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On non σ-shortness of Axiom A posets with frame systems

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1 Introduction

In [4], we introduced the notion of σ-shortness for Booolean algebras and partially ordered sets. We say that a subset D of a Boolean algebra $\mathbb{B}$ is σ-short if every strictly descending sequence $\{b_n\}$ has no positive lower bound in $\mathbb{B}$. A Boolean algebra $\mathbb{B}$ is σ-short if it has a σ-short dense subset. Cohen algebras and measure algebras are typical examples of σ-short Boolean algebras. Cohen algebras satisfy more strong property. That is, those are strongly σ-short. We say that $\mathbb{B}$ is strongly σ-short if it has a $\wedge$-closed σ-short dense subset. σ-short posets and strongly σ-short posets are similarly defined as Boolean algebras(see [4]). Strongly σ-short posets have a good characterization as follows.

Theorem[2]. *A poset $\mathbb{P} = (P, \leq)$ is strongly σ-short if and only if there exists a sequence $\{X_n\}_{n\in\omega}$ of subsets of P which satisfies the following conditions:*

(1) X_n is a pairwise incomparable subset of P.

(2) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(3) $\{y \in X_m | y \geq x\}$ is finite for every $m < n$ and $x \in X_n$.

(4) $\bigcup_{n\in\omega} X_n$ is a dense subset of P.

In [2], we left the characterization problem open for σ-short posets, that is, for every σ-short poset $\mathbb{P} = (P, \leq)$, does there exist a sequence $\{X_n\}_{n\in\omega}$ of subsets of $\mathbb{P}$ which satisfies the following conditions:

(1) X_n is a pairwise incomparable subset of P.

(2) If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.

(3) $\bigcup_{n\in\omega} X_n$ is a dense subset of P

Though this problem is still open, we gave a sufficient condition in [3] that above conditions are satisfied. And using this sufficient condition , we reported that some Axiom A posets ,e.g., Sacks forcing , Mathias forcing are not σ-short. We also conjectured that non-ccc Axiom A posets are not σ-short. To concern this problem , in this paper we shall show that Axiom A posets with frame systems are not σ-short. Moreover, we shall show that Axiom A posets which satisfy Mildenberger's finiteness property with some additional conditions and Hechler forcing which adds a strictly increasing function from ω to ω are not σ-short.

2 Preliminaries

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set (poset). We say that $p, q \in P$ are compatible ($p \uparrow q$) if there exists an r such that $r \leq p$ and $r \leq q$. If p and q are not compatible, then we say that they are incompatible ($p \perp q$). A poset $(P, \leq)$ is separative if for every $p, q \in P$, $p \not\leq q$ implies that there exists an r such that $r \leq p$ and $r \perp q$. For a set X, we denote by $|X|$ the cardinality of X. Let $\sim$ be an equivalence relation on a set X. Then we denote the quotient of X by the equivalence relation $\sim$ by $X/\sim$. In this paper, we assume that posets are non-atomic and separative.

A poset $(P, \leq)$ satisfies Axiom A if there are partial orderings $\{\leq_n\}_{n\in\omega}$ such that

(A1): If $p \leq_0 q$ then $p \leq q$;

(A2): If $p \leq_{n+1} q$, then $p \leq_n q$;

(A3): If $\{p_n\}_{n\in\omega}$ is a fusion sequence; i.e., if $p_{n+1} \leq_n p_n$ for every $n \in \omega$, then there is q such that $q \leq_n p_n$ for all $n \in \omega$;

(A4): If $p \in P$ and W is a partition of p, then for every n there is $q \leq_n p$ such that q is compatible with at most countably many $x \in W$.

We say that a poset $(P, \leq)$ with partial orderings $\{\leq_n\}_{n\in\omega}$ is a fusion poset if it satisfies (A1),(A2),(A3). First of all, we consider the following condition (C1) for fusion posets.

(C1): $\quad \forall n \in \omega \forall p \in P \exists p^* \geq_n p \forall p' \geq_n p [p^* \geq_n p']$

For $n \in \omega$ and $p \in P$, we denote p^* in (C1) by $stem_n(p)$. If a fusion poset P satisfies (C1), then the relation $\sim_n$ on P defined by $p \sim_n q \overset{\text{def}}{\iff} stem_n(p) = stem_n(q)$ is an equivalence relation on P. Using this equivalence relation, we consider conditions (C2) and (C3) as follows.

(C2): $\forall n \in \omega[\,|P/\sim_n| \leq \omega]$

(C3): $\forall n \in \omega \forall p, q \in P[p \sim_n q \ \wedge \ p \geq q \Rightarrow p \geq_n q]$

Lemma 2.1. *Suppose that a fusion poset P is σ-short and D is a σ-short dense subset of P. Then*

$$D_0 = \{d \in D \mid \exists m \in \omega \forall x \in P[d >_m x \Longrightarrow x \notin D]\}$$

is a dense subset of P.

