数理解析研究所講究録
第 1989 巻 2016 年 25-33

The stability of an unbalanced rotating cylindrical vessel with a small amount of fluid

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Abstract

This paper is concerned with the dynamics of a cylindrical vessel containing a small amount of liquid which, during rotation, is spun out to form a thin liquid layer on the outermost inner surface of the vessel. The liquid is able to counteract unbalanced mass in an elastically mounted rotor. Hence the name 'fluid balancer'. The paper discusses the equations of motion for the coupled fluid-structure system, their solution in terms of a perturbation approach, and the stability of this solution.

1 Introduction

The dynamics, and possible unstable motion (whirl), of rotating machinery has been of concern for more than 100 years [1, 2]. The influence of a small amount of fluid trapped inside a cylindrical rotating, whirling vessel was investigated for the first time (theoretically) by Schmidt [3] in 1958 and followed up (theoretically and experimentally) by Kollmann [4] in 1961. These pioneering papers initiated many detailed investigations which mainly are concerned with the stability of the rotor motion, and how the fluid may cause instability.

But it has been known even longer that a small amount of trapped fluid also can stabilize an unbalanced, whirling rotor, in the sense that the fluid can act as a counterbalance and thus limit the whirl amplitude [5]. This is a topic that has enjoyed renewed interest in recent years, with applications in household washing machines as one example. It was the aim of a recent paper [6] to explain the basic mechanism behind this application, known as a 'fluid balancer'. The modeling of the fluid layer followed the shallow water wave approach of Berman et al. [7]. The scaling used in the perturbation analysis was also similar to the one used in that paper, with the ratio between the fluid mass and the mass of the empty rotor playing the role of the basic small parameter in the problem. However, this parameter is not necessarily so small in a typical modern washing machine. In the present formulation we use a different and more appropriate scaling, assuming that the basic small parameter is of the order (fluid layer thickness)/(vessel radius).

The equations of motion for the rotor are given in section 2. The equations describing the fluid layer are given in section 3. The perturbation-based solution of these equations is described in section 5. The coupling between fluid and structure is discussed in section 6. A stability analysis is discussed in section 7. Finally, concluding remarks are made in section 8.

2 The rotor equations

We consider a rotating vessel (rotating fluid chamber) of mass M equipped with a small unbalanced mass m located a distance s from the geometric center, and containing a small amount of liquid, as sketched in Fig. 1. The inner radius of the vessel is R. The rotor is supported by springs with spring constants K_x and K_y, in the $\bar{X}$ and $\bar{Y}$ directions, respectively, of the space-fixed coordinate system $(\bar{X}, \bar{Y})$. The structural damping forces in both of these directions are proportional to the parameter C.

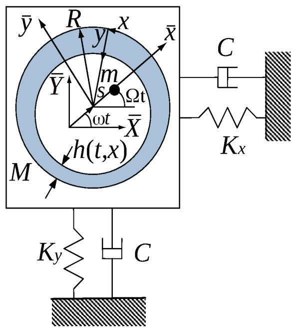

Figure 1: Sketch of the basic configuration, with definition of the space-fixed coordinate system $(\bar{X}, \bar{Y})$, the rotating (rotor-fixed) coordinate system $(\bar{x}, \bar{y})$, and some of the fundamental symbols.

The matrix equation of motion in terms of a coordinate system $(\bar{x}, \bar{y})$ fixed to the rotor is

$$\begin{bmatrix} M+m & 0 \\ 0 & M+m \end{bmatrix} \begin{Bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{Bmatrix} + \begin{bmatrix} C & -2(M+m)\Omega \\ 2(M+m)\Omega & C \end{bmatrix} \begin{Bmatrix} \dot{x}_r \\ \dot{y}_r \end{Bmatrix}$$
$$+ \begin{bmatrix} K_x - (M+m)\Omega^2 & -C\Omega \\ C\Omega & K_y - (M+m)\Omega^2 \end{bmatrix} \begin{Bmatrix} x_r \\ y_r \end{Bmatrix} = \begin{Bmatrix} ms\Omega^2 \\ 0 \end{Bmatrix} + \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}. \quad (1)$$

where x_r and y_r are the rotor deflections in the $\bar{x}$ and $\bar{y}$ direction, respectively, Ω is the angular velocity of the rotor, and t is the time. An overdot denotes differentiation with respect to t. In the rotating coordinate system the unbalanced mass introduces a time-independent force proportional to Ω^2, acting in the $\bar{x}$-direction.

