Conditions for $k$-connected graphs to have a contractible edge

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Abstract

An edge of $k$-connected graph is said to be $k$-contractible if the contraction of it results in a $k$-connected graph. When $k \leq 3$, each $k$-connected graph on more then $k + 1$ vertices has a $k$-contractible edge. When $k \geq 4$, there are infinitely many $k$-connected graphs with no $k$-contractible edge for each $k$. Hence, if $k \geq 4$, we can not expect the existence of a $k$-contractible edge in a $k$-connected graph with no condition. In this note, we give a brief survey on sufficient conditions for $k$-connected graphs to have a $k$-contractible edge. We also give some recent new results on this topic.

1 Introduction

In this note, we deal with finite undirected graphs with neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices of $G$ and the set of edges of $G$, respectively. For an edge $e \in E(G)$, let $V(e)$ stand for the set of end vertices of $e$. For a vertex $x \in V(G)$, we write $N_G(x)$ for the neighborhood of $x$. Moreover, for a subset $X \subset V(G)$, let $N_G(X) = \cup_{x \in X} N_G(x) - X$. We denote the degree of $x \in V(G)$ by $deg_G(x)$, namely $deg_G(x) = |N_G(x)|$. We denote the set of vertices of degree $i$ by $V_i(G)$. We denote the minimum degree of $G$ by $\delta(G)$. For a subset $X$ of $V(G)$, the subgraph induced by $X$ is denoted by $G[X]$. If there is no ambiguity, we write $X$ for $G[X]$. For a subgraph $A \subset G$, if there is no ambiguity, we write $A$ for $V(A)$. Let $K_n$, $P_n$ and $C_n$ be the complete graph on $n$ vertices, the path on $n$ vertices and the cycle on $n$ vertices, respectively. Let $G$ and $H$ be two graphs. Let $G \cup H$ and $G + H$ denote the union of $G$ and $H$, and the join of $G$ and $H$, respectively. Let $K_4^{-}$ be the graph obtained from $K_4$ by removing one edge, that is $K_4^- = K_2 + 2K_1$. If $G$ has no $H$ as a subgraph, then $G$ is said to be $H$-off. If $G$ has no $H$ as an induced subgraph, then $G$ is said to be $H$-free. If $H$ is a complete graph, then $H$-off and $H$-free are equivalent. For a connected
graph $G$, a subset $S \subset V(G)$ is said to be a cutset if $G - S$ is disconnected. A cutset $S$ is said to be an $i$-cutset if $|S| = i$.

Let $k$ be an integer such that $k \geq 2$. Let $G$ be a $k$-connected graph and let $e = xy$ be an edge of $G$. We consider the following operation on $G$. Delete both $x$ and $y$, and add new vertex $z$ and join $z$ to each vertex in $N_G(x) \cup N_G(y) - \{x, y\}$. We call this operation contraction of $e = xy$ and we write the resulting graph by $G/e$. An edge $e$ of $G$ is said to be $k$-contractible if the contraction of it results in a $k$-connected graph. If an edge $e$ of $G$ is not contractible, then it is said to be $k$-noncontractible. We observe that an edge $e \in E(G)$ is $k$-noncontractible if and only if there is a $k$-cutset $S$ such that $V(e) \subset S$. A $k$-connected graph $G$ is said to be contraction critically $k$-connected if $G$ has no $k$-contractible edge. A $k$-cutset $S$ is said to be trivial if there is a vertex $x \in V_k(G)$ such that $S = N_G(x)$.

![Fig.1: Contractible edge](image)

In Fig. 1, $G$ is 3-connected, $e$ is a 3-noncontractible edge of $G$ and $f$ is a 3-contractible edge of $G$. If $k \geq 4$, then we cannot expect the existence of a contractible edge in a $k$-connected graph with no condition. We consider conditions for a $k$-connected graph to have a $k$-contractible edge. We say ‘$k$-sufficient condition’ for a condition for a $k$-connected graph to have a $k$-contractible edge.