Proof. Suppose not. Then there exists $d' \in D$ such that $d \notin D_0$ for every $d \leq d'$. Hence it holds that $\forall d \in D[d \leq d' \Longrightarrow \forall m \in \omega \exists x \in D[d >_m x]]$. Now we define a strictly decreasing fusion sequence $\{p_n\}$ in D as follows. Let $p_0 = d'$ and p_{n+1} be an element $x \in D$ such that $p_n >_n x$. Since P satisfies the condition (A3) and D is dense , there exists $q \in P$ and $r \in D$ such that $\forall n \in \omega[p_n \geq_n q \geq r]$. This contradicts that D is σ-short. □

Lemma 2.2. *Suppose that a fusion poset P satisfies conditions* (C1),(C2) *and* (C3). *If P is σ-short, then there exists a family $\{X_n\}_{n\in\omega}$ which satisfy the following three condition.*

(1) *X_n is a pairwise incomparable subset of P.*

(2) *If $x \in X_n, y \in X_m$ and $n < m$, then $y \not\geq x$.*

(3) *$\bigcup_{n\in\omega} X_n$ is a dense subset of P.*

Proof. Suppose that P is σ-short and D is a σ-short dense subset of P. Let D_0 be the dense subset of P as in Lemma 2.1. Then for every $d \in D_0$, there exists $m \in \omega$ such that $\forall x \in P[d >_m x \Rightarrow x \notin D]$. Let m_d be the minimum number of $\{m \in \omega \mid \forall x \in P[d >_m x \Rightarrow x \notin D]\}$. Then put $X_d = \{d' \in D_0 \mid d \sim_{m_d} d', m_d = m_{d'}\}$ for every $d \in D$. Then X_d is a pairwise incomparable subset of P. It holds that $X_d = X_{d'}$ if and only if $d \sim_{m_d} d', m_d = m_{d'}$, so that $\{X_d \mid d \in D\}$ is at most countable by (C2). Let $\{X'_{d_n} \mid n \in \omega\}$ be an enumeration of $\{X_d \mid d \in D\}$. We define $\{X_{d_n} \mid n \in \omega\}$ inductively by $X_n = X'_{d_n} \backslash \{d \in X'_{d_n} \mid \exists k < n \exists d' \in X_k[d' \leq d]\}$. Then it holds the second condition. Since $D_0 = \bigcup_{n\in\omega} X'_{d_n}$ is a dense subset of P, $\bigcup_{n\in\omega} X_n$ is also a dense subset of P. □

Theorem 2.3. *Suppose that a fusion poset P satisfies conditions* (C1),(C2) *and* (C3). *If P satisfies the following condition* (C4), *then P is not σ-short.*

(C4): *If $p \in P$ and X is a pairwise incomparable subset of P, then for every n there is $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.*

Proof. Suppose that P is σ-short. Then, there exists a family $\{X_n\}$ which satisfy the conditions as in Lemma 2.2. We define a fusion sequence $\{p_n\}_{n\in\omega}$ inductively as follows. Put $p_0 = p$. Suppose that p_n is already defined. There exists $q \leq_n p_n$ such that $r \not\leq q$ forall $r \in X_n$ by (C4). Let p_{n+1} be such an element q. Then $\{p_n\}_{n\in\omega}$ is a fusion sequence, so that there exists a fusion p_ω of $\{p_n\}_{n\in\omega}$. Since $\bigcup_{n\in\omega} X_n$ is a dense subset of P, there exists $n \in \omega$ and $r \in X_n$ such that $r \leq p_\omega$. On the other hand, since $p_\omega \leq p_{n+1}$, we have $r' \not\leq p_\omega$ for all $r' \in X_n$ by virtue of the definition of p_{n+1}. This contradicts that $r \in X_n$ and $r \leq p_\omega$. □

3 Frame system

Definition 3.1. *Let $(P, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset which satisfies (C1),(C2) and (C3). And let f be a map from $P \times \omega$ to ω. We denote $\{k \in \omega \mid 0 \leq k \leq f(p,n)\}$ by $I_{p,n}$ and $\{k \in \omega \mid 0 \leq k \leq f(stem_n(p), n)\}$ by $I^*_{p,n}$. We say that $\{a_{p,n,k} \in P \mid n \in \omega, p \in P, 0 \leq k \leq f(p,n)\}$ is a frame system for P if it satisfies the following conditions.*

(FS1): $\forall n \in \omega \forall p, q \in P[p \sim_n q \Rightarrow f(p,n) = f(q,n)]$

(FS2): $\forall n \in \omega \forall p \in P\, \big[\{a_{p,n,k}\}_{0\leq k\leq f(p,n)} \textit{is a partition of } p \textit{ and}$
$$\{a_{p,n+1,j}\}_{0\leq j\leq f(p,n+1)} \textit{is a refinement of } \{a_{p,n,k}\}_{0\leq k\leq f(p,n)}.\big]$$

(FS3): $\forall n \in \omega \forall p, q \in P\, [p \geq_n q \Rightarrow \forall k \in I_{p,n}\, [a_{p,n,k} \geq_0 a_{q,n,k}]]$

(FS4): $\forall p, r \in P\, [p \geq r \Rightarrow \exists n \in \omega \exists k \in I_{p,n}\, [a_{p,n,k} \geq_0 r]]$

(FS5): $\forall n \in \omega \forall p, r \in P\, [p \geq r \Rightarrow$
$$\exists q \leq_n p[q \geq r \wedge \forall r' \in P[q \geq r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']]$$

(FS6): $\forall n \in \omega \forall p, r \in P\, [p \geq r \Rightarrow \exists r' \in P\, [r > r' \wedge r \sim_{p,n+1} r']]$

(FS7): $\forall n \in \omega \forall p, r \in P\, [a_{p,n,k} \geq_0 r \Rightarrow a_{p,n,k} \sim_{p,n+1} r]$

where $r \sim_{p,n+1} r'$ is defined by $r \uparrow a_{stem_n(p),n+1,j} \Leftrightarrow r' \uparrow a_{stem_n(p),n+1,j}$ for all $j \in I^*_{p,n+1}$.

