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PROