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MODIFIED KDV AND ASYMPTOTICS OF SOLUTIONS

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1. Introduction

We survey the large time asymptotics of solutions to the Cauchy problem for the modified Korteweg-de Vries (KdV) equation

$$\begin{cases} \partial_t u - \frac{1}{3}\partial_x^3 u = \lambda \partial_x \left(u^3\right), \ t > 0, \ x \in \mathbf{R}, \\ u\left(0, x\right) = u_0\left(x\right), \ x \in \mathbf{R} \end{cases} \tag{1.1}$$

with mass condition $\int_{\mathbf{R}} u_0\left(x\right) dx \neq 0, \lambda \in \mathbf{R}$

Large time asymptotics of solutions to the generalized Korteweg-de Vries equation

$$\partial_t u - \frac{1}{3} u_{xxx} = \partial_x \left(|u|^{\rho - 1} u \right)$$

was studied by W. A. Strauss [16], [17], S. Klainerman [11], S. Klainerman and G. Ponce[12], J. Shatah [15] G. Ponce and L. Vega [14], F.M. Christ and M.I. Weinstein [2], and ours [5] for different values of ρ in the super critical region $\rho > 3$. Stability of solutions in the neighborhood of solitary waves was shown by T. Mizumachi [13].

For the KdV equation and the modified KdV equation (1.1), the Cauchy problem was solved by the Inverse Scattering Transform method and thus the large time asymptotic behavior of solutions was studied (see [1], [3]). This method depends on the nonlinearity in the equation and note that if we replace nonlinear term by $a\left(t\right) \partial_x \left(u^3\right)$ with $|a\left(t\right)| \leq C$ it does not work. Therefore it is important to develop alternative methods for studying the large time asymptotics of solutions to the Cauchy problem (1.1).

To state our results precisely we introduce *Notation and Function Spaces.* The weighted Sobolev space is

$$\mathbf{H}^{k,s} = \left\{ \varphi \in \mathbf{S}'; \|\phi\|_{\mathbf{H}^{k,s}} = \left\| \langle x \rangle^s \langle i\partial_x \rangle^k \phi \right\|_{\mathbf{L}^2} < \infty \right\},$$

$k, s \in \mathbf{R}, 1 \leq p \leq \infty$, $\langle x \rangle = \sqrt{1 + x^2}$, $\langle i\partial_x \rangle = \sqrt{1 - \partial_x^2}$. We also use the notation $\mathbf{H}^k = \mathbf{H}^{k,0}$ shortly.

2. Zero total mass case

In [6], we showed the large time asymptotics of solutions to (1.1) in the case of small real-valued initial data $u_0 \in \mathbf{H}^{1,1}$ with zero total mass assumption $\int_{\mathbf{R}} u_0(x)dx = 0$. We have the asymptotics

$$\begin{aligned} u(t,x) &= \sqrt{2\pi} t^{-\frac{1}{3}} \mathrm{Re} Ai \left(x t^{-\frac{1}{3}} \right) \widehat{u_+}\left(\varkappa\right) \exp\left(-3i\pi \left|\widehat{u_+}\left(\varkappa\right)\right|^2 \log t \right) \\ &\quad + O\left(t^{-\frac{1}{2} - \lambda} \right) \end{aligned} \tag{2.1}$$

for large time t, where $0 < \lambda < \frac{1}{21}$, $\varkappa = (x/t)^{\frac{1}{2}}$, $\widehat{u_+} \in \mathbf{L}^\infty$ is uniquely defined by the data u_0, is such that $\widehat{u_+}(0) = 0$, and

$$Ai(x) = \frac{1}{\pi}\int_0^\infty e^{ix\xi - \frac{1}{3}\xi^3} d\xi$$

is the Airy function.

It is known that the following asymptotics for the Airy function

$$Ai(\eta) = C|\eta|^{-\frac{1}{4}} \exp\left(\frac{2}{3}i|\eta|^{\frac{3}{2}} + i\frac{\pi}{4}\right) + O\left(|\eta|^{-7/4}\right) \text{ as } \eta = xt^{-\frac{1}{3}} \to \infty$$

is valid. Airy function oscillates rapidly and decays slowly as $x \to \infty$, When $x \to -\infty$, $Ai(\eta)$ decays exponentially as

$$Ai(\eta) = C|\eta|^{-\frac{1}{4}} e^{-\frac{2}{3}|\eta|^{\frac{3}{2}}} + O\left(|\eta|^{-7/4} e^{-\frac{2}{3}|\eta|^{\frac{3}{2}}}\right) \text{ as } \eta = xt^{-\frac{1}{3}} \to -\infty.$$

These asymptotics are obtained by the stationary method (see [4]).

3. Stability of the self similar solution I

In [7] we showed

Proposition 3.1. *Assume hat the initial data $u_0 \in \mathbf{H}^{1,1}$ are real - valued functions with sufficiently small norm $\|u_0\|_{\mathbf{H}^{1,1}} = \varepsilon$. Then there exists a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,1}\right)$ of the Cauchy problem for (1.1) such that*

$$\langle t\rangle^{\frac{1}{3}-\frac{1}{3\beta}} \|u(t)\|_{\mathbf{L}^p} \le C\varepsilon$$

for all $t \in \mathbf{R}$, where $4 < \beta \le \infty$.

We denote by

$$v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right)$$

the self similar solution of (1.1). Note that if the function $f_m(\eta)$ satisfies the second Peinleve equation

$$\frac{d^2}{d\eta^2} f_m + \eta f_m - 3f_m^3 = 0,$$

then v_m satisfies (1.1).

The next result from [7] says the asymptotic stability of solutions in the neighborhood of the self similar solution.

Proposition 3.2. *Let $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{1,1}\right)$ be the solution of (1.1) constructed in Proposition 3.1 and $\int f_m(x)\,dx = \int u_0(x)\,dx$. Then for any $u_0 \in \mathbf{H}^{1,1}$, there exist unique functions H_j and $B_j \in \mathbf{L}^\infty$ (B_j are real-valued), $j = 1,2$, such that the following asymptotic formula is valid for large time $t \ge 1$*

$$\begin{aligned} u(t,x) &= t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right) \\ &\quad + \sqrt{2\pi}t^{-\frac{1}{3}} Re Ai\left(xt^{-\frac{1}{3}}\right)\left(H_1(\varkappa)\exp\left(iB_1(\varkappa)\log|x|t^{-\frac{1}{3}}\right)\right. \\ &\quad \left. + H_2(\varkappa)\exp\left(iB_2(\varkappa)\log|x|t^{-\frac{1}{3}}\right)\right) \\ &\quad + O\left(\varepsilon t^{4\gamma-\frac{5}{12}}\left(1+|x|t^{-\frac{1}{3}}\right)^{-1/4}\right), \end{aligned} \tag{3.1}$$

where $\gamma \in \left(0, \frac{1}{50}\right)$ and $\varkappa = (x/t)^{\frac{1}{2}}$.

Since H_j in the second term of the right-hand side of (3.1) are not necessarily zero at the origin, and asymptotic property of solutions to the second Peinleve equation is not stated explicitly in [7], therefore it is not determined which one is the leading term $f_m(\eta)$ or $Ai(\eta)$ from the previous work. In the recent work [8], we proved that the leading term of $f_m(\eta)$ as $\eta = xt^{-\frac{1}{3}} \to \infty$ is similar to the leading term of $Ai(\eta)$ for $\eta > 0$. Thus the previous work says that the main term consists of the first and the second terms of the right-hand side of (3.1). In [8], we developed the factorization technique to obtain the sharp time decay estimate of solutions and make an improvement of the previous result from [7].

4. Stability of the self similar solution II

We are now in a position to state our first result from [8].

Theorem 4.1. *Assume that the initial data*

$$u_0 \in \mathbf{H}^s \cap \mathbf{H}^{0,1}, s > \frac{3}{4}$$

are real-valued with a sufficiently small norm

$$\|u_0\|_{\mathbf{H}^s \cap \mathbf{H}^{0,1}} \leq \varepsilon.$$

Then there exists a unique global solution

$$\mathcal{F} e^{-\frac{it}{3}\partial_x^3} u \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty \cap \mathbf{H}^{0,1}\right)$$

of the Cauchy problem (1.1). Furthermore the estimate

$$\sup_{t>0}\left(\left\|\mathcal{F} e^{-\frac{it}{3}\partial_x^3} u(t)\right\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{1}{6}} \left\|x e^{-\frac{it}{3}\partial_x^3} u(t)\right\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{1}{3}\left(1-\frac{1}{p}\right)} \|u(t)\|_{\mathbf{L}^p}\right) \leq C\varepsilon$$

is true, where $p > 4$.

In order to state the stability of global solutions in the neighborhood of the self similar solution

$$v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right),$$

we need

Theorem 4.2. *Assume that m is sufficiently small real number. Then there exists a unique real-valued solution of the Cauchy problem (1.1) in the form $v_m(t,x) = t^{-\frac{1}{3}} f_m\left(xt^{-\frac{1}{3}}\right)$, such that*

$$\int f_m(x)\,dx = m,$$

$$\mathcal{F} e^{-\frac{it}{3}\partial_x^3} v_m \in \mathbf{C}\left([1,\infty); \mathbf{L}^\infty\right), \; x e^{-\frac{it}{3}\partial_x^3} v_m \in \mathbf{C}\left([1,\infty); \mathbf{L}^2\right).$$

Furthermore the estimates

$$\sup_{t>1}\left(\left\|\mathcal{F} e^{-\frac{it}{3}\partial_x^3} v_m(t)\right\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}} \left\|x e^{-\frac{it}{3}\partial_x^3} v_m(t)\right\|_{\mathbf{L}^2}\right) \leq C|m|$$

and

$$\frac{1}{2}|m|\, t^{-\frac{1}{3}\left(1-\frac{1}{p}\right)} \leq \|v_m(t)\|_{\mathbf{L}^p} \leq 2|m|\, t^{-\frac{1}{3}\left(1-\frac{1}{p}\right)}$$

are true, where $p > 4$.

Theorem 4.3. *Suppose that*

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} f_m(x)\,dx = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} u_0(x)\,dx = m \neq 0.$$

Let $u(t,x)$ and $v_m(t,x)$ be the solutions constructed in Theorem 4.1 and Theorem 4.2, respectively. Then there exists a $\gamma > 0$ such that the asymptotics

$$|u(t,x) - v_m(t,x)| \leq C\varepsilon t^{-\frac{1}{2}+\gamma} \tag{4.1}$$

for $x > 0$ and

$$|u(t,x) - v_m(t,x)| \leq C\varepsilon t^{-\frac{1}{2}+\gamma}\left\langle xt^{-\frac{1}{3}}\right\rangle^{-\frac{3}{4}} \tag{4.2}$$

for $x \leq 0$ are true for large $t \geq 1$. Also the sharp time decay estimate of solutions is valid, namely there exist positive constants C_1, C_2 such that

$$C_1\varepsilon t^{-\frac{1}{3}\left(1-\frac{1}{q}\right)} \leq \|u(t)\|_{\mathbf{L}^q} \leq C_2\varepsilon t^{-\frac{1}{3}\left(1-\frac{1}{q}\right)}$$

for $4 < q < \infty$.

5. Strategy of proofs in [8]

Local existence and uniqueness of solutions to the Cauchy problem (1.1) was shown when $u_0 \in \mathbf{H}^s$, $s > \frac{3}{4}$ and the estimate of solutions such that $\int_0^T \|\partial_x u(t)\|_{\mathbf{L}^\infty}^4 dt \leq C$ for some time T was also shown by C.E. Kenig-G. Ponce-L. Vega [9], [10]. By using the local existence result, we have

Theorem 5.1. *Assume that the initial data*

$$u_0 \in \mathbf{H}^s \cap \mathbf{H}^{0,1}, s > \frac{3}{4}.$$

Then there exists a unique local solution u of the Cauchy problem (1.1) such that

$$\mathcal{U}(-t)\,u \in \mathbf{C}\left([0,T];\mathbf{H}^s \cap \mathbf{H}^{0,1}\right).$$

We can take $T > 1$ if the data are small in $\mathbf{H}^s \cap \mathbf{H}^{0,1}$ and we may assume that

$$\|\mathcal{F}\mathcal{U}(-t)\,u(1)\|_{\mathbf{L}^\infty} + \|\mathcal{J}u(1)\|_{\mathbf{L}^2} + \|u(1)\|_{\mathbf{L}^p} \leq \varepsilon, \tag{5.1}$$

where $p > 4$. To get the result, in [8] we showed a priori estimates of solutions under the following norm

$$\|u\|_{\mathbf{X}_T} = \sup_{t\in[1,T]}\left(\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{L}^\infty} + t^{-\frac{1}{6}}\|\mathcal{J}u(t)\|_{\mathbf{L}^2} + t^{\frac{1}{3}\left(1-\frac{1}{p}\right)}\|u(t)\|_{\mathbf{L}^p}\right),$$

where $\mathcal{J} = x - t\partial_x^2 = \mathcal{U}(t)\,x\,\mathcal{U}(-t)$. In particular, we use the factorization method to get a priori estimates of $\|\mathcal{F}\mathcal{U}(-t)\,u(t)\|_{\mathbf{L}^\infty}$ in Theorem 4.1 and $\left\|\mathcal{F}e^{-\frac{it}{3}\partial_x^3}v_m(t)\right\|_{\mathbf{L}^\infty}$ in 4.2. In order to prove these estimates we introduce the free evolution group

$$\mathcal{U}(t) = \mathcal{F}^{-1}e^{-\frac{it}{3}\xi^3}\mathcal{F},$$

dilation operator

$$\mathcal{D}_t\phi = |t|^{-\frac{1}{2}}\phi\left(xt^{-1}\right),$$

scaling operator

$$(\mathcal{B}\phi)(x) = \phi\left(x|x|^{-\frac{1}{2}}\right).$$

Define the cut off function $\chi(\xi) \in \mathbf{C}^2(\mathbf{R})$ such that

$$\chi(\xi) = 0 \text{ for } \xi \leq -\frac{1}{3}, \chi(\xi) = 1 \text{ for } \xi \geq \frac{1}{3}$$

and

$$\chi(\xi) + \chi(-\xi) \equiv 1.$$

Then we write

$$\begin{aligned} &\mathcal{U}(t)\mathcal{F}^{-1}\phi \\ = \; &\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(\xi x^{-1})\,d\xi \\ &+\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi - \frac{1}{3}\xi^3)}\phi(-\xi)\chi(\xi x^{-1})\,d\xi \end{aligned}$$

for $x > 0$. Also we have

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi + \frac{1}{3}\xi^3)}\phi(\xi)\,d\xi$$

for $x \leq 0$. Since $u = \mathcal{U}(t)\mathcal{F}^{-1}\phi$ is a real-valued function, we have $\phi(-\xi) = \overline{\phi(\xi)}$, hence

$$\begin{aligned} &\mathcal{U}(t)\theta(x)\mathcal{F}^{-1}\phi \\ = \; &\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{it(x^2\xi - \frac{1}{3}\xi^3)}\phi(\xi)\chi(\xi x^{-1})\,d\xi \\ &+\mathcal{D}_t\mathcal{B}\frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-it(x^2\xi - \frac{1}{3}\xi^3)}\overline{\phi(\xi)}\chi(\xi x^{-1})\,d\xi \\ = \; &\mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\phi + \overline{M\mathcal{V}\phi}\right) \end{aligned}$$

with $\theta(x) = 0$ for $x \leq 0$, and $\theta(x) = 1$ for $x > 0$, where the multiplication factor

$$M(t,x) = e^{\frac{2it}{3}x^3},$$

the phase function

$$S(x,\xi) = \frac{2}{3}x^3 - x^2\xi + \frac{1}{3}\xi^3,$$

and the operator

$$\mathcal{V}\phi = \frac{|t|^{\frac{1}{2}}\theta(x)}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS(x,\xi)}\phi(\xi)\chi(\xi x^{-1})\,d\xi.$$

Also we have

$$\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t\mathcal{B}\mathcal{W}\phi$$

for $x \leq 0$, where the operator

$$\mathcal{W}\phi = \frac{|t|^{\frac{1}{2}}(1-\theta(x))}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-itS_0(x,\xi)}\phi(\xi)\,d\xi,$$

and the phase function $S_0(x,\xi) = x^2\xi + \frac{1}{3}\xi^3$. If we define the new dependent variable $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$, then we obtain the representation

$$u(t) = \mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi} = \mathcal{D}_t\mathcal{B}\left(M\mathcal{V}\widehat{\varphi} + \overline{M\mathcal{V}\widehat{\varphi}}\right) + \mathcal{D}_t\mathcal{B}\mathcal{W}\widehat{\varphi} \tag{5.2}$$

The first term of the right-hand side of (5.2) is the main term comparing with the second one.

We also need the representation for the inverse evolution group $\mathcal{FU}(-t)$

$$\mathcal{FU}(-t)\,\phi = \mathcal{Q}\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi + \mathcal{R}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\phi, \tag{5.3}$$

where

$$\mathcal{D}_t^{-1}\phi = |t|^{\frac{1}{2}}\,\phi(xt)\,,\ \left(\mathcal{B}^{-1}\phi\right)(x) = \phi(x\,|x|)$$

and the operators

$$\mathcal{Q}\phi = \frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{itS(x,\xi)}\phi(x)\,xdx,$$

and

$$\mathcal{R}\phi = -\frac{2t\,|t|^{-\frac{1}{2}}}{\sqrt{2\pi}}\int_{-\infty}^0 e^{itS_0(x,\xi)}\phi(x)\,xdx.$$

Since $\mathcal{FU}(-t)\,\mathcal{L} = \partial_t\mathcal{FU}(-t)$, with $\mathcal{L} = \partial_t - \frac{1}{3}\partial_x^3$, applying the operator $\mathcal{FU}(-t)$ to equation (1.1) we get with $\widehat{\varphi} = \mathcal{FU}(-t)\,u$

$$\partial_t\widehat{\varphi} = \partial_t\mathcal{FU}(-t)\,u = \mathcal{FU}(-t)\,\mathcal{L}u = \mathcal{FU}(-t)\,\partial_x\left(u^3\right) = 3\mathcal{FU}(-t)\left(u^2u_x\right).$$

Then by (5.2) we find the following factorization property

$$\begin{aligned}&\partial_t\widehat{\varphi} = 3\mathcal{FU}(-t)\left(u^2u_x\right)\\ =\ & 3t^{-1}\mathcal{Q}\overline{M}\left(M\mathcal{V}\widehat{\varphi} + \overline{M\mathcal{V}\widehat{\varphi}}\right)^2\left(M\mathcal{V}i\xi\widehat{\varphi} + \overline{M\mathcal{V}i\xi\widehat{\varphi}}\right) + R.\end{aligned}$$

Note that

$$\begin{aligned}&\overline{M}\left(M\mathcal{V}\widehat{\varphi} + \overline{M\mathcal{V}\widehat{\varphi}}\right)^3 = M^2\left(\mathcal{V}\widehat{\varphi}\right)^3 + 3\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\\ &+3\overline{M}^2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2 + \overline{M}^4\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^3\end{aligned}$$

and for $\alpha \neq -1$

$$\mathcal{Q}(t)\,M^\alpha\phi = E^{-\frac{\alpha(2+\alpha)}{(1+\alpha)^2}}\mathcal{D}_{1+\alpha}\mathcal{Q}(t(1+\alpha))\,\phi, E = e^{-\frac{it}{3}\xi^3}.$$

Thus we obtain the equation for the new dependent variable $\widehat{\varphi}(t,\xi) = \mathcal{FU}(-t)\,u(t)$

$$\begin{aligned}&\partial_t\widehat{\varphi}(t,\xi)\\ =\ & 3t^{-1}E^{-\frac{8}{9}}\mathcal{D}_3\mathcal{Q}(3t)\left(\mathcal{V}\widehat{\varphi}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right)\\ &+3t^{-1}\mathcal{Q}(t)\left(2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\mathcal{V}i\xi\widehat{\varphi}\right) + \left(\mathcal{V}\widehat{\varphi}\right)^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\ &+3t^{-1}\mathcal{D}_{-1}\mathcal{Q}(-t)\left(\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\mathcal{V}i\xi\widehat{\varphi}\right) + 2\left(\mathcal{V}\widehat{\varphi}\right)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right)\\ &+3t^{-1}E^{-\frac{8}{9}}\mathcal{D}_{-3}\mathcal{Q}(-3t)\left(\overline{\mathcal{V}\widehat{\varphi}}\right)^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right) + R.\end{aligned} \tag{5.4}$$

Now we explain how to use equation (5.4) for estimating $|\widehat{\varphi}(t,\xi)|$ uniformly with respect to ξ. For the real-valued solution u, we have $\overline{\widehat{\varphi}(t,\xi)} = \widehat{\varphi}(t,-\xi)$, hence it is sufficient to consider the case $\xi > 0$ only.

The second term of the right hand side. of (5.4) is a main term. We have

$$
\begin{aligned}
& 3t^{-1}\mathcal{Q}(t)\left(2(\mathcal{V}\widehat{\varphi})\left(\overline{\mathcal{V}\widehat{\varphi}}\right)(\mathcal{V}i\xi\widehat{\varphi})+(\mathcal{V}\widehat{\varphi})^2\left(\overline{\mathcal{V}i\xi\widehat{\varphi}}\right)\right) \\
\simeq\ & \frac{3}{2}t^{-1}\xi^{\frac{1}{2}}\left(t^{\frac{1}{6}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-\frac{1}{2}}\right)^2 t^{-\frac{1}{6}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{\frac{1}{2}}|\widehat{\varphi}|^2\widehat{\varphi}(\xi) \\
\simeq\ & \frac{3}{2}t^{-1}t^{\frac{1}{3}}\xi\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-1}|\widehat{\varphi}|^2\widehat{\varphi}(\xi),
\end{aligned}
$$

where

$$
\mathcal{Q}(t)f(x)\simeq\xi^{\frac{1}{2}}f(\xi),\mathcal{V}\widehat{\varphi}\simeq t^{\frac{1}{6}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-\frac{1}{2}},\mathcal{V}i\xi\widehat{\varphi}\simeq t^{-\frac{1}{6}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{\frac{1}{2}}.
$$

The main term for the second summand of the right-hand side of (5.4) will be

$$
\frac{3}{2}it^{-1}\xi t^{\frac{1}{3}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-1}|\widehat{\varphi}|^2\widehat{\varphi}(\xi)\ .
$$