Let $n \in \omega, p, r \in P$ and $p \geq r$. Then by (FS5), we can find $q \leq_n p$ such that $q \geq r$ and $\forall r' \in P[q \geq r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']$. We denote such element q by $p|r$ and call it the n-amalgamation of r into p.

Example 3.2. *In the following examples, we consider a canonical enumeration of $2^{<\omega}$ or $\omega^{<\omega}$. And, when we enumerate elements of a subset of those sets, we use this canonical enumeration. If t appears in an enumeration after s, then we denote it by $s \prec t$.*

Sacks forcing: $(P_S, \leq)$ is defined as follows.

$P_S = \{p \mid p \text{ is a perfect tree of } 2^{<\omega}\}$ and $p \geq q$ iff $p \supseteq q$.

We define a partial order $\leq_n$ by $p \geq_n q \Leftrightarrow p \geq q$ and $B_n(p) = B_n(q)$ where $B_n(p)$ is a set of the $(n+1)$-st branching points of p. For $p \in P_S$ and $n \in \omega$, put $p^* = \{t \in 2^{<\omega}\} \mid \exists s \in B_n(p)\,[t \subseteq s \text{ or } s \subseteq t]\}$. Then $p^* \geq_n p$ and $p' \geq_n p$ implies $p^* \geq_n p'$. Hence P_S satisfies (C1). It holds that $p \sim_n q$ iff $B_n(p) = B_n(q)$. So P_S satisfies (C2) and (C3).

For $p \in P_S$ and $n \in \omega$, let $f(p,n) = 2^n - 1$ and $B_n(p) = \{s_0, \ldots, s_{2^n-1}\}$.
Put $a_{p,n,k} = p{\restriction} s_k = \{t \in p \mid t \subseteq s_k \text{ or } s_k \subseteq t\}$. Then $\{a_{p,n,k} \in P_S \mid n \in \omega, p \in P_S, 0 \leq k \leq f(p,n)\}$ is a frame system for P_S.

Prikry-Silver forcing: $(P_{PS}, \leq)$ is defined as follows.

$P_{PS} = \{p \mid p : \text{dom}(p) \to \{0,1\}, \text{dom}(p) \text{ is a co-infinite subset of } \omega\}$ and $p \geq q \Leftrightarrow p \subseteq q$.

We define a partial order $\leq_n$ by $p \geq_n q$ iff $p \geq q$ and $[\omega \backslash \text{dom}(p)]_n = [\omega \backslash \text{dom}(q)]_n$ where $[\omega \backslash \text{dom}(p)]_n$ is a set of the first n elements of $\omega \backslash \text{dom}(p)$. For $p \in P_{PS}$ and $n \in \omega$, let k be an n-th element of $\omega \backslash \text{dom}(p)$ and put $\text{dom}(p^*) = \{m \in \text{dom}(p) \mid m < k\}$ and $p^*(m) = p(m)$ for every $m \in \text{dom}(p^*)$. Then $p^* \geq_n p$ and $p' \geq_n p$ implies $p^* \geq_n p'$. Hence P_{PS} satisfies (C1). It holds that $p \sim_n q$ iff $[\omega \backslash \text{dom}(p)]_n = [\omega \backslash \text{dom}(q)]_n$ and $p \restriction \text{dom}(p) \cap [0,k] = q \restriction \text{dom}(q) \cap [0,k]$. So P_{PS} satisfies (C2) and (C3).

For $p \in P_{PS}$ and $n \in \omega$, let $f(p,n) = 2^n - 1, [\omega \backslash \text{dom}(p)]_n = \{\ell_0, \ldots, \ell_{n-1}\} (\ell_0 < \ell_1 < \cdots < \ell_{n-1})$ and $\{0,1\}^n = \{s_0, \ldots, s_{2^n-1}\}$. Put $a_{p,n,k} = p \cup \{\langle \ell_i, s_k(i) \rangle \mid 0 \leq i < n\}$. Then $\{a_{p,n,k} \in P_{PS} \mid n \in \omega, p \in P_{PS}, 0 \leq k \leq f(p,n)\}$ is a frame system for P_{PS}.

Mathias forcing: $(P_M, \leq)$ is defined as follows.

$P_M = \{(s,S) \mid s \in \omega^{<\omega} \text{ is increasing}, S \text{ is an infinite subset of } \omega \backslash \max(s)\}$ and $(s,S) \geq (t,T) \Leftrightarrow t \supseteq s, T \subseteq S$ and $\text{range}(t) \backslash \text{range}(s) \subseteq S$.