3 Fluid modeling by the shallow water equations

The fluid motion in the rotating vessel will be described by a shallow water wave approximation of the Navier-Stokes equations, and in terms of a coordinate system (x, y) attached to the wall of the rotor, as shown in Fig. 1. x and y are rectangular (Cartesian) coordinates, indicating that curvature effects will be ignored. This is permissible when the fluid layer thickness $h(t, x)$ is sufficiently small in comparison with the vessel radius R, i.e., $|h(t, x)|/R \ll 1$ for all x, t. Ignoring gravitational forces also, the fluid equations of motion can be written as [7, 8]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} + \ddot{\mathfrak{X}}_r \sin\left(\frac{x}{R}\right) - \ddot{\mathfrak{Y}}_r \cos\left(\frac{x}{R}\right). \quad (2)$$

$$\frac{\partial v}{\partial t} + 2\Omega u + R\Omega^2 = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \quad (3)$$

Here u and v are the fluid velocity components in the x and y directions, p is the fluid pressure, ρ is the fluid density, and ν is the kinematic viscosity of the fluid. The body force $\mathfrak{F} = \ddot{\mathfrak{X}}_r \sin(x/R) - \ddot{\mathfrak{Y}}_r \cos(x/R)$ is given relative to the rotor-fixed coordinate system $(\bar{x}, \bar{y})$, where the acceleration vector $\{\ddot{\mathfrak{X}}_r \ \ddot{\mathfrak{Y}}_r\}^T$ is given by

$$\begin{Bmatrix} \ddot{\mathfrak{X}}_r \\ \ddot{\mathfrak{Y}}_r \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_r \\ \ddot{y}_r \end{Bmatrix} + 2\Omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_r \\ \dot{y}_r \end{Bmatrix} - \Omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_r \\ y_r \end{Bmatrix}. \quad (4)$$

Strictly speaking, a body force term on the form $\mathfrak{G} = \ddot{\mathfrak{X}}_r \cos(x/R) + \ddot{\mathfrak{Y}}_r \sin(x/R)$ is present on the right hand side of (3), as is a viscous term on the form $\nu\partial^2 v/\partial y^2$. These terms have been dropped here, however, since the fluid layer is thin (in comparison with R).

The continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{5}$$

The boundary conditions are

$$u=v=0 \quad \text{at} \quad y=0, \qquad \left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\right)=v, \quad p=0, \quad \text{at} \quad y=h, \tag{6}$$

where, again, $h(t,x)$ specifies the free surface of the fluid layer.

In the shallow water approximation it is assumed [8] that

$$v(t,x,y)=\frac{y}{h_0}\frac{\partial h}{\partial t}, \tag{7}$$

where h_0 is the mean fluid depth. Using this relation (3) can be written as

$$\frac{y}{h_0}\frac{\partial^2 h}{\partial t^2}+2\Omega u+R\Omega^2=-\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{8}$$

This equation can be integrated (with respect to y), to give

$$\frac{1}{\rho}p=\frac{1}{2h_0}\left(h_0^2-y^2\right)\frac{\partial^2 h}{\partial t^2}+2\Omega\int_y^h u\,dy+R\Omega^2(h-y). \tag{9}$$

Inserting (9) into (2) we get

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= -R\Omega^2\frac{\partial h}{\partial x}+2\Omega\frac{\partial h}{\partial t} \\ &\quad -\frac{1}{2h_0}(h_0^2-y^2)\frac{\partial^3 h}{\partial x\partial t^2}+\nu\frac{\partial^2 u}{\partial y^2}+\ddot{\mathfrak{X}}_r\sin\left(\frac{x}{R}\right)-\ddot{\mathfrak{Y}}_r\cos\left(\frac{x}{R}\right).\end{aligned} \tag{10}$$

Let

$$U=\frac{1}{h}\int_0^h u\,dy \tag{11}$$

denote the mean flow velocity in the x-direction. Applying this 'operator' to (10) we get

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\frac{h_0}{3}\frac{\partial^3 h}{\partial x\partial t^2}+R\Omega^2\frac{\partial h}{\partial x}-2\Omega\frac{\partial h}{\partial t}-\eta U^2-\nu_{\text{ev}}\frac{\partial^2 U}{\partial x^2}=\ddot{\mathfrak{X}}_r\sin\left(\frac{x}{R}\right)-\ddot{\mathfrak{Y}}_r\cos\left(\frac{x}{R}\right), \tag{12}$$

where

$$\beta U^2 \doteqdot -\frac{\nu}{h_0}\left[\frac{\partial u}{\partial y}\right]_{y=0}, \qquad \nu_{\text{ev}}\frac{\partial^2 U}{\partial x^2}\doteqdot -\frac{\partial}{\partial x}\frac{1}{h_0}\int_0^{h_0}(u-U)^2 dy \tag{13}$$

are models for dissipation due to wall friction and internal fluid friction, respectively. In the first equation β is a friction coefficient (known from head loss in pipe flow) and ν_{ev} in the second equation is a so-called eddy viscosity coefficient. Applying (11) to the continuity equation (5), the latter can be written as

