THE ITERATED REMAINDERS OF THE RATIONALS

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ABSTRACT. Repeat taking remainders of Stone-Čech compactifications of the rationals

\[ Q^{(1)} = Q^* = \beta Q \backslash Q, \quad Q^{(2)} = \beta Q^{(1)} \backslash Q^{(1)}, \quad Q^{(3)} = \beta Q^{(2)} \backslash Q^{(2)}, \quad Q^{(4)} \quad \ldots. \]

We point out that they have similar structures, but, are topologically different. In particular we prove here that \( Q^{(1)} \not\approx Q^{(3)} \). This result will be generalized to show that \( Q^{(n)} \not\approx Q^{(n+2)} \) for any \( n \geq 1 \) in the forthcoming paper [4].

1. INTRODUCTION

Consider the space of rationals \( \mathbb{Q} \), and repeat taking its remainders of Stone-Čech compactifications \( \mathbb{Q}^{(n+1)} = (\mathbb{Q}^{(n)})^* = \beta \mathbb{Q}^{(n)} \backslash \mathbb{Q}^{(n)} \) (\( n \geq 0 \)) where \( \mathbb{Q}^{(0)} = \mathbb{Q} \), i.e.,

\[ Q^{(1)} = Q^*, \quad Q^{(2)} = Q^{**}, \quad Q^{(3)} = Q^{***}, \ldots. \]

Van Douwen [2] asked whether or not \( Q^{(n)} \approx Q^{(n+2)} \) for \( n \geq 1 \), remarking that \( Q^{(m)} \) for even \( m \) is never homeomorphic to \( Q^{(n)} \) for odd \( n \), because the former is \( \sigma \)-compact but the latter is not.

In this paper we point out that both \( Q^{(n)} \) and \( Q^{(n+2)} \) have a similar structure of “fiber bundle” for every \( n \geq 1 \), but they are topologically different. In particular we here show that \( Q^{(1)} \not\approx Q^{(3)} \), which we can generalize in the forthcoming paper [4] to show that \( Q^{(n)} \not\approx Q^{(n+2)} \) for any \( n \geq 1 \), answering van Douwen’s question.

The precise connections of the remainders can be seen by the following construction. Viewing \( \beta \mathbb{Q} \) as a compactification of \( Q^{(1)} \), let

\[ \Phi_0 : \beta Q^{(1)} = Q^{(1)} \cup Q^{(2)} \to Q \cup Q^{(1)} = \beta Q \]

be the Stone extension of the identity map \( id : Q^{(1)} \to Q^{(1)} \). Denote by

\[ \phi_0 : Q^{(2)} \to Q^{(0)} \]

the restriction of \( \Phi_0 \). Next let

\[ \Phi_1 : \beta Q^{(2)} = Q^{(2)} \cup Q^{(3)} \to Q^{(1)} \cup Q^{(2)} = \beta Q^{(1)} \]

be the Stone extension of the identity map \( id : Q^{(2)} \to Q^{(2)} \), and let

\[ \phi_1 : Q^{(3)} \to Q^{(1)} \]

2000 Mathematics Subject Classification. 54C45, 54C10.

Key words and phrases. Stone-Čech compactification, \( C^* \)-embedded.
denote the restriction of $\Phi_1$. In this way, for every $n \geq 0$ we can generally get the Stone extension

$$\Phi_n : \beta \mathbb{Q}^{(n+1)} = \mathbb{Q}^{(n+1)} \cup \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} = \beta \mathbb{Q}^{(n)}$$

of the identity map $id : \mathbb{Q}^{(n+1)} \rightarrow \mathbb{Q}^{(n+1)}$, and its restriction map

$$\phi_n : \mathbb{Q}^{(n+2)} \rightarrow \mathbb{Q}^{(n)}.$$

Since every $\Phi_n (n \in \omega)$ is perfect, so is every $\phi_n$. Hence every $\mathbb{Q}^{(n)} (n \in \omega)$ is Lindelöf since both $\mathbb{Q}^{(0)} = \mathbb{Q}, \mathbb{Q}^{(1)}$ are Lindelöf. We can also see that $\mathbb{Q}^{(n)}$ is $\sigma$-compact for even $n$, but $\mathbb{Q}^{(n)}$ is not for odd $n$, because $\mathbb{Q}^{(0)}$ is $\sigma$-compact but $\mathbb{Q}^{(1)}$ is not since $\mathbb{Q}^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ as we see below.