We define a partial order $\leq_n$ by $(s,S) \geq_n (t,T)$ iff $(s,S) \geq (t,T), s = t$ and $[S]_n = [T]_n]$. For $p = (s,S) \in P_M$ and $n \in \omega$, put $p^* = (s, \omega \backslash \max(s))$. Then $p^* \geq_n p$ and $p' \geq_n p$ implies $p^* \geq_n p'$. Hence P_M satisfies (C1). It holds that $(s,S) \sim_n (t,T)$ iff $s = t$ and $[S]_n = [T]_n$. So P_M satisfies (C2) and (C3).

For $p = (s,S) \in P_M$ and $n \in \omega$, let $f(p,n) = 2^n - 1$ and $\mathcal{P}([S]_n) = \{\tau \in \omega^{<\omega} \mid \tau \text{ is increasing }, \text{range}(\tau) \subseteq [S]_n\} = \{\tau_0, \ldots, \tau_{2^n-1}\}$. Put $a_{p,n,k} = (s^\frown \tau_k, S \backslash [S]_n)$. Then $\{a_{p,n,k} \in P_M \mid n \in \omega, p \in P_M, 0 \leq k \leq f(p,n)\}$ is a frame system for P_M.

Laver forcing: $(P_L, \leq)$ is defined as follows.

$P_L = \{p \mid p \text{ is a tree of } \omega^{<\omega} \text{ which has a stem } s \text{ such that } \forall t \supseteq s[S(t) = \{k \in \omega \mid t ^\frown k \in p\} \text{ is infinite}]\}$ and $p \geq q \Leftrightarrow p \supseteq q$.

For $p \in P_L$, let $s_0^p = \text{stem}(p), s_1^p, \ldots, s_m^p, \ldots$ be an enumeration of $\{t \in p \mid t \supseteq \text{stem}(p)\}$. We define a partial order $\leq_n$ by $p \geq_n q$ iff $p \geq q$ and $s_i^p = s_i^q$ for all $i = 0, \ldots n$. For $p \in P_L$ and $n \in \omega$, $p^* = \{t \in p \mid t \subseteq s_0^p\} \cup \{s_1^p, \ldots, s_n^p\} \cup \{t \in$

$\omega^{<\omega} \mid t$ appears in an enumeration of $\omega^{<\omega}$ after $s_n^p\}$. Then $p^* \geq_n p$ and $p' \geq_n p$ implies $p^* \geq_n p'$. Hence P_L satisfies (C1). It holds that $p \sim_n q$ iff $s_i^p = s_i^q$ for all $i = 0, \ldots n$. So P_L satisfies (C2) and (C3).

For $p \in P_L$ and $n \in \omega$, let $f(p,n) = n$ and $K = \{s_0^p, s_1^p, \ldots, s_n^p\}$. If s_k^p is a $\subseteq$-maximal node among K, then put $a_{p,n,k} = \{t \in p \mid t \subseteq s_k^p \text{ or } t \supseteq s_k^p\}$. Otherwise, put $a_{p,n,k} = \{t \in p \mid t \subseteq s_k^p \text{ or } [t \supseteq s_k^p \text{ and } \forall j > k[s_j^p \not\subseteq t]]\}$. Then $\{a_{p,n,k} \in P_L \mid n \in \omega, p \in P_L, 0 \leq k \leq f(p,n)\}$ is a frame system for P_L.

Lemma 3.3. *Let $(P, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset with a frame system which satisfies* (C1),(C2) *and* (C3). *If $n \in \omega, p, r \in P$ and $r \leq_0 a_{p,n,k}$, then we have $a_{p|r,n,k} = r$.*

Proof. Suppose that $n \in \omega, p, r \in P$ and $r \leq_0 a_{p,n,k}$. Let $q = p|r$. Then we have

$$(*) \quad \forall r' \in P[q \geq r' \wedge r \sim_{p,n+1} r' \Rightarrow r \geq r']$$

Since $\{a_{p,n,k}\}_{0\leq k \leq f(p,n)}$ is a partition of p and $r \leq a_{p,n,k}$, r is not compatible with $a_{p,n,j}$ for all $j \neq k$. By virtue of (FS3), we have $a_{p,n,j} \geq a_{q,n,j}$. So r is not compatible with $a_{q,n,j}$ for all $j \neq k$. Since $q = p|r \geq r$, we have $a_{q,n,k} \geq r$. On the other hand, we have $a_{p,n,k} \geq_0 a_{q,n,k}$ by (FS3) and $a_{p,n,k} \geq_0 r$ by assumption, so that we have $r \sim_0 a_{q,n,k}$. Hence by virtue of (FS3), $a_{q,n,k} \geq_0 r$. Therefore $a_{q,n,k} \sim_{q,n+1} r$ by (FS7). Since $p \geq_n q$, we have $a_{q,n,k} \sim_{p,n+1} r$. So we have $r \geq a_{q,n,k}$ by $(*)$. Thus $a_{q,n,k} = a_{p|r,n,k} = r$. □

Lemma 3.4. *Let $(P, \leq, \{\leq_n\}_{n\in\omega})$ be a fusion poset with a frame system which satisfies* (C1),(C2) *and* (C3). *Suppose that W is a partition of P and $p \in P$. Then there exists $q \leq_0 p$ such that q is compatible with at most countably many $r \in W$.*