$$\frac{\partial h}{\partial t}=-\frac{\partial(hU)}{\partial x}. \tag{14}$$

At this point we introduce the nondimensional 'traveling wave' variable

$$\xi=\frac{x}{R}-(\omega-\Omega)t, \tag{15}$$

where ω is the angular whirling velocity of the vessel, which is assumed to be close, but not equal, to the imposed angular velocity Ω. The body force, that is, the right hand side of (12), needs to be transformed into this 'coordinate system'. To this end we write

$$\begin{aligned}\sin\left(\frac{x}{R}\right) &= \sin\left[\left\{\frac{x}{R}-(\omega-\Omega)\,t\right\}+(\omega-\Omega)\,t\right]=\sin\left(\xi+(\omega-\Omega)\,t\right) \\ &= \sin\xi\cos(\omega-\Omega)t-\cos\xi\sin(\omega-\Omega)t,\end{aligned} \tag{16}$$

and similarly for the $\cos(x/R)$ term. This expresses the body force in terms of ξ but it is still given in terms of the rotor-fixed coordinate system $(\bar{x}, \bar{y})$. It will be shown in section 5 that $\omega < \Omega$. Then, $\xi = x/R - (\omega - \Omega)t$ represents a backward traveling wave. Thus, in order to transform the rotor-fixed body force components $(\mathfrak{F}, \mathfrak{G})^T$ *backwards*, into the traveling wave-fixed components $(\mathfrak{F}_w, \mathfrak{G}_w)^T$, we need the transformation†

$$\left\{ \begin{array}{c} \mathfrak{F}_w \\ \mathfrak{G}_w \end{array} \right\} = \left[\begin{array}{cc} \cos(\omega - \Omega)t & -\sin(\omega - \Omega)t \\ \sin(\omega - \Omega)t & \cos(\omega - \Omega)t \end{array} \right] \left\{ \begin{array}{c} \mathfrak{F} \\ \mathfrak{G} \end{array} \right\}, \tag{17}$$

which gives the simple expressions $\mathfrak{F}_w = \ddot{\mathfrak{X}}_r \sin\xi - \ddot{\mathfrak{Y}}_r \cos\xi$ and $\mathfrak{G}_w = \ddot{\mathfrak{X}}_r \cos\xi + \ddot{\mathfrak{Y}}_r \sin\xi$. Writing $U = U(\xi)$, $h = h_0 + h'(\xi)$, (12) can now be written as

$$\begin{aligned} \Omega(2\omega - \Omega)\frac{\partial h'}{\partial \xi} - (\omega - \Omega)\frac{\partial U}{\partial \xi} &= -\frac{U}{R}\frac{\partial U}{\partial \xi} + \frac{\nu_{ev}}{R^2}\frac{\partial^2 U}{\partial \xi^2} \\ &+ \beta U^2 - \frac{h_0}{3}\frac{(\omega - \Omega)^2}{R}\frac{\partial^3 h'}{\partial \xi^3} + \ddot{\mathfrak{X}}_r \sin\xi - \ddot{\mathfrak{Y}}_r \cos\xi, \end{aligned} \tag{18}$$

where h' is the fluid layer thickness perturbation. The continuity equation (14) can be written as

$$-(\omega - \Omega)\frac{\partial h'}{\partial \xi} + \frac{h_0}{R}\frac{\partial U}{\partial \xi} = -\frac{U}{R}\frac{\partial h'}{\partial \xi} - \frac{1}{R}h'\frac{\partial U}{\partial \xi}. \tag{19}$$

4 Nondimensionalization

In order to recast the governing equations into nondimensional form we introduce the parameters

$$\begin{aligned} &\delta = \left(\frac{h_0}{R}\right)^{\frac{1}{2}}, \; h_* = \frac{h'}{R}, \; \omega_* = \frac{\omega}{\omega_s}, \; \omega_s = \left(\frac{K_x}{M}\right)^{\frac{1}{2}}, \; \Omega_* = \frac{\Omega}{\omega_s}, \; t_* = \Omega t, \; \bar{\omega}_s = \frac{\omega_s}{\Omega} = \frac{1}{\Omega_*}, \\ &c_0 = R\Omega\left(\frac{h_0}{R}\right)^{\frac{1}{2}}, \; p_* = \frac{p}{\frac{1}{2}\rho c_0^2}, \; U_* = \frac{U}{c_0}, \; \beta_* = R\beta, \; \nu_* = \frac{\nu_{ev}}{Rc_0}, \; \alpha^2\ddot{\mathfrak{X}}_* = \frac{\ddot{\mathfrak{X}}}{h_0\Omega^2}, \; \alpha^2\ddot{\mathfrak{Y}}_* = \frac{\ddot{\mathfrak{Y}}}{h_0\Omega^2}, \\ &x_* = \frac{x_r}{h_0}, \; y_* = \frac{y_r}{h_0}, \; \mu = \frac{m}{M}, \; \zeta = \frac{C}{(MK_x)^{\frac{1}{2}}}, \; F_{x*} = \frac{F_x}{M\Omega^2 h_0}, \; F_{y*} = \frac{F_y}{M\Omega^2 h_0}, \; \sigma = \frac{s}{R}, \; \chi = \frac{K_y}{K_x}. \end{aligned} \tag{20}$$