$$\begin{align*}
\mathbb{Q}^{(0)} & \rightarrow \Phi_0 \rightarrow \mathbb{Q}^{(1)} & \Phi_1 \rightarrow \mathbb{Q}^{(2)} & \Phi_2 \rightarrow \mathbb{Q}^{(3)} \\
\Phi_0 & \rightarrow \mathbb{Q}^{(0)} & \Phi_1 & \rightarrow \mathbb{Q}^{(1)} & \Phi_2 & \rightarrow \mathbb{Q}^{(2)} & \Phi_3 & \rightarrow \mathbb{Q}^{(3)} \\
\phi_0 & \rightarrow \mathbb{Q}^{(0)} & \phi_1 & \rightarrow \mathbb{Q}^{(1)} & \phi_2 & \rightarrow \mathbb{Q}^{(2)} & \phi_3 & \rightarrow \mathbb{Q}^{(3)}
\end{align*}$$

**FIG. 1**

A collection $B$ of nonempty open sets of $X$ is called a $\pi$-base for $X$ if every nonempty open set in $X$ includes some member of $B$. The minimal cardinality of such a $\pi$-base is called the $\pi$-weight of $X$. Note that any dense subspace of $X$ has the same $\pi$-weight as $X$, and any space of countable $\pi$-weight is separable. Consequently, any dense subset of a space of countable $\pi$-weight is also of countable $\pi$-weight, and hence separable. So, all of $\beta \mathbb{Q}^{(n)}, \mathbb{Q}^{(n)} (n \in \omega)$ are of countable $\pi$-weight, and hence separable.

Recall that an onto map $g : X \rightarrow Y$ is called irreducible if every nonempty open subset $U$ of $X$ includes some fiber $g^{-1}(y)$, and it is well known and easy to see that

(1) every extension of a homeomorphism is irreducible, and

(2) the restriction of a closed irreducible map to any dense subset is irreducible.
Therefore we can see that all of the maps $\Phi_n, \phi_n$ ($n \in \omega$) are perfect irreducible. Consider the partition of the closed interval $[0,1] = Q \cup P$ where

$$Q = [0,1] \cap Q \approx Q \text{ and } P = [0,1]\setminus Q \approx P,$$

and let $f : \beta Q \to [0,1]$ be the Stone extension of the homeomorphism $Q \approx Q$. Then the restriction $f_0 = f \upharpoonright Q^{(1)} : Q^{(1)} \to P \approx P$ is perfect irreducible. Thus we get the following sequence of perfect irreducible maps:

$$Q \leftarrow Q^{(2)} \leftarrow Q^{(4)} \leftarrow \cdots ; \quad P \leftarrow Q^{(1)} \leftarrow Q^{(3)} \leftarrow Q^{(5)} \leftarrow \cdots.$$

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union." For a subset $A$ of some compact space $K$ we use the notation $A^*$ to denote the remainder $c_K A \setminus A$ when $K$ is clear from the context. Our terminologies are based upon [3].

2. Similar Structures

We first show that both $Q^{(n)}$ and $Q^{(n+2)}$ have a similar structure for every $n \geq 1$. In general, for any space $Y$ let us denote by $H(Y)$ the collection of all homeomorphisms $h : Y \approx Y$. Let $X$ be a nowhere compact, dense-in-itself space, where nowhere compact (or nowhere locally compact) means that $X$ contains no compact neighborhood, or equivalently, that $X$ is a dense subset of some/any compact space $K$ such that the remainder $K \setminus X$ is also dense in $K$. Let $cX$ be some compactification of $X$ and let $\mathcal{H}_* \subseteq H(X)$ denote the collection of all $h \in H(X)$ such that

\[(*) \quad h \text{ is extendable to } c(h) \in H(cX).\]

