) \geq \deg_G(x_i) - k$$

and

$$\deg_D(w_i)^- = \deg_G(y_i) - \deg_{Y\cup\{x_i\}}(y_i) \geq \deg_G(y_i) - k.$$

We next check the out-degree and in-degree sum condition in D. If $z_1z_2 \notin A(D)$ for $z_1, z_2 \in Z$, then $z_1z_2 \notin E(G)$, and hence it follows from the degree sum condition that

$$\begin{aligned}\deg_D(z_1)^+ + \deg_D(z_2)^- &\geq \deg_G(z_1) - k + \deg_G(z_2) - k \\ &\geq |G| + k - 2k = |D|.\end{aligned}$$

If $zw_i \notin A(D)$ for $z \in Z$ and $w_i \in W$, then $zy_i \notin E(G)$, and hence

$$\begin{aligned}\deg_D(z)^+ + \deg_D(w_i)^- &\geq \deg_G(z) - k + \deg_G(y_i) - k \\ &\geq |G| + k - 2k = |D|.\end{aligned}$$

If $w_iz \notin A(D)$ for $w_i \in W$ and $z \in Z$, then $x_iz \notin E(G)$, and so

$$\begin{aligned}\deg_D(w_i)^+ + \deg_D(z)^- &\geq \deg_G(x_i) - k + \deg_G(z) - k \\ &\geq |G| + k - 2k = |D|.\end{aligned}$$

If $w_iw_j \notin A(D)$ for $w_i \in W$ and $w_j \in W$, then $x_iy_j \notin E(G)$, and

$$\begin{aligned}\deg_D(w_i)^+ + \deg_D(w_j)^- &\geq \deg_G(x_i) - k + \deg_G(y_j) - k \\ &\geq |G| + k - 2k = |D|.\end{aligned}$$

Thus, D satisfies the degree sum condition of Theorem 8, and so D has a directed Hamiltonian cycle. By putting w_i back to x_i and y_i, we can obtain spanning k directed vertex-disjoint paths from vertices of X to vertices of Y in D^* (see Fig. 7).

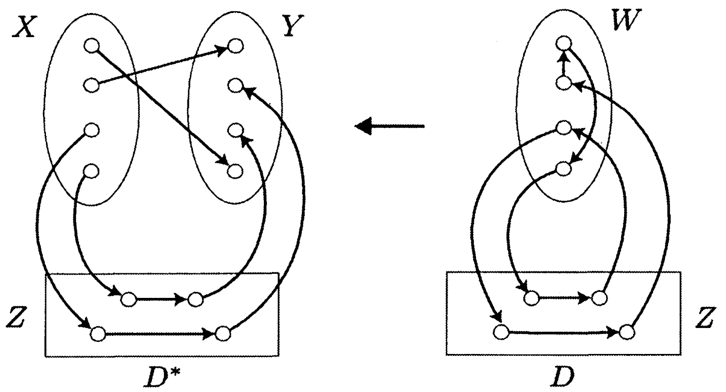

Figure 7: The transform from a directed Hamiltonian cycle in D to spanning k directed vertex-disjoint paths in D^*

Furthermore, by putting the arcs of directed paths back to edges, we can obtain spanning k vertex-disjoint (X, Y)-paths in G. □

4 Conclusion remarks

In this paper, we proved known results on spanning trees and vertex-disjoint paths by using known results on Hamiltonian cycles. As far as I know, these facts were not written anywhere. It is interesting to investigate whether there exist any other results which can be proved by using other results. Also, it is interesting to determine a maximal subgraph which is guaranteed by a degree sum condition.

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