Here (in the second line of (20)) and in the following, α is a small 'bookkeeping' parameter, assumed to be of the order (fluid layer thickness)/(vessel radius), that is, $\alpha = O(h_0/R) = O(\delta^2)$. It is noted that, in terms of standard shallow water wave theory [8] the parameter c_0 corresponds to the shallow water wave speed $(gh_0)^{\frac{1}{2}}$, but here the gravity acceleration g is replaced by the centrifugal acceleration $R\Omega^2$.

4.1 The fluid equations

In the following it will be assumed that the variables in the traveling wave frame depends on the *slow time* $\tau = \alpha t_*$ only‡. A nondimensional version of (18) can then be obtained as

$$\begin{aligned} -(1 - 2\tilde{\omega})\frac{\partial h_*}{\partial \xi} + \delta^{-1}(1 - \tilde{\omega})\frac{\partial U_*}{\partial \xi} &= -U_*\frac{\partial U_*}{\partial \xi} + \nu_*\frac{\partial^2 U_*}{\partial \xi^2} + \beta_* U_*^2 - \tfrac{1}{3}\delta^2(1 - \tilde{\omega})^2\frac{\partial^3 h_*}{\partial \xi^3} \\ &+ \alpha^2\left(\alpha^2\frac{\partial^2 x_*}{\partial \tau^2} - 2\alpha\frac{\partial y_*}{\partial \tau} - x_*\right)\sin\xi - \alpha^2\left(\alpha^2\frac{\partial^2 y_*}{\partial \tau^2} + 2\alpha\frac{\partial x_*}{\partial \tau} - y_*\right)\cos\xi, \end{aligned} \tag{21}$$

where $\tilde{\omega} = \omega/\Omega = \omega_*/\Omega_*$. It is noted that $\ddot{\mathfrak{X}}_* = -x_* + O(\alpha)$, $\ddot{\mathfrak{Y}}_* = -y_* + O(\alpha)$ in the traveling wave frame.

†It is noted that although $\mathfrak{G}$ is ignored in (3) we still need to consider it here in order to get $\mathfrak{F}$ correctly transformed into $\mathfrak{F}_w$.

‡This time dependence applies thus to (18), but not to the rotor matrix equation (1).

The nondimensional version of the continuity equation (19) is

$$(1-\tilde{\omega})\delta^{-1}\frac{\partial h_*}{\partial \xi}+\frac{\partial U_*}{\partial \xi}=-U_*\frac{\partial h_*}{\partial \xi}+h_*\frac{\partial U_*}{\partial \xi}. \tag{22}$$

4.2 The rotor matrix equation

Applying (20) to (1) we obtain

$$\begin{bmatrix} 1+\mu & 0 \\ 0 & 1+\mu \end{bmatrix}\begin{Bmatrix} x_*'' \\ y_*'' \end{Bmatrix}+\begin{bmatrix} \zeta\bar{\omega}_s & -2(1+\mu) \\ 2(1+\mu) & \zeta\bar{\omega}_s \end{bmatrix}\begin{Bmatrix} x_*' \\ y_*' \end{Bmatrix} \tag{23}$$
$$+\begin{bmatrix} \bar{\omega}_s^2-(1+\mu) & -\zeta\bar{\omega}_s \\ \zeta\bar{\omega}_s & \chi\bar{\omega}_s^2-(1+\mu) \end{bmatrix}\begin{Bmatrix} x_* \\ y_* \end{Bmatrix}=\begin{Bmatrix} \mu\sigma\delta^{-1} \\ 0 \end{Bmatrix}+\begin{Bmatrix} F_{*x} \\ F_{*y} \end{Bmatrix},$$

where a dash denotes differentiation with respect to the nondimensional time t_*.

5 Perturbation solution of the fluid equations

Let

$$h_*(\xi)=\alpha h_1(\xi)+\alpha^2 h_2(\xi)+\cdots,\quad U_*(\xi)=\alpha U_1(\xi)+\alpha^2 U_2(\xi)+\cdots,\quad \tilde{\omega}=\tilde{\omega}_0+\alpha\tilde{\omega}_1+\cdots. \tag{24}$$

Also, let $\nu_*=\alpha\nu_1$, and assume that $\delta^2/\alpha=O(1)$. Collecting the terms of order α^1 then gives

$$-(1-2\tilde{\omega}_0)\frac{\partial h_1}{\partial \xi}+\delta^{-1}(1-\tilde{\omega}_0)\frac{\partial U_1}{\partial \xi}=0,\qquad (1-\tilde{\omega}_0)\delta^{-1}\frac{\partial h_1}{\partial \xi}+\frac{\partial U_1}{\partial \xi}=0. \tag{25}$$

These equations only have non-trivial solutions if

$$\begin{vmatrix} -(1-2\tilde{\omega}_0) & \delta^{-1}(1-\tilde{\omega}_0) \\ \delta^{-1}(1-\tilde{\omega}_0) & 1 \end{vmatrix}=0, \tag{26}$$

which gives $\tilde{\omega}_0=1+\delta^2\pm\delta(1+\delta^2)^{\frac{1}{2}}$. The traveling wave definition (15) gives that these frequencies correspond to the possible wave speeds $c_\pm=c_0\left[\delta\pm(1+\delta^2)^{\frac{1}{2}}\right]$. The '+ solution' corresponds to a progressive (forward traveling) wave and the '− solution' to a retrograde (backward traveling) wave. Experiments show that only the latter type exists [7], that is, is stable; accordingly the '− solution' is used in the following. [This means that the whirling frequency ω is slightly lower than the rotor frequency Ω, i.e., that $\tilde{\omega}<1$, as was mentioned in Section 3.] The continuity equation (second equation in (25)) now gives $U_1=h_1c_-/c_0=\left[\delta-(1+\delta^2)^{\frac{1}{2}}\right]h_1$. Employing this expression, the terms of order α^2 in the expansions of (21) and (22) can be combined, to eliminate h_2 and U_2 and to give

$$\mathcal{A}_1\frac{\partial h_1}{\partial \xi}-\mathcal{B}_1h_1\frac{\partial h_1}{\partial \xi}-\mathcal{C}_1\frac{\partial^3 h_1}{\partial \xi^3}-\mathcal{D}_1\frac{\partial^2 h_1}{\partial \xi^2}+\mathcal{E}_1h_1^2=x_*\sin\xi-y_*\cos\xi, \tag{27}$$

where

$$\mathcal{A}_1=-2\tilde{\omega}_1\frac{(1+\delta^2)^{\frac{1}{2}}}{\delta},\ \mathcal{B}_1=3\left(\frac{c_-}{c_0}\right)^2,\ \mathcal{C}_1=\tfrac{1}{3}\delta^2\left(\frac{c_-}{c_0}\right)^2,\ \mathcal{D}_1=-\nu_1\frac{c_-}{c_0},\quad \mathcal{E}_1=\beta_*\left(\frac{c_-}{c_0}\right)^2. \tag{28}$$

Equation (27) is a forced Korteweg-de Vries-Burgers equation. Without dissipation ($\mathcal{D}_1=\mathcal{E}_1=0$) and external forcing ($x_*=y_*=0$) it reduces to the Korteweg-de Vries equation. The Burgers equation is obtained with $\mathcal{C}_1=\mathcal{E}_1=0$ and $x_*=y_*=0$.

5.1 A boundary layer problem

In the following we will assume that $\mathcal{E}_1 = 0$, that is, boundary friction is ignored. Then, one integration gives

$$-\mathcal{A}_1 h_1 + \tfrac{1}{2}\mathcal{B}_1 h_1^2 + \mathcal{C}_1 \frac{\partial^2 h_1}{\partial \xi^2} + \mathcal{D}_1 \frac{\partial h_1}{\partial \xi} = x_* \cos\xi + y_* \sin\xi + \mathfrak{C}, \tag{29}$$

where $\mathfrak{C}$ is an integration constant. The periodicity conditions which must be satisfied are

$$h_1(0) = h_1(2\pi), \quad \frac{\partial h_1}{\partial \xi}(0) = \frac{\partial h_1}{\partial \xi}(2\pi). \tag{30}$$

In the light of these conditions we will set $\mathfrak{C} = 0$ in (29). The constant $\mathcal{C}_1$ is now assumed to be small (as it is in the case of a typical washing machine). Thus, let $\epsilon = \mathcal{C}_1$ be a small parameter and write (29) on the form

$$\epsilon \frac{\partial^2 h_1}{\partial \xi^2} + d_1 \frac{\partial h_1}{\partial \xi} + a_1 h_1 + b_1 h_1^2 = \epsilon\,(x\cos\xi + y\sin\xi)\,, \tag{31}$$

where

$$d_1 = \mathcal{D}_1, \quad a_1 = -\mathcal{A}_1, \quad b_1 = \tfrac{1}{2}\mathcal{B}_1, \quad x = \frac{x_*}{\epsilon}, \quad y = \frac{y_*}{\epsilon}. \tag{32}$$

Here it is assumed that $x_*/\epsilon,\ y_*/\epsilon = O(1)$. Let the 'outer variable' $h_1(\xi)$, away from possible boundary layers, be expanded as

$$h_1(\xi) = \epsilon\kappa_1(\xi) + \epsilon^2\kappa_2(\xi) + \cdots \tag{33}$$

The terms of order ϵ^1 are

$$d_1 \frac{d\kappa_1}{d\xi} + a_1\kappa_1 = x\cos\xi + y\sin\xi. \tag{34}$$

The complete solution to this equation is

$$\kappa_1(\xi) = \frac{x}{a_1^2 + d_1^2}\{a_1\cos\xi + d_1\sin\xi\} + \frac{y}{a_1^2 + d_1^2}\{a_1\sin\xi - d_1\cos\xi\} + \mathfrak{C}e^{-(a_1/d_1)\xi}, \tag{35}$$

where $\mathfrak{C}$ is again a constant. The periodicity conditions (30) can only be satisfied if $\mathfrak{C} = 0$. On the other hand, this choice is not associated with any problems. This is thus a boundary value problem without a boundary layer![§]

Returning to the original variables, we get the fluid layer thickness perturbation, described to leading order, as

$$h_1(\xi) = \frac{x_*}{\mathcal{A}_1^2 + \mathcal{D}_1^2}\{-\mathcal{A}_1\cos\xi + \mathcal{D}_1\sin\xi\} - \frac{y_*}{\mathcal{A}_1^2 + \mathcal{D}_1^2}\{\mathcal{A}_1\sin\xi + \mathcal{D}_1\cos\xi\}. \tag{36}$$

The determination of $\mathcal{A}_1$ necessitates consideration of the next order in the expansion in α, due to the still undetermined $\tilde{\omega}_1$, see (28). This problem will not be considered in the present paper; we will be content with a qualitative solution of the coupled fluid-structure problem.

6 Coupling with the rotor equation

The nondimensional version of the pressure equation (9), evaluated on the vessel surface $y = 0$, takes the form

$$p_*(0) = \alpha\left[\delta^4\left(\frac{c_-}{c_0}\right)^2\frac{\partial^2 h_1}{\partial \xi^2} + 2\left(1 + 2\delta\frac{c_-}{c_0}\right)h_1\right] + O(\alpha^2) \approx \alpha 2\left(1 + 2\delta\frac{c_-}{c_0}\right)h_1, \tag{37}$$

[§]Such cases are not uncommon, see e.g. the book by Bender & Orszag [9].

where the last approximation is made on the assumption that δ is small.

The fluid force components, acting in the radial and the tangential direction relative to the traveling wave, are given by

$$\mathfrak{F}_{\mathrm{r}*} = \frac{\Gamma}{\delta^2}\int_0^{2\pi} p_*(\xi,0)\cos\xi\, d\xi, \quad \mathfrak{F}_{\mathrm{t}*} = \frac{\Gamma}{\delta^2}\int_0^{2\pi} p_*(\xi,0)\sin\xi\, d\xi, \tag{38}$$

where $\Gamma = \frac{1}{4\pi}M_{\text{fluid}}/M$, where $M_{\text{fluid}} = \rho 2\pi R h_0 w$ is the mass of the contained fluid, w being the width of the vessel (the height 'out of the paper' in Fig. 1). Evaluation gives

$$\mathfrak{F}_{\mathrm{r}*} \approx -\Gamma\frac{\alpha}{\delta^2}2\pi\left(1+2\delta\frac{c_-}{c_0}\right)\frac{\mathcal{A}_1 x_* + \mathcal{D}_1 y_*}{\mathcal{A}_1^2+\mathcal{D}_1^2}, \quad \mathfrak{F}_{\mathrm{t}*} \approx \Gamma\frac{\alpha}{\delta^2}2\pi\left(1+2\delta\frac{c_-}{c_0}\right)\frac{\mathcal{D}_1 x_* - \mathcal{A}_1 y_*}{\mathcal{A}_1^2+\mathcal{D}_1^2}. \tag{39}$$

Let

$$a = \Gamma\frac{\alpha}{\delta^2}2\pi\left(1+2\delta\frac{c_-}{c_0}\right)\frac{\mathcal{A}_1}{\mathcal{A}_1^2+\mathcal{D}_1^2}, \quad d = \Gamma\frac{\alpha}{\delta^2}2\pi\left(1+2\delta\frac{c_-}{c_0}\right)\frac{\mathcal{D}_1}{\mathcal{A}_1^2+\mathcal{D}_1^2}. \tag{40}$$

Then we can write

$$\mathfrak{F}_{\mathrm{r}*} = -a x_* - d y_*, \quad \mathfrak{F}_{\mathrm{t}*} = d x_* - a y_*. \tag{41}$$

In Section 5 it was assumed that $\delta^2/\alpha = O(1)$. Thus, in (39) we must likewise assume that $\alpha/\delta^2 = O(1)$.