## 2 Degree $k$-sufficient conditions

A fragment $A$ is a non-empty union of components of $G - S$, where $S$ is a $k$-cutset of $G$ for which $V(G) - (A \cup S) \neq \emptyset$. Mader [11], [12] showed that every contraction critically $k$-connected graph has a fragment $A$ whose neighbourhood contains an edge for which $|A| \leq \frac{1}{4}(k - 1)$.

By generalizing methods of Mader, Egawa [4] proved that every contraction critically $k$-connected graph has a fragment $A$ such that $|A| \leq \frac{1}{4}k$, and gave the following minimum-degree $k$-sufficient condition. He also showed that the bound is sharp.

**Theorem 1** Let $k \geq 2$ be an integer, and let $G$ be a $k$-connected graph with $\delta(G) \geq \lfloor \frac{3}{4}k \rfloor$. Then $G$ has a $k$-contractible edge, unless $k = 2$ or $3$ and $G$ is isomorphic to $K_{k+1}$.

Kriesell [8] extended Theorem 1 and proved the following degree-sum $k$-sufficient condition.
Theorem 2 Let $G$ be a $k$-connected graph for which $\deg(v) + \deg(w) \geq 2\lceil \frac{5}{4}k \rceil - 1$, for any pair $v, w$ of distinct vertices of $G$. Then $G$ contains a $k$-contractible edge.

Solving a conjecture in [8], Su and Yuan [13] proved the following stronger result.

Theorem 3 Let $G$ be a contraction-critical $k$-connected graph with $k \geq 8$. Then $G$ has two adjacent vertices $v, w$ for which $\deg(v) + \deg(w) \leq 2\lceil \frac{5}{4}k \rceil - 2$.

The following interesting result is also due to Kriesell [10].

Theorem 4 For every $k \geq 1$, there exists a number $f(k)$ for which every $k$-connected graph with average degree at least $f(k)$ has a $k$-contractible edge.

In [10], he showed that $f(k) \leq ck^2 \log k$ for some constant $c$, and posed the following conjecture.

Conjecture A There exists a constant $c$ for which every finite $k$-connected graph with average degree at least $ck^2$ has a $k$-contractible edge.

Another problem is to determine the best value of $f(k)$, for a given value of $k$. Up to now, we have no 5-contraction-critical graph whose average degree exceeds $\frac{25}{2}$. Hence we pose the following.

Conjecture B The average degree of every 5-contraction-critical graph is less than $\frac{25}{2}$.

If Conjecture B is true, the bound $\frac{25}{2}$ is sharp. We construct a series of 5-connected graphs whose average degree tend to $\frac{25}{2}$. To construct the 5-connected graphs, we introduce $K_4^-$-configuration. Let $S = \{a_1, a_2, v, b_1, b_2\}$ be a 5-cutset of a 5-connected graph $G$, and let $A$ be a component of $G - S$ such that $V(A) \subseteq V_6(G)$, $|V(A)| = 4$, and $G[A] \cong K_4^-$ say, $A = \{x_1, x_2, y_1, y_2\}$, with edges within $A$ and between $A$ and $S$ exactly as in Fig. 2; there may be edges between the vertices of $S$. We call this configuration, $G[V(A) \cup S]$, a $K_4^-$-configuration with centre $v$. Note that $\{x_1, x_2, y_1, y_2\} \subseteq V_5(G)$ and that the edges in Fig. 2 other than $vx_1$ and $vy_1$ are all trivially 5-noncontractible. Moreover, we can find two nontrivial 5-cutsets, $\{x_1, x_2, v, b_1, b_2\}$ and $\{y_1, y_2, v, a_1, a_2\}$ that contain $V(vx_1)$ and $V(vy_1)$, respectively. Hence all edges in Fig. 2 are 5-noncontractible. Note finally that if there is an edge between vertices of $S$, then it is 5-noncontractible, since $S$ is a 5-cutset of $G$.