To justify the above procedure, we need the estimates of the derivatives $\partial_\xi\mathcal{W}$ and $\partial_\xi\mathcal{V}$, for details, see [8]. We have the desired a priori estimate of $\|\mathcal{FU}(-t)\,u(t)\|_{\mathbf{L}^\infty}=\|\widehat{\varphi}(t)\|_{\mathbf{L}^\infty}$. In the similar way we have the result for the self-similar solution. Therefore Theorem 4.1 and Theorem 4.2 follow. To obtain Theorem 4.3 we consider the estimates for the difference of two solutions u_j with the same mass. Define

$$
\|u_1-u_2\|_{\mathbf{Y}_T}=\sup_{t\in[1,T]}\left(t^{\frac{1}{2}-\gamma}\|u_1-u_2\|_{\mathbf{L}^\infty}+t^{-\gamma}\|\mathcal{J}(u_1-u_2)\|_{\mathbf{L}^2}\right)
$$

with a small $\gamma>0$. Then we have Theorem 4.3 by the following lemma if we put $u_1=u$ and $u_2=v_m=t^{-\frac{1}{3}}f_m\left(xt^{-\frac{1}{3}}\right)$.

Lemma 5.2. *Suppose that $\|u_j\|_{\mathbf{X}_T}\le C\varepsilon$, $j=1,2$, where ε is sufficiently small. Let $\widehat{\varphi_1}(t,0)=\widehat{\varphi_2}(t,0)$ for $j=1,2$, $t\ge1$, where $\widehat{\varphi_j}(t,\xi)=\mathcal{FU}(-t)\,u_j(t)$. Let $u_2=t^{-\frac{1}{3}}f\left(xt^{-\frac{1}{3}}\right)$ be a self-similar solution. Then the estimate*

$$
\|u_1-u_2\|_{\mathbf{Y}_T}<C\varepsilon
$$

is true for all $T>1$.

6. Asymptotics of the self similar solution

Let us consider the asymptotics of the self similar solutions. We assume that $\widehat{\varphi}=\mathcal{F}e^{\frac{it}{3}\partial_x^3}u$ satisfies

$$
\begin{aligned}
\partial_t\widehat{\varphi} &= \frac{3}{2}it^{-1}\xi t^{\frac{1}{3}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-1}|\widehat{\varphi}(t,\xi)|^2\widehat{\varphi}(t,\xi)+R \\
&= \frac{3}{2}it^{-1}\xi t^{\frac{1}{3}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-1}|\widehat{\varphi}(t,0)|^2\widehat{\varphi}(t,\xi)+R \\
&= \frac{3}{2}i|m|^2\xi t^{-\frac{2}{3}}\left\langle\xi t^{\frac{1}{3}}\right\rangle^{-1}\widehat{\varphi}(t,\xi)+R
\end{aligned}
$$

which suggests the self-similar solution is

$$
\psi_m\left(\xi t^{\frac{1}{3}}\right)=me^{\frac{9}{2}i|m|^2\log\left\langle\xi t^{\frac{1}{3}}\right\rangle}.
$$

Indeed

$$\partial_t \psi_m\left(\xi t^{\frac{1}{3}}\right) = \frac{3}{2} i\,|m|^2\, m \left\langle \xi t^{\frac{1}{3}}\right\rangle^{-1} \xi t^{-\frac{2}{3}} e^{\frac{9}{2} i |m|^2 \log\left\langle \xi t^{\frac{1}{3}}\right\rangle}$$

$$= \frac{3}{2} i t^{-1} \xi t^{\frac{1}{3}} \left\langle \xi t^{\frac{1}{3}}\right\rangle^{-1} \left|\psi_m\left(\xi t^{\frac{1}{3}}\right)\right|^2 \psi_m\left(\xi t^{\frac{1}{3}}\right)$$