Proof. Let $\{a_{p,n,k}\}$ be a frame system for P. We construct a fusion sequence $\{p_n\}_{n\in\omega}$ and a sequence of at most countable sets $\{A_n\}_{n\in\omega}$ inductively as follows. Let $p_0 = p$ and $A_0 = \emptyset$. Given p_n and A_n, we shall define $\{q_n^k\}_{0\leq k\leq f(p,n)}$ and $\{A_n^k\}_{0\leq k \leq f(p,n)}$ as follows. Let $q_n^0 = p_n, A_n^0 = A_n$. Suppose that we already have q_n^k, and A_n^k. If there exists p' and $r \in W$ such that $a_{q_n^k,n,k} \geq_0 p'$ and $r \geq p'$, then we put $q_n^{k+1} = q_n^k|p'$ and $A_n^{k+1} = A_n^k \cup \{r\}$. Otherwise let $q_n^{k+1} = q_n^k$ and $A_n^{k+1} = A_n^k$. Finally, we put $p_{n+1} = q_n^{f(p,n)+1}$ and $A_{n+1} = A_n^{f(p,n)+1}$. Since $p_n \geq_n p_{n+1}$, $\{p_n\}_{n\in\omega}$ is a fusion sequence. Thus, $\{p_n\}_{n\in\omega}$ has a fusion p_ω by (A3). $p_\omega \leq_n p_n$ for all $n \in \omega$. Let $A = \bigcup_{n=0}^\infty A_n$. A is at most countable by the construction of A_n. We shall show that p_ω is compatible with at most countably many $r \in W$. Since A is at most countable and W is a partition of P, it suffices to show that for all $q \leq p_\omega$ there exists $r \leq q$ such that $r \in W \cap A$. Let $q \leq p_\omega$. Since W is a partition of P, we can find q' and $r \in W$ such that $q' \leq q$ and $q' \leq r$. Since $q' \leq q \leq p_\omega$, $q' \leq_0 a_{p_\omega,n,k}$ for some $n, k \in \omega$ by (FS4). Then we have $q' \leq_0 a_{p_\omega,n,k} \leq_0 a_{q_n^k,n,k}$ by (FS3). Hence by the construction of q_n^k, and A_n^k, $q_n^{k+1} = q_n^k|q^*$ and $A_n^{k+1} = A_n^k \cup \{r^*\}$ for some q^*

and $r^* \in W$ such that $a_{q_n^k,n,k} \geq_0 q^*$ and $r \geq q^*$. Then we have $r^* \in W \cap A$. By Lemma 3.3, $a_{q_n^k|q^*,n,k} = q^*$, so that $q' \leq_0 a_{p_\omega,n,k} \leq_0 a_{q_n^{k+1},n,k} = a_{q_n^k|q^*,n,k} = q^* \leq r^*$. Hence r and r^* are compatible. Hence we have $r = r^* \in W \cap A$. □

Lemma 3.5. *If* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *is a fusion poset with a frame system which satisfies* (C1),(C2) *and* (C3), *then* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *satisfies* (A4).

Proof. Let $\{a_{p,n,k}\}$ be a frame system for P, W be a partition of P and $p \in P$. We shall show that there exists $q \leq_n p$ such that q is compatible with at most countably many $r \in W$. We construct a sequence $\{q_k\}_{0\leq k\leq f(p,n)+1}$ inductively such that $q_{k+1} \leq_n q_k$ for all k. Let $q_0 = p$. Suppose that we already have q_k. By virtue of Lemma 3.4, there exists $p_k \leq_0 a_{q_k,n,k}$ such that p_k is compatible with at most countably many $r \in W$. Then put $q_{k+1} = q_k|p_k$. $a_{q_{k+1},n,k} = p_k$ by Lemma 3.3. Finally we put $q = q_{f(p,n)+1}$. Then we have $q \leq_n p$ and $a_{q,n,k} \leq a_{q_{k+1},n,k} = p_k$for all k. If $r \in W$ is compatible with q, then r is compatible with $a_{q,n,k}$ for some k, so that r is compatible with p_k for some k. Since p_k is compatible with at most countable many $r \in W$, q is also compatible with at most countable many $r \in W$. □

Lemma 3.6. *If* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *is a fusion poset with a frame system which satisfies* (C1),(C2) *and* (C3), *then* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *satisfies* (C4).

Proof. Let $\{a_{p,n,k}\}$ be a frame system for P, X be a pairwise incomparable subset of P and $p \in P$. We shall show that there exists $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$. If there exists no $r \in X$ such that $r \leq p$, then we put $q = p$. So we assume that there exists $r \in X$ such that $r \leq p$. Let $\ell = f(stem_n(p), n+1)$ and $\mathcal{P}(I^*_{p,n+1}) = \{t_1, ..., t_{2^{\ell+1}}\}$. We construct a $\{q_k\}_{0\leq k\leq 2^{\ell+1}+1}$ inductively such that $q_{k+1} \leq_n q_k$ for all k. Put $q_0 = p$. Suppose that we already have q_k. In the following, we denote $\{j \mid r \uparrow a_{stem_n(p),n+1,j}\}$ by $C(r)$. If there exists $r \in X$ such that $r \leq q_k$ and $C(r) = t_k$. We pick such an element r and take $\tilde{r} < r$ such that $r \sim_{p,n+1} \tilde{r}$ by (FS6). Then put $q_{k+1} = q_k|\tilde{r}$. If there exists no $r \in X$ such that $r \leq q_k$ and $C(r) = t_k$, then put $q_{k+1} = q_k$. Finally we put $q = q_{2^{\ell+1}+1}$. By virtue of the definition, we have $q \leq_n p$. So we shall show that $q \not\geq r$ for all $r \in X$. Suppose that $q \geq r$ for some $r \in X$. Put $t = C(r)$. Then $t = t_k$ for some k. Thus we have $q_k \geq q \geq r$ and $C(r) = t_k$. So, by the definition of the sequence $\{p_k\}$, we have defined $q_{k+1} = q_k|\tilde{r}$ where $\tilde{r} < r^*, \tilde{r} \sim_{p,n+1} r^*$and $C(r^*) = t_k$ for some $r^* \in X$. Then $C(\tilde{r}) = C(r^*) = t_k = C(r)$. Since $q_k|\tilde{r} = q_{k+1} \geq q \geq r$, $\tilde{r} \geq r$ by (FS5). Hence we have $r^* > \tilde{r} \geq r$ and $r^*, r \in X$. This contradicts that X is a pairwise incomparable subset of P. □