![Fig. 2. A $K_4^-$-configuration with centre $x$](image-url)
Let \( \ell \) be a suitable integer which is divisible by 10. Then, by the results due to Hanani (1961), we can find \( \ell (\ell - 1) / 20 \) many copies of \( K_5 \)'s whose edge sets partitions \( E(K_5) \). We apply \( K_4^- \)-attaching on each \( K_5 \) and let \( G_\ell \) be the resulting graph. Then \( G_\ell \) is contraction-critically 5-connected. Let \( \xi \) denote the number of \( K_4^- \)-attachings in \( G_\ell \). Then we observe that \( \xi = \frac{1}{10} \ell (\ell - 1) \approx \frac{\ell^2}{20} \) and \( |V(G_\ell)| = 4 \times \xi + \ell \approx \frac{\ell^2}{5} \). Moreover we observe that \( |E(G_\ell)| = |E(K_5)| + 15 \times \xi \approx \frac{5\ell^2}{4} \). Hence we have \( \lim_{\ell \to \infty} \text{Average degree of } G_\ell = \lim_{\ell \to \infty} \frac{2|E(G_\ell)|}{|V(G_\ell)|} = \frac{25}{2} \). This series of graphs shows that \( f(5) \) does not exceed the value \( \frac{25}{2} \).

3 Forbidden-subgraph \( k \)-sufficient conditions

From Mader’s result that every contraction critically \( k \)-connected graph has a fragment \( A \) whose neighbourhood contains an edge for which \( |A| \leq \frac{1}{2} (k - 1) \), we can see that ‘triangle-free’ is a \( k \)-sufficient condition; Thomassen [14] pointed out this condition.

**Theorem 5** Every \( k \)-connected triangle-free graph has a \( k \)-contractible edge.

The following result due to Egawa, Enomoto and Saito [5] gives a bound on the number of \( k \)-contractible edges in a \( k \)-connected triangle-free graph.

**Theorem 6** Every \( k \)-connected triangle-free graph with \( n \) vertices and \( m \) edges contains \( \min \{n + \frac{3}{2} k^2 - 3k, m\} \) \( k \)-contractible edges.

From Theorem 5 we know that every contraction critically \( k \)-connected graph has triangles. The following result due to Kriesell [9] is a substantial improvement on a bound \( \frac{2}{3} n \) found by Mader [12].

**Theorem 7** Let \( G \) be a \( k \)-connected graph of order \( n \) with no contractible edges. Then \( G \) contains at least \( \frac{2}{3} n \) triangles.

In view of Theorem 6, a \( k \)-connected ‘triangle-free’ graph has many \( k \)-contractible edges, indicating the possible existence of a weaker \( k \)-sufficient condition involving forbidden subgraphs.

In this direction, Kawarabayashi [7] showed the following.

**Theorem 8** For an odd integer \( k \geq 3 \), every \( K_1 + (K_2 \cup P_3) \)-off \( k \)-connected graph has a \( k \)-contractible edge.

Since \( K_1 + (K_2 \cup P_3) \) contains \( K_3 \), Theorem 8 is an extension of Theorem 5 when \( k \) is odd.

Recall \( K_4^- = K_2 + 2K_1 \). We call the graph \( K_1 + 2K_2 \) a bowtie.

Ando, Kaneko, Kawarabayashi and Yoshimoto [2] proved that every \( k \)-connected bowtie-off graph has a \( k \)-contractible edge.

**Theorem 9** Every \( k \)-connected bowtie-off graph has a \( k \)-contractible edge.
Theorem 9 is also an extension of Theorem 5. Since $K_1 + P_4$ contains a bowtie, the following Theorem 10 is an extension of Theorem 9 (see [6]).

**Theorem 10** Let $k \geq 5$, and let $G$ be a $k$-connected $(K_1 + P_4)$-off graph. If $G[V_k(G)]$ is bowtie-off, then $G$ has a $k$-contractible edge.

The following Theorem 11 is another extension of Theorem 5 (see [3]). Note that if $s = t = 1$, then Theorem 11 is equivalent to Theorem 5. Also note that $'K_4^- \cong K_2 + 2K_1'$ and $'bowtie \cong K_1 + 2K_2'$; hence $K_2 + sK_1$ and $K_1 + tK_2$ may be regarded as a generalized $K_4^-$ and a generalized bowtie, respectively.

**Theorem 11** For $k \geq 5$, take two positive integers $s$ and $t$ with $s(t - 1) < k$. If a $k$-connected graph $G$ contains neither $K_2 + sK_1$ nor $K_1 + tK_2$, then $G$ contains a $k$-contractible edge.