It is possible to consider the difference between $\psi_m\left(\xi t^{\frac{1}{3}}\right)$ and $\widehat{\varphi}(t,\xi)$. Therefore it suggests the self similar solution is

$$t^{-\frac{1}{3}} f_m\left(x t^{-\frac{1}{3}}\right)$$

$$= \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{3}} \int e^{i\left(x t^{-\frac{1}{3}}\eta - \frac{1}{3}\eta^3\right)} \psi_m(\eta)\, d\eta$$

$$= t^{-\frac{1}{3}} m e^{\frac{9}{2} i |m|^2 \log\left\langle x t^{-\frac{1}{3}}\right\rangle} \sqrt{\pi} \left\langle x t^{-\frac{1}{3}}\right\rangle^{-\frac{1}{4}}$$

$$\times \exp\left(\frac{2}{3} i \left|x t^{-\frac{1}{3}}\right|^{\frac{3}{2}} + i\frac{\pi}{4}\right) + O\left(t^{-\frac{1}{3}} \left\langle x t^{-\frac{1}{3}}\right\rangle^{-7/4}\right).$$

However it is not stated in [8] since the estimate of $\psi_m\left(\xi t^{\frac{1}{3}}\right) - \widehat{\varphi}(t,\xi)$ is not enough to show the leading term of $t^{-\frac{1}{3}} f_m\left(x t^{-\frac{1}{3}}\right)$ is the first term of the right hand side of the above.

References

[1] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia 1981.

[2] F.M. Christ and M.I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation.* J. Funct. Anal. **100** (1991), pp. 87-109.

[3] P. Deift and X. Zhou, *A steepest descent method for oscillatory Riemann - Hilbert problems.* Asymptotics for the MKdV equation, Ann. Math., **137** (1993), pp. 295-368.

[4] M.V. Fedoryuk, Asymptotics: integrals and series, Mathematical Reference Library, "Nauka", Moscow, 1987. 544 pp. Encycl. of Math.Sciences, Springer-Verlag, New York, **13** (1989), pp. 83-191.

[5] N. Hayashi and P.I. Naumkin, *Large time asymptotics of solutions to the generalized Korteweg-de Vries equation,* J. Funct. Anal. **159** (1998), pp. 110-136.

[6] N. Hayashi and P. I. Naumkin, *Large time behavior of solutions for the modified Korteweg-de Vries equation,* International Mathematics Research Notices, **8** (1999), pp. 395-418.

[7] N.Hayashi and P.I.Naumkin, *On the modified Korteweg - de Vries equation,* Mathematical Physics, Analysis and Geometry **4** (2001), pp. 197-227.

[8] N.Hayashi and P.I.Naumkin, *Factorization technique for the modified Korteweg-de Vries equation,* preprint, 2015.

[9] C.E. Kenig, G. Ponce and L. Vega, *On the (generalized) Korteweg-de Vries equation.* Duke Math.J. **59** (1989), pp. 585-610.

[10] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg - de Vries equation via contraction principle.* Comm. Pure Appl. Math. **46** (1993), pp. 527-620.

[11] S. Klainerman, *Long time behavior of solutions to nonlinear evolution equations.* Arch. Rat. Mech. Anal. **78** (1982), pp. 73-89.

[12] S. Klainerman and G. Ponce, *Global small amplitude solutions to nonlinear evolution equations.* Comm. Pure Appl. Math. **36** (1983), pp. 133-141.

[13] T. Mizumachi, *Large time asymptotics of solutions around solitary waves to the generalized Korteweg - de Vries equations,* SIAM J. Math. **32**, no. 5, pp. 1050-1080.

[14] G. Ponce and L. Vega, *Nonlinear small data scattering for the generalized Korteweg - de Vries equation,* J. Funct. Anal. **90** (1990), pp. 445-457.

[15] J. Shatah, *Global existence of small solutions to nonlinear evolution equations,* J. Diff. Eqs. **46** (1982), pp. 409-425.

[16] W.A. Strauss, *Dispersion of low-energy waves for two conservative equations.* Arch. Rat. Mech. Anal. **55** (1974), pp. 86-92.

[17] W.A. Strauss, *Nonlinear scattering theory at low energy,* J. Funct. Anal. **41** (1981), pp. 110-133.

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