In order to transform these force components forward to the rotor-fixed coordinate system, we need the transformation inverse to (17), which in terms of nondimensional parameters is given by

$$\begin{Bmatrix} F_{*x} \\ F_{*y} \end{Bmatrix} = \begin{bmatrix} \cos(\tilde{\omega}-1)t_* & \sin(\tilde{\omega}-1)t_* \\ -\sin\tilde{\omega}-1)t_* & \cos(\tilde{\omega}-1)t_* \end{bmatrix}\begin{Bmatrix} \mathfrak{F}_{\mathrm{r}*} \\ \mathfrak{F}_{\mathrm{t}*} \end{Bmatrix}. \tag{42}$$

Since $\tilde{\omega}$ is close to 1 the coefficients in the matrix are slowly varying parameters, and we will write $\cos(\tilde{\omega}-1)t_* = 1-[1-\cos(\tilde{\omega}-1)t_*] \doteqdot 1-\alpha[1-\cos(\tilde{\omega}-1)t_*]$ and similarly $\sin(\tilde{\omega}-1)t_* \doteqdot \alpha\sin(\tilde{\omega}-1)t_*$, which is true for non-large times t_*. Thus, inserting (39) and (42) into (23), we get

$$\begin{aligned}&\begin{bmatrix} 1+\mu & 0 \\ 0 & 1+\mu \end{bmatrix}\begin{Bmatrix} x_*'' \\ y_*'' \end{Bmatrix} + \left(\begin{bmatrix} 0 & -2(1+\mu) \\ 2(1+\mu) & 0 \end{bmatrix} + \alpha\begin{bmatrix} \zeta\bar{\omega}_s & 0 \\ 0 & \zeta\bar{\omega}_s \end{bmatrix}\right)\begin{Bmatrix} x_*' \\ y_*' \end{Bmatrix} \\ &+\left(\begin{bmatrix} \bar{\omega}_s^2-(1+\mu)+a & d \\ -d & \chi\bar{\omega}_s^2-(1+\mu)+a \end{bmatrix} + \alpha\begin{bmatrix} a\mathfrak{c}-d\mathfrak{s} & d\mathfrak{c}+a\mathfrak{s} \\ -d\mathfrak{c}-a\mathfrak{s} & a\mathfrak{c}-d\mathfrak{s} \end{bmatrix}\right)\begin{Bmatrix} x_* \\ y_* \end{Bmatrix} \\ &= \begin{Bmatrix} \mu\sigma\delta^{-1} \\ 0 \end{Bmatrix},\end{aligned} \tag{43}$$

where the shorthand notation $\mathfrak{c} = 1-\cos(\tilde{\omega}-1)t_*$ and $\mathfrak{s} = \sin(\tilde{\omega}-1)t_*$ has been used. In (43) it has been used as well that the damping parameter ζ is of order of magnitude α; thus $\alpha\zeta$ has been written in place of ζ.

7 Investigation of stability

Limiting the following discussion to the symmetrix case $\chi = 1$ only, (43) can be reduced to a single complex equation. Let $z_* = x_* + \mathrm{i}y_*$; we get

$$\begin{aligned}&(1+\mu)z_*'' + \Big[\mathrm{i}2(1+\mu)+\alpha\zeta\bar{\omega}_s\Big]z_*' \\ &+\Big[\bar{\omega}_s^2-(1+\mu)+a-\mathrm{i}d+\alpha\Big\{(a-\mathrm{i}d)\left(e^{\mathrm{i}(\tilde{\omega}-1)t_*}-1\right)+\mathrm{i}\zeta\bar{\omega}_s\Big\}\Big]z_* = \mu\sigma\delta^{-1}.\end{aligned} \tag{44}$$