By virtue of Theorem 2.3 and Theorem 3.6, we have

Theorem 3.7. *If* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *is a fusion poset with a frame system which satisfies* (C1),(C2) *and* (C3), *then* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *is not* σ*-short.*

4 Mildenberger's finiteness property

In [1], Mildenberger defined the finiteness property for Axiom A posets. It is defined as follows.

Definition 4.1. *An Axiom A poset* $(P, \leq, \{\leq_n\}_{n\in\omega})$ *whose elements are subsets of* $2^{<\omega}$ *or of* $\omega^{<\omega}$ *has the finiteness property iff*

(1) $p \geq q$ *implies* $p \supseteq q$,

(2) *there is a function* $f : P \times \omega \longrightarrow \omega$ *such that for every* n, p, q,

$$p \geq_n q \textit{ iff } p \geq q \textit{ and } q \cap f(p,n)^{f(p,n)} = p \cap f(p,n)^{f(p,n)}.$$

In the case of $2^{<\omega}$, *we can write* $2^{f(p,n)}$ *instead of* $f(p,n)^{f(p,n)}$.

We assume that elements of P are trees. We say that P has the uniform finiteness property if it has the finiteness property and for every $n \in \omega, p, q \in P$, $p \geq_n q$ implies $f(p,n) = f(q,n)$. For $p \in P$, $s \in p$ is called the stem of p if (i): for every $t \in p$, $s \subseteq t$ or $t \subseteq s$, and (ii): p is a branching point, i.e., s has at least two successors in p. We denote the stem of p as $st(p)$. If σ is a finite subtree of p, we denote it by $\sigma \Subset p$. We say that $t \in \sigma$ is a σ-branching point of p if there exists $k \in \omega$ such that $t^\frown\langle k\rangle \in p$ and $t^\frown\langle k\rangle \notin \sigma$. We denote the set of σ-branching points of p by σ^b. Let $\sigma_p^n = \{t \in \omega^{<\omega} \mid \exists s \in p \cap f(p,n)^{f(p,n)} \,[t \subseteq s]\}$. Then every element of $p \cap f(p,n)^{f(p,n)}$ is a σ_p^n-branching point of p. Let $p \geq r$ and $t \in \sigma^b$. Then we say that t is a r-σ-branching point of p if there exists $s \in r$ such that $t \subsetneq s$ and $\forall k \in \omega \left[t^\frown\langle k\rangle \subseteq s \Rightarrow t^\frown\langle k\rangle \notin \sigma\right]$. We denote the set of r-σ-branching points of p by $\sigma^{b,r}$. For $p \geq r, r'$ and $\sigma \Subset p$, we define $r \approx_\sigma r'$ if and only if $r \cap \sigma = r' \cap \sigma$ and $\sigma^{b,r} = \sigma^{b,r'}$.

We say that P has enough elements if P satisfies the following

(1) $I = 2^{<\omega}$ or $\omega^{<\omega} \in P$,

(2) for every $r \in P$, there exists $r' \in P$ such that $r > r'$ and $st(r) = st(r')$,

(3) for every $p \in P$,

$$p^* = I \setminus \{t \in I \mid t \notin p, \exists s \in (f(p,n)^{f(p,n)} \setminus p) \;\; [s \subseteq t \text{ or } t \subseteq s]\} \in P,$$

(4) for every $p \in P$ and $s \in p$,

$$p\restriction s = \{t \in p \mid t \subseteq s \text{ or } s \subseteq t\} \in P,$$

(5) for every $p \in P$ and $r \leq p$,

$$p|r = r \cup \{t \in p \mid t \not\subseteq st(r) \text{ and } st(r) \not\subseteq t\} \in P.$$

Lemma 4.2. *Let $(P, \leq, \{\leq_n\}_{n\in\omega})$ be an Axiom A poset with uniform finiteness property which has enough elements. Then for every $n \in \omega, p \in P$ and $p \geq r$, there exists $r' < r$ such that $r \approx_{\sigma_p^n} r'$.*

Proof: If $r \cap f(p,n)^{f(p,n)} = \emptyset$, then pick any r' such that $r > r'$. Otherwise, let $s \in r \cap f(p,n)^{f(p,n)}$. Pick r_0 such that $r{\restriction}s > r_0$ and $st(r{\restriction}s) = st(r_0)$ and let $r' = r|r_0$. Then we have $r \approx_{\sigma_p^n} r'$.