We cannot replace the condition $s(t - 1) < k$ by $s(t - 1) \leq k$ in Theorem 11.

## 4 Some $k$-sufficient conditions involving degree and forbidden-subgraph

Theorems 5, 10 and 11 deal with forbidden-subgraph $k$-sufficient conditions. On the other hand, Theorem 1 gives a minimum-degree $k$-sufficient condition. However, if we restrict ourselves to a class of graphs that satisfy some forbidden-subgraph conditions, then we may relax the minimum-degree bound in Theorem 1. The following forbidden-subgraph condition relaxes the minimum-degree bound (see [3]). Let $K_5^-$ be the graph obtained from $K_5$ by removing one edge.

**Theorem 12** For $k \geq 5$, let $G$ be a $k$-connected graph which contains neither $K_5^-$ nor $5K_1 + P_3$. If $\delta(G) \geq k + 1$, then $G$ has a $k$-contractible edge.

Note that if $k \geq 5$, then $\lfloor \frac{5}{k}k \rfloor \geq k + 1$. Since there is a $k$-regular contraction critically $k$-connected graph which contains neither $K_5^-$ nor $5K_1 + P_3$, we cannot replace $\delta(G) \geq k + 1$ by $\delta(G) \geq k$ in Theorem 12. In this sense, the minimum-degree bound in Theorem 12 is sharp.

Yang and Sun [15] present the following as a counter part of a Kawarabayashi’s result.
Theorem 13 Let $G$ be a $K_{\overline{4}}$-off $k$-connected graph with $k \geq 5$. If $V_k(G)$ is independent, then $G$ has a $k$-contractible edge.

The degree condition of Theorem 13, 'V_k(G) is independent' is equivalent to the condition that $d_G(W) \geq 2k + 1$ for any connected subgraph $W$ of $G$ with $|W| = 2'$. Here we consider the more general degree condition that $d_G(W) \geq f(k)$ for any connected subgraph $W$ of $G$ with $|W| = m$.

Using a condition of this type, we have the following (see [1]).

Theorem 14 Let $G$ be a $k$-connected graph with $k \geq 5$ having neither $K_1 + C_4$ nor $K_2 + (K_1 \cup K_2)$. If $d_G(W) \geq 3k + 1$ for any connected subgraph $W$ of $G$ with $|W| = 3$, then $G$ has a $k$-contractible edge.

Note that both $K_1 + C_4$ and $K_2 + (K_1 \cup K_2)$ contains $K_{\overline{4}}$. And the degree condition $d_G(W) \geq 3k + 1$ for any connected subgraph $W$ of $G$ with $|W| = 3'$ is much weaker than the degree condition that 'V_k(G) is independent'. Hence Theorem 14 extends both degree and forbidden-subgraph conditions of Theorem 13.

Yang and Sun [16] also present the following result.

Theorem 15 Let $k$ be an integer such that $k \geq 5$, and let $G$ be a $(K_1 + C_4)$-off $k$-connected graph. If $d_G(W) \geq 2k + 2$ for any connected subgraph $W$ of $G$ with $|W| = 2$, then $G$ has a $k$-contractible edge.

We think that we can relax the degree bound of Theorem 15 by one.
**Conjecture C** Let $k$ be an integer such that $k \geq 5$, and let $G$ be a $(K_1 + C_4)$-off $k$-connected graph. If $d_G(W) \geq 2k + 1$ for any connected subgraph $W$ of $G$ with $|W| = 2$, then $G$ has a $k$-contractible edge.

The conjecture C is still open, however we obtain the following weaker result, which is an extension of Theorem 15.

**Theorem 16** Let $k$ be an integer such that $k \geq 5$, and let $G$ be a $(K_1 + C_4)$-off $k$-connected graph. If $d_G(W) \geq 3k + 2$ for any connected subgraph $W$ of $G$ with $|W| = 3$, then $G$ has a $k$-contractible edge.

**References**

[1] K. Ando, Some degree and forbidden subgraph conditions for a graph to have a $k$-contractible edge, *Discrete Math.* in print.