This equation is solved approximately, by applying the method of multiple scales. Let

$$z_*(t_*) = z_0(t_0, t_1) + \alpha z_1(t_0, t_1) + \cdots , \tag{45}$$

where $t_0 = t_*$ and $t_1 = \tau = \alpha t_*$. Due to lack of space details will not be given; we will just state the final result, which is

$$z_0(t_0, t_1) = +\frac{\mu\sigma\delta^{-1}}{\bar{\omega}_s^2 - (1+\mu) + a - id} + e^{-it_0}\left[\mathcal{F}_0(T_1)e^{is_0t_0} + \mathcal{G}_0(T_1)e^{-is_0t_0}\right] \tag{46}$$

where

$$s_0 = \left[1 + \frac{\bar{\omega}_s^2 - (1+\mu) + a - id}{1+\mu}\right]^{\frac{1}{2}} = \left[\frac{\bar{\omega}_s^2 + a - id}{1+\mu}\right]^{\frac{1}{2}} \tag{47}$$

and

$$\mathcal{F}_0(t_1) = \mathcal{A}_0 \exp\left(-\frac{d + \zeta\bar{\omega}_s s_0 + ia}{2s_0(1+\mu)}t_1\right), \quad \mathcal{G}_0(t_1) = \mathcal{B}_0 \exp\left(\frac{d - \zeta\bar{\omega}_s s_0 + ia}{2s_0(1+\mu)}t_1\right). \tag{48}$$

From these expressions it is clear that the structural damping (with coefficient ζ) stabilizes the dynamic part of z_0 (the send term in (46)), while the fluid viscosity (represented by d) has a destabilizing effect. In fact, if $\zeta = 0$ and $d > 0$ then the solution z_0 is always unstable.

This fact can easily be verified by considering the characteristic polynomial for the linearized version of (44). Writing this as

$$(a_0 + ib_0)s^2 + (a_1 + ib_1)s + (a_2 + ib_2) = 0, \tag{49}$$

we have

$$\begin{aligned} a_0 &= 1+\mu, \quad & b_0 &= 0, \quad & a_1 &= 0, \\ b_1 &= 2(1+\mu), \quad & a_2 &= \bar{\omega}_s^2 - (1+\mu) + a, \quad & b_2 &= -d. \end{aligned} \tag{50}$$

The polynomial (49) with complex coefficients can be written as a one with real coefficients, as [10]

$$c_0 s^4 + c_1 s^3 + c_2 s^2 + c_3 s + c_4 = 0, \tag{51}$$

with

$$\begin{aligned} &c_0 = a_0^2 + b_0^2 = (1+\mu)^2, \qquad c_1 = 2(a_0a_1 + b_0b_1) = 0, \\ &c_2 = 2(a_0a_2 + b_0b_2) + a_1^2 + b_1^2 = 2(1+\mu)\left\{\bar{\omega}_s^2 - (1+\mu) + a\right\} + 4(1+\mu)^2, \\ &c_3 = 2(a_1a_2 + b_1b_2) = -4(1+\mu)d, \quad c_4 = a_2^2 + b_2^2 = \left\{\bar{\omega}_s^2 - (1+\mu) + a\right\}^2 + d^2. \end{aligned} \tag{52}$$

From the Routh-Hurwitz conditions [11] it is known that a necessary (but not sufficient) condition for stability is that all coefficients have the same sign. Since $c_3 < 0$ and all other $c_i > 0$ it is clear that the solution z_0 is unstable. If the structural damping terms are moved back to order α^0 then $b_2 = \zeta\bar{\omega}_s - d$, and the stabilizing effect of damping ($\zeta\bar{\omega}_s$) may possibly keep the destabilizing effect of viscosity (d) in check.

8 Concluding remarks

In this paper the dynamics of the so-called fluid balancer has been investigated based on a model of a rotor containing a small amount of liquid. The thin internal fluid layer, which forms due to the rotation, is described in terms of shallow water wave theory. A perturbation approach gives that the fluid layer thickness variation is described by a forced Korteweg-de Vries-Burgers equation. An approximate solution to this equation is given. The form of this equation, where the term of the highest derivative vanishes in the lowest order approximation, suggests application of boundary layer theory. Nonetheless, the lowest order approximation is able to satisfy all boundary conditions which, in the present case, are periodicity conditions. Thus, a boundary layer does not come into play.

The stability of the solution has been investigated. It was found that the structural damping stabilizes the dynamic part of the solution (i.e. the rotor deflection), while the fluid viscosity destabilizes it. This is actually not surprising, since it is known that damping of motion with respect to stationary axes (as the present damping model) always tend to stabilize the motion, while damping with respect to rotating axes (as the present viscous dissipation model) tend to destabilize it [12].

The working principle of the fluid balancer can be explained explicitly/analytically in terms of the solution (46); but this has been done in earlier publications [6, 13] and will thus not be repeated here.

Finally, it is noted that, rather than the fluid balancer itself, perhaps the most interesting aspect of the present work is the investigation of solutions to the forced Korteweg de Vries-Burgers equation. While analytical solutions to the unforced (homogeneous) problem are well known, the forced problem is, except for a few special cases, still unsolved [14].

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