Theorem 4.3. *Let $(P, \leq, \{\leq_n\}_{n\in\omega})$ be an Axiom A poset with uniform finiteness property which has enough elements. Then we have*

(1) *P satisfies* (C1),(C2) *and* (C3).

(2) *If $(P, \leq, \{\leq_n\}_{n\in\omega})$ satisfies the following strong amalgamation property* (AP), *then P is not σ-short.*

$$(\mathrm{AP}) : \forall n \in \omega \forall p \in P \forall r \in P\,[p \geq r \Rightarrow \exists q \leq_n p\,[q \geq r \wedge \forall r' \in P[q \geq r' \wedge r \approx_{\sigma_p^n} r' \Rightarrow r \geq r']]$$

Proof. (1): Since P has enough elements, P satisfies (C1). Then $p \sim_n q$ if and only if $f(p,n) = f(q,n)$ and $p \cap f(p,n)^{f(p,n)} = q \cap f(q,n)^{f(q,n)}$ by (C1) and the finiteness property. So it is easy to show that P satisfies (C2) and (C3).

(2): Suppose that P satisfies (AP). By virtue of (1), P satisfies (C1), (C2) and (C3). Hence by Theorem 2.3, it is sufficient to show that P satisfies (C4). At first, we shall show that the following claim

Claim 4.4. *If $p \in P$ and X is a pairwise incomparable subset of P with same stem, then for every n there is $q \leq_n p$ such that $r \not\leq q$ for all $r \in X$.*

Proof: Let $p \in P, n \in \omega$ and $\forall r \in X\ [st(r) = t^*]$. If $p \not\geq r$ for all $r \in X$, then put $q = p$. We assume that $p \geq r$ and $r \in X$. If $st(r) \notin \sigma_p$, then put $q = p|r'$ for some r' such that $r > r'$ and $st(r) = st(r')$. Then $q \leq_n p$ and $q \not\geq r^*$ for all $r^* \in X$. So we assume that $st(r) \in \sigma_p$. Put $K = \{(\rho, \tau) \mid \tau \subseteq \rho \subseteq \sigma_p\} = \{(\rho_1, \tau_1), \ldots, (\rho_\ell, \tau_\ell)\}$. We define inductively $\{q_k\}_{0\leq k\leq \ell+1}$ such that $q_k \leq_n p$ as follows: Let $q_0 = p$. Suppose that q_k is already defined. If there exists $r \in X$ such that $q_k \geq r, \rho_k = r \cap f(p,n)^{f(p,n)}$ and $\tau_k = \sigma^{b,r}$, let r' be $r > r'$ and $r \approx_{\sigma_p^n} r'$, and q_{k+1} be defined such that $q_{k+1} \geq r' \wedge \forall r'' \in P[q_k \geq r'' \wedge r' \approx_{\sigma_p} r'' \Rightarrow r' \geq r'']$ by (AP). Otherwise, $q_{k+1} = q_k$. Finally, put $q = q_{\ell+1}$. We shall show that $q \not\geq r$ for all $r \in X$. Suppose that $q \geq r$ for some $r \in X$. Put $\rho = r \cap f(p,n)^{f(p,n)}$ and $\tau = (\sigma_p^n)^{b,r}$. Then there exists $(\rho_k, \tau_k) = (\rho, \tau)$ for some k. Since $q_k \geq q \geq r, \rho_k = r \cap f(p,n)^{f(p,n)}$ and $\tau_k = \sigma^{b,r}$, q_{k+1} is defined from r^* and r' such that $\rho_k = r^* \cap f(p,n)^{f(p,n)}, \tau_k = \sigma^{b,r^*}, r^* > r', r^* \approx_{\sigma_p^n} r', q_k \geq r'$ and $\forall r'' \in P[q_{k+1} \geq r'' \wedge r' \approx_{\sigma_p} r'' \Rightarrow r' \geq r'']$. Since $q_{k+1} \geq q \geq r$ and

$r \approx_{\sigma_p} r^* \approx_{\sigma_p} r'$, we have $r^* > r' \geq r$. This contradict that X is a pairwise incomparable subset of P. Hence $q \not\geq r$ for all $r \in X$. □

By using Claim 4.4, we can easily show that P satisfies (C4), since P satisfies (A3) and $X = \bigcup_{s \in \omega^{<\omega}} \{r \in X \mid st(r) = s\}$. □

5 Hechler forcing

In this section we show that Hechler forcing which adds a strictly increasing function from ω to ω is not σ-short.

The Hechler forcing P is defined as follows.

$$(s, f) \in P \Longleftrightarrow s \in \omega^{<\omega} \wedge f \in \omega^\omega \wedge s \subseteq f \wedge f \text{strictlyincreasing}$$

$$(s, f) \leq (t, g) \Longleftrightarrow s \supseteq t \wedge \forall n \in \omega[f(n) \geq g(n)]$$

To prove that P is not σ-short, we need the following lemma proved by Todorčević.