(Of course, $\mathcal{H}_* = H(cX)$ if $cX = \beta X$.) Let $X^{(1)} = cX \setminus X$ be the remainder, and for every $h \in \mathcal{H}_*$ define $h^{(1)} \in H(X^{(1)})$ to be the restriction of $c(h)$ to $X^{(1)}$. Next consider the Stone-Čech compactification $\beta X^{(1)}$ of $X^{(1)}$ and the Stone extension $\beta h^{(1)} \in H(\beta X^{(1)})$ of $h^{(1)}$. Let $X^{(2)} = \beta X^{(1)} \setminus X^{(1)}$ be the remainder, and define $h^{(2)} \in H(X^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $X^{(2)}$; hence

$$h : X \approx X, \quad h^{(1)} : X^{(1)} \approx X^{(1)}, \quad h^{(2)} : X^{(2)} \approx X^{(2)}.$$

Note that $X^{(1)}$ is dense in $\beta X$, and $X^{(2)}$ is dense in $\beta X^{(1)}$, since we assume that $X$ is nowhere compact. Viewing that $\beta X$ is a compactification of $X^{(1)}$, we can consider the Stone extension $\Phi : \beta X^{(1)} \to \beta X$ of the identity map $id_{X^{(1)}} : X^{(1)} = X^{(1)}$. Let $\phi : X^{(2)} \to X$ be the restriction of $\Phi$. Then both $\Phi$ and $\phi$ are perfect irreducible maps. We can show that the correspondence $H(X) \supseteq \mathcal{H}_* \ni h \mapsto h^{(2)} \in H(X^{(2)})$ is compatible with the perfect irreducible map $\phi$, i.e.,

Lemma 2.1. $h \circ \phi = \phi \circ h^{(2)} : X^{(2)} \to X$. 

Proof. To show this equality, it suffices to prove the equality
\[ c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta X^{(1)} \to cX, \]
which follows from the obvious equality
\[ h^{(1)} \circ id_{X^{(1)}} = id_{X^{(1)}} \circ h^{(1)} : X^{(1)} \to X^{(1)} \]
on the dense subset $X^{(1)}$ of $\beta X^{(1)}$. \qed

Corollary 2.2. If $h(x) = y$ for $x, y \in X$, then $h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y)$.

Proof. The inclusion $h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y)$ follows from 2.1. Since $h$ is a homeomorphism, we can replace $h$ by $h^{-1}$ to get the reverse inclusion. \qed

Taking $X = \mathbb{Q}$, $cX = \beta \mathbb{Q}$, $\mathcal{H}_\star = H(\mathbb{Q})$, we can deduce from 2.1 that
\begin{equation}
(2-1) \quad h \circ \phi_0 = \phi_0 \circ h^{(2)} : \mathbb{Q}^{(2)} \to \mathbb{Q} \quad \text{for every } h \in H(\mathbb{Q}).
\end{equation}
Let $[0, 1] = Q \cup P$, $Q \approx \mathbb{Q}$, $P \approx \mathbb{P}$ be as at the end of §1, and take $X = P$, $cX = [0, 1]$; then $X^{(1)} = Q$, $X^{(2)} = Q^{(1)}$, and the corresponding map $\phi$ in Fig. 2 is identical to the map $f_0 : Q^{(1)} \to P$ at the end of §1. Note that $\mathcal{H}_\star \subseteq H(P)$ is the collection of all homeomorphisms of $P$ extendable to homeomorphisms of $[0, 1]$. Then we can deduce from 2.1 that
\begin{equation}
(2-2) \quad h \circ f_0 = f_0 \circ h^{(2)} : Q^{(1)} \to P \quad \text{for every } h \in \mathcal{H}_\star.
\end{equation}
Note that for every pair of irrationals $p_1 < p_2$ in $P = [0, 1] \setminus \mathbb{Q}$ we can find an $h \in \mathcal{H}_\star$ such that $h(p_1) = p_2$; for example, we can take as $c(h)$ in (⋆) a strictly increasing function $c(h) : [0, 1] \to [0, 1]$ such that $c(h)(Q) = Q$, $c(h)(0) = 0$, $c(h)(p_1) = p_2$, $c(h)(1) = 1$. For $m \geq 1$ define $g_{2m}$ and $f_{2m-1}$ by
\[ g_{2m} = \phi_0 \circ \phi_2 \circ \cdots \circ \phi_{2m-2} : \mathbb{Q}^{(2m)} \to \mathbb{Q}, \]
\[ f_{2m-1} = f_0 \circ \phi_1 \circ \phi_3 \circ \cdots \circ \phi_{2m-3} : \mathbb{Q}^{(2m-1)} \to P. \]
Then, using 2.1 we can extend the above (2-1), (2-2) to the followings, respectively, for \( m \geq 1 \).

\[
(2-3) \quad h \circ g_{2m} = g_{2m} \circ h^{(2m)} : \mathbb{Q}^{(2m)} \to \mathbb{Q} \quad \text{for every } h \in H(\mathbb{Q}),
\]

\[
(2-4) \quad h \circ f_{2m-1} = f_{2m-1} \circ h^{(2m-1)} : \mathbb{Q}^{(2m-1)} \to P \quad \text{for every } h \in H_{*},
\]

where \( h^{(n)} \in H(\mathbb{Q}(n)) \). Combining these results with 2.2 we can summarize that

**Theorem 2.3.** Let \( m \geq 1 \). Then every \( \mathbb{Q}^{(2m)} \) admits a perfect irreducible projection \( g_{2m} \) onto \( \mathbb{Q} \), and every \( \mathbb{Q}^{(2m-1)} \) admits a perfect irreducible projection \( f_{2m-1} \) onto \( P \approx \mathbb{P} \), with the additional property that they are "fiberwise" homogeneous in the following sense:

1. For any \( q_{1} < q_{2} \in \mathbb{Q} \) there exists a homeomorphism of \( \mathbb{Q}^{(2m)} \), induced by a homeomorphism of \( \mathbb{Q} \), carrying the fiber \( g_{2m}^{-1}(q_{1}) \) to \( g_{2m}^{-1}(q_{2}) \).
2. For any \( p_{1} < p_{2} \in P \) there exists a homeomorphism of \( \mathbb{Q}^{(2m-1)} \), induced by a homeomorphism of \( P \), carrying the fiber \( f_{2m-1}^{-1}(p_{1}) \) to \( f_{2m-1}^{-1}(p_{2}) \).

Moreover, under CH (=the Continuum Hypothesis) every fiber \( g_{2m}^{-1}(q) \) of \( q \in \mathbb{Q} \) as well as every fiber \( f_{2m-1}^{-1}(p) \) of \( p \in P \) is homeomorphic to \( \omega^{*} = \beta \omega \backslash \omega \).

This last assertion follows from the well-known

**Fact 2.4.** (see 1.2.6 in [8] or 3.37 in [9]) (CH) Let \( Y \) be a 0-dimensional, locally compact, \( \sigma \)-compact, non-compact space of weight at most \( c \). Then \( Y^{*} = \beta Y \backslash Y \) and \( \omega^{*} \) are homeomorphic.

Indeed, put \( Z = g_{2m}^{-1}(q) \) and \( Y = \beta \mathbb{Q}^{(2m-1)} \backslash Z \). Then \( Z \) is a zero-set of the 0-dimensional \( \beta \mathbb{Q}^{(2m-1)} \) included in the remainder \( \mathbb{Q}^{(2m)} = \beta \mathbb{Q}^{(2m-1)} \backslash \mathbb{Q}^{(2m-1)} \), so that \( Y^{*} = \beta Y \backslash Y = Z \). Since \( Y \) is a cozero-set and separable, \( Y \) satisfies the condition in 2.4. Hence \( Z \approx \omega^{*} \). Similarly we can prove that \( f_{2m-1}^{-1}(p) \approx \omega^{*} \).

3. Remote Points and Extremally Disconnected Points

To analyze further the structure of \( \mathbb{Q}^{(n)}'s \), we need the notion of remote points and extremally disconnected points. A point \( p \in \beta X \backslash X \) is called a remote point of \( X \) if \( p \notin \text{cl}_{\beta X} F \) for every nowhere dense closed subset \( F \) of \( X \). Van Douwen [2], Chae, Smith [1], showed

**Fact 3.1.** Every non-pseudocompact space of countable \( \pi \)-weight has \( 2^{c} \) many remote points.

An easy consequence of this fact is

**Fact 3.2.** Let \( X \) be a non-compact, Lindelöf space of countable \( \pi \)-weight. Then remote points of \( X \) form a \( G_{\delta} \)-dense subset of \( X^{*} = \beta X \backslash X \).
Proof. Choose any point \( p \in X^* \) and a zero-set \( Z \) of \( \beta X \) containing \( p \). Since \( X \) is Lindelöf, we can suppose that \( Z \) misses \( X \). Put \( Y = \beta X \setminus Z \); then \( \beta Y = \beta X \), and \( Y \) is of countable \( \pi \)-weight since \( X \) is. Hence 3.1 implies that \( Y^* = Z \) contains remote points of \( Y \), which are also remote points of \( X \). \( \square \)

A space \( T \) is said to be \textit{extremally disconnected at a point} \( p \in T \) (see [2]) if \( p \notin cl_T U_1 \cap cl_T U_2 \) for every pair of disjoint open sets \( U_1, U_2 \) in \( T \). Let us call such a point \( p \) as an \textit{extremally disconnected point} of \( T \), or simply, an \( e.d. \) point of \( T \), and denote the set of all such \( e.d. \) points by \( Ed(T) \). A space \( T \) is \textit{extremally disconnected} if every point of \( T \) is an \( e.d. \) point, i.e., \( Ed(T) = T \). If \( S \) is dense in \( T \), we always have \( cl_T U = cl_T (U \cap S) \) for every open set \( U \) of \( T \); hence a point \( p \in S \) is an \( e.d. \) point of \( S \) if and only if it is an \( e.d. \) point of \( T \), i.e., \( Ed(S) = S \cap Ed(T) \).

**Fact 3.3.** ([2])

1. Any remote point of \( X \) is an \( e.d. \) point of \( \beta X \).
2. Suppose \( X \) is first countable and hereditarily separable, and \( p \in \beta X \setminus X \). Then \( p \) is a remote point of \( X \) if and only if \( p \) is an \( e.d. \) point of \( \beta X \).

Let us call a point \( p \in T \) a \textit{common boundary} point of \( T \) if \( p \) is not an \( e.d. \) point of \( T \), i.e., if \( p \in cl_T U_1 \cap cl_T U_2 \) for some pair of disjoint open sets \( U_1, U_2 \) in \( T \). Similarly, we call a subset \( A \subseteq T \) a \textit{common boundary} set in \( T \) if \( A \subseteq cl_T U_1 \cap cl_T U_2 \) for some pair of disjoint open sets \( U_1, U_2 \) in \( T \). We abbreviate “common boundary” to “co-boundary.” (Such \( p, A \) are called “2-point,” “2-set,” respectively, in [2].) Note that any co-boundary set in \( T \) is nowhere dense in \( T \), but the converse need not be true. Let \( Cob(T) = T \setminus Ed(T) \) denote the set of all co-boundary points of \( T \). Note also that if \( A \) is a co-boundary set, then every point of \( A \) is obviously a co-boundary point, but the converse need not be true except the case \( A \) is a countable discrete subset:

**Lemma 3.4.** Suppose \( A \) is a countable discrete subset consisting of co-boundary points of \( T \). Then \( A \), and hence also \( cl_T A \), is a co-boundary set in \( T \). Therefore, if \( T \) is compact, \( Cob(T) \) is always countably compact in the strong sense that every countable discrete subset has compact closure in \( Cob(T) \).

**Proof.** Let \( A = \{ a_n \}_{n \in \omega} \subseteq Cob(T) \) be discrete in \( T \), and choose disjoint open sets \( \{ W_n \}_{n \in \omega} \) in \( T \) such that \( a_n \in W_n \). In each \( W_n \), choose disjoint open sets \( U_n, V_n \) with \( a_n \in cl_T U_n \cap cl_T V_n \). Put \( U = \bigcup_{n \in \omega} U_n \) and \( V = \bigcup_{n \in \omega} V_n \). Then these disjoint open sets \( U, V \) satisfy \( A \subseteq cl_T U \cap cl_T V \), and hence \( cl_T A \subseteq cl_T U \cap cl_T V \). \( \square \)

For an open set \( U \subseteq X \) its maximal open extension \( Ex(U) \subseteq \beta X \) is defined by

\[
Ex(U) = \beta X \setminus cl_{\beta X} (X \setminus U).
\]

Suppose \( W \) is an open set in \( \beta X \); then

\[
cl_{\beta X} W = cl_{\beta X} (W \cap X) = cl_{\beta X} Ex(W \cap X).
\]
Therefore we see

**Fact 3.5.** Suppose \( p \in \beta X \setminus X \). Then \( p \) is a co-boundary point of \( \beta X \) if and only if \( p \in \text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) \) for some disjoint open sets \( U, V \) in \( X \).

We denote the boundary of a subset \( W \) in \( Y \) by \( \text{Bd}_Y W \) so that \( \text{Bd}_Y W = \text{cl}_Y W \setminus W \) if \( W \) is open in \( Y \). Van Douwen [2] proved the equality

\[
(\ast) \quad \text{Bd}_{\beta X} \text{Ex}(U) = \text{cl}_{\beta X} \text{Bd}_X(U)
\]

for every open set \( U \) of \( X \). (Note that 3.3 (1) follows from this equality since \( \text{Bd}_X(U) \) is a nowhere dense subset of \( X \).) Using this (\( \ast \)) and 3.5 we get an "inner" characterization of co-boundary points, hence of e.d. points also, of \( \beta X \) for a normal space \( X \):

**Lemma 3.6.** Assume \( X \) is normal, and \( p \in \beta X \setminus X \). Then \( p \) is a co-boundary point of \( \beta X \) if and only if \( p \in \text{cl}_{\beta X} F \) for some co-boundary set \( F \) in \( X \). In other words, \( p \) is an e.d. point of \( \beta X \) if and only if

\[
p \notin \text{cl}_{\beta X} F \text{ for every co-boundary set } F \text{ in } X.
\]

**Proof.** By 3.5 it suffices to show the equality

\[
\text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) = \text{cl}_{\beta X} (\text{cl}_X U \cap \text{cl}_X V)
\]

for disjoint open sets \( U, V \) in \( X \), since \( \text{cl}_X U \cap \text{cl}_X V \) is a co-boundary set in \( X \). Using (\( \ast \)) we get

\[
\text{cl}_{\beta X} \text{Ex}(U) \cap \text{cl}_{\beta X} \text{Ex}(V) = \text{Bd}_{\beta X} \text{Ex}(U) \cap \text{Bd}_{\beta X} \text{Ex}(V) = (\text{cl}_{\beta X} \text{Bd}_X U) \cap (\text{cl}_{\beta X} \text{Bd}_X V).
\]

Since \( X \) is normal, this set is equal to \( \text{cl}_{\beta X} (\text{Bd}_X U \cap \text{Bd}_X V) \), where \( \text{Bd}_X U \cap \text{Bd}_X V = \text{cl}_X U \cap \text{cl}_X V \).

**Lemma 3.7.** Suppose \( A \) is a closed subset of a normal space \( X \). Then \( A \subseteq \text{Ed}(X) \) implies \( \text{cl}_{\beta X} A \subseteq \text{Ed}(\beta X) \).

**Proof.** Let \( A \) be a closed subset of a normal space \( X \), and that \( A \subseteq \text{Ed}(X) \). Let \( F \) be any co-boundary closed set in \( X \). By 3.6 it suffices to show that \( \text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset \). Since \( F \subseteq \text{Cob}(X) \) and \( A \subseteq \text{Ed}(X) \), we know that \( F, A \) are disjoint closed subsets of \( X \). Hence the normality of \( X \) implies that \( \text{cl}_{\beta X} F \cap \text{cl}_{\beta X} A = \emptyset \).

The next lemma shows how co-boundary points or e.d. points behave w.r.t. closed irreducible maps. Let \( g \) be a map from \( X \) onto \( Y \). For a subset \( U \subseteq X \) define \( g^o(U) \subseteq Y \), a small image of \( U \), by

\[
y \in g^o(U) \quad \text{if and only if } \quad g^{-1}(y) \subseteq U,
\]

i.e., \( g^o(U) = Y \setminus g(X \setminus U) \subseteq g(U) \); so, \( g \) is irreducible if \( g^o(U) \neq \emptyset \) for every non-empty open set \( U \). Note an obvious useful formula

\[
g^o(U \cap V) = g^o(U) \cap g^o(V)
\]
for any sets \( U, V \subseteq X \), which especially implies that \( g^o(U) \cap g^o(V) = \emptyset \) whenever \( U \cap V = \emptyset \). Suppose \( g \) is closed irreducible. Then it is well known that \( g^o(U) \) is non-empty and open whenever \( U \) is, and
\[
\text{cl}_Y g^o(U) = \text{cl}_Y g(U) = g(\text{cl}_X U)
\]
for every open subset \( U \subseteq X \).

**Lemma 3.8.** Let \( g : X \to Y \) be any closed irreducible map. Then \( g \) maps co-boundary points to co-boundary points, i.e., \( g(\text{Cob}(X)) \subseteq \text{Cob}(Y) \). Furthermore, for every \( x \in X \)
\[
g(x) \in \text{Cob}(Y) \text{ if and only if } x \in \text{Cob}(X) \text{ or } |g^{-1}(g(x))| > 1, \text{ i.e.,}
\]
g(x) \in \text{Ed}(Y) \text{ if and only if } x \in \text{Ed}(X) \text{ and } g^{-1}(g(x)) = \{x\}.

Consequently, \( g^{-1}(\text{Ed}(Y)) \subseteq \text{Ed}(X) \), and the restriction of \( g \) to
\[
g^{-1}(\text{Ed}(Y)) \to \text{Ed}(Y)
\]
is a homeomorphism.

**Proof.** Let \( U_1, U_2 \) be any disjoint open sets in \( X \). Then
\[
g(\text{cl}_X U_1 \cap \text{cl}_X U_2) \subseteq g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^o(U_1) \cap \text{cl}_Y g^o(U_2),
\]
and \( g^o(U_1) \), \( g^o(U_2) \) are disjoint open. Hence \( g \) maps co-boundary points to co-boundary points. Similarly, we can show that
\[
|g^{-1}(g(x))| > 1 \text{ implies } g(x) \in \text{Cob}(Y).
\]
Indeed, if we take two points \( x_1 \neq x_2 \) in \( g^{-1}(g(x)) \), we can choose disjoint open sets \( U_1, U_2 \) in \( X \) such that \( x_1 \in U_1 \) and \( x_2 \in U_2 \) (using the Hausdorff-ness of \( X \)), getting \( g(x) \in g(\text{cl}_X U_1) \cap g(\text{cl}_X U_2) = \text{cl}_Y g^o(U_1) \cap \text{cl}_Y g^o(U_2) \).

So, to complete our proof, assume \( g(x) \in \text{Cob}(Y) \) and \( |g^{-1}(g(x))| = 1 \); then we need to show \( x \in \text{Cob}(X) \). The condition \( g(x) \in \text{Cob}(Y) \) implies that
\[
g(x) \in \text{cl}_Y V_1 \cap \text{cl}_Y V_2 \text{ for some disjoint open sets } V_1, V_2 \text{ in } Y.
\]
Since \( g \) is a closed map, \( g(x) \in \text{cl}_Y V_i \text{ implies } g^{-1}(g(x)) \cap \text{cl}_X g^{-1}(V_i) \neq \emptyset \text{ for } i = 1, 2 \). Hence the condition \( g^{-1}(g(x)) = \{x\} \) implies \( x \in \text{cl}_X g^{-1}(V_1) \cap \text{cl}_X g^{-1}(V_2) \), showing \( x \in \text{Cob}(X) \). \( \square \)

### 4. Topological Difference of \( Q^{(1)} \) and \( Q^{(3)} \)

Now let us apply the general theory in §3 to our spaces
\[
\beta Q^{(n)} = Q^{(n)} \cup Q^{(n+1)} \quad (n \geq 0).
\]
Recall that every \( Q^{(n)} \) is of countable \( \pi \)-weight and Lindelöf, hence normal. Put \( C_n = \text{Cob}(Q^{(n)}) \) and \( E_n = \text{Ed}(Q^{(n)}) \); then this gives a partition of \( Q^{(n)} \)
\[
Q^{(n)} = C_n \cup E_n.
\]
It is obvious that \( E_0 = \emptyset \), i.e., \( Q^{(0)} = C_0 \). Lemma 3.4 implies that each \( C_n \) \( (n \geq 1) \) is dense in \( Q^{(n)} \), and Fact 3.2 with 3.3 (1) implies that each
$E_n$ ($n \geq 1$) is dense in $\mathbb{Q}^{(n)}$. Note in particular that $E_1$ coincides with the set of all remote points of $\mathbb{Q}$, by 3.3 (2).