Lemma 5.1 (Todorčević[5]). *Suppose $\{a_\alpha \mid \alpha < \theta\} \subseteq \omega^\omega$ is $<^*$-increasing and $<^*$-unbounded in ω^ω and that each a_α is an increasing function. Then there exists $\alpha < \beta < \theta$ such that $a_\alpha \leqq a_\beta$.*

Theorem 5.2. *If P is σ-short , then $(\omega^\omega, \leq)$ is σ-short.*

Proof: Let D be a σ-short dense subset of P. For $(s, f) \in D$, put $f_s(0) = |s|, f_s(n) = |s| + 1 + f(n-1)$. Since f is strictly increasing, f_s is also strictly increasing. Then put $D_0 = \{f_s \mid (s, f) \in D\}$. We shall show that D_0 is a dense σ-short subset of $(\omega^\omega, \leq)$. Let $g \in \omega^\omega$. W.l.o.g we may assume that g is strictly increasing. Put $g^*(n) = g(n) + g(n+1)$. Then g^* is strictly increasing. Put $t = g^* \restriction g^*(0)$. Since D is dense in P, there exists $(s, f) \in D$ such that $(t, g^*) \geq (s, f)$. Then $f_s(0) = |s| \geq |t| \geq g(0)$ and $f_s(n) = |s| + 1 + f(n-1) \geq g^*(n-1) = g(n) + g(n-1) \geq g(n)$. So we have $f_s \geq g$. Hence D_0 is a dense subset of $(\omega^\omega, \leq)$. We shall show that D_0 is σ-short. Suppose that $\{f^n_{s_n}\}$ is strictly increasing sequence in D_0. We show that $\{f^n_{s_n}(i)\}$ is unbounded for some i. If $\lim_{n\to\infty} |s_n| = \infty$, then $\{f^n_{s_n}(0)\}$ is unbounded in ω. So w.l.o.g we assume that $|s_n| = k$ for all $n \in \omega$. If $\{s_n(i)\}$ is unbounded for some $i < k$, then $\{f^n_{s_n}(i+1)\}$ is unbounded in ω. Hence we assume that $\{s_n(i)\}$ is bounded in ω for all $i < k$. Then there exists s such that $\{s_n \mid s = s_n\}$ is infinite. W.l.o.g we assume that $s = s_n$ for all $n \in \omega$. Since $f^n(i) = f^n(i+1) - |s_n| - 1 = f^n(i+1) - |s| - 1$, we have $(s, f^1) \gneq (s, f^2) \gneq \cdots$. Since $(s, f^i) \in D$ and D is σ-short, $\{f^n_{s_n}(i)\}$ is unbounded for some i. □

Put $D_1 = \{f \in D_0 \mid \exists n \forall g \in \omega^\omega [f \restriction n = g \restriction n \wedge f \lneq g \Longrightarrow g \notin D_0\}$.

Claim 5.3. *D_1 is a dominating family of $(\omega^\omega, \leq)$.*

Proof. Suppose not. Then there exists $f \in \omega^\omega$ such that $g \notin D_1$ for every $f \leq g$. That is , it holds that for every $g \geq f$

$$\forall n \exists h \in \omega^\omega [g \restriction n = h \restriction n \wedge g \lneq h \wedge h \in D_0]$$

Let $g_0 \in D_0$ be such that $f \lneq g_0$. Then we have $g_0 \notin D_1$. Hence there exists $g_1 \in D_0$ such that $g_1 \restriction 1 = g_0 \restriction 1 \wedge g_0 \lneq g_1$. Since $f \lneq g_0 \lneq g_1$, we have $g_1 \notin D_1$. So Hence there exists $g_2 \in D_0$ such that $g_2 \restriction 1 = g_1 \restriction 1 \wedge g_1 \lneq g_2$. Continuing this construction, we have $\{g_n\}$ such that $\forall n \in \omega [g_n \restriction n+1 = g_{n+1} \restriction n+1 \wedge g_n \lneq g_{n+1}$. There exists g_ω such that $g_n \leq g_\omega$ for all $n \in \omega$. Since D_0 is dense, there exists $h \in D_0$ such that $g_\omega \leq h$. But this contradicts that D_0 is σ-short. □

For $n \in \omega, t \in \omega^{<\omega}$, put

$$D_t^n = \{f \in D_1 \mid t = f \restriction n \wedge \forall g \in \omega^\omega [t = g \restriction n \wedge f \lneq g \Longrightarrow g \notin D_0\}.$$

Since $D_1 = \bigcup_{n \in \omega} \bigcup_{t \in \omega^{<\omega}} D_t^n$ and D_1 is a dominating family of $(\omega^\omega, \leq)$, D_t^n is a dominating family of $(\omega^\omega, \leq^*)$ for some $n \in \omega$ and $t \in \omega^{<\omega}$. Let D_t^n be such a dominating family.

Claim 5.4. *Elements of D_t^n are mutually incomparable.*

Proof. Suppose that $f, g \in D_t^n$ and $f \neq g$. Since $t = f \restriction n = g \restriction n$ and $f, g \in D_0$, we have $f \nleq g$ and $g \nleq f$. □

On the other hand, every dominating family of $(\omega^\omega, \leq^*)$ has a $<^*$-increasing and $<^*$-unbounded subset, so that D_t^n has comparable different elements by Lemma 5.1. This contradict to Claim 5.4. □

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