**Property 4.1.** Let $A$ be any countable discrete subset of $E_2$ which is closed in $\mathbb{Q}^{(2)}$. Then

1. $\text{cl} \ A \subseteq E_2 \cup C_1$ in $\beta \mathbb{Q}^{(1)}$, while
2. $\text{cl} \ A \subseteq E_2 \cup E_3$ in $\beta \mathbb{Q}^{(2)}$.

**Proof.** (2) follows from 3.7. To prove (1), let $A$ be as above. Then, since $\phi_0 : \mathbb{Q}^{(2)} \rightarrow \mathbb{Q}^{(0)}$ is perfect, $\phi_0(A)$ is also a countable discrete closed subset of $\mathbb{Q}^{(0)} = C_0$. Since $C_0 \cup C_1 = \text{Cob}(\beta \mathbb{Q}^{(0)})$ is countably compact in the strong sense as stated in 3.4, we have $\text{cl} \ \phi_0(A) \subseteq C_0 \cup C_1$ in $\beta \mathbb{Q}^{(0)}$. Pulling back by the map $\Phi_0$, we get $\text{cl} \ A \subseteq E_2 \cup C_1$ in $\beta \mathbb{Q}^{(1)}$. This is the same as the assertion (1) since $A \subseteq E_2$. \hfill $\square$

Now we can prove the following strong assertion which in particular implies that $\mathbb{Q}^{(1)} \not\simeq \mathbb{Q}^{(3)}$.

**Theorem 4.2.** $\mathbb{Q}^{(1)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(3)}$.

**Proof.** Suppose there existed a perfect irreducible map $\psi : \mathbb{Q}^{(1)} \rightarrow \mathbb{Q}^{(3)}$. Then, since $\beta \mathbb{Q}^{(2)}$ can be seen as a compactification of $\mathbb{Q}^{(3)}$, $\psi$ extends to a perfect irreducible map

$$\Psi : \beta \mathbb{Q}^{(1)} = \mathbb{Q}^{(1)} \cup \mathbb{Q}^{(2)} \rightarrow \beta \mathbb{Q}^{(2)} = \mathbb{Q}^{(3)} \cup \mathbb{Q}^{(2)}.$$

Lemma 3.8 implies then that

$$E_2 \cup E_1 \supseteq \Psi^{-1}(E_2 \cup E_3) \approx E_2 \cup E_3.$$

Choose any countable discrete subset $B \subseteq E_2 \subseteq \mathbb{Q}^{(2)} \subseteq \beta \mathbb{Q}^{(2)}$ which is closed in $\mathbb{Q}^{(2)}$. (We can do this because $E_2$ is dense in $\mathbb{Q}^{(2)}$, and $\mathbb{Q}^{(2)}$ is
Lindelöf.) Put $A = \Psi^{-1}(B)$, then this $A$ is also a countable discrete subset of $E_2$ which is closed in $\mathbb{Q}^{(2)}$. Property 4.1 (2) shows $\text{cl } B \subseteq E_2 \cup E_3$ in $\beta \mathbb{Q}^{(2)}$, and so, pulling back by $\Psi$, we get

$$\text{cl } A \subseteq \Psi^{-1}(E_2 \cup E_3) \subseteq E_2 \cup E_1$$

in $\beta \mathbb{Q}^{(1)}$. But this contradicts 4.1 (1). \hfill \Box

We will be able to show in [4] that for any $n \geq 1$, $\mathbb{Q}^{(n)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(n+2)}$ by analyzing further the behavior of limit points of countable discrete subsets in $\mathbb{Q}^{(m)}$. Some of the basic techniques in this paper can be found also in [5, 6, 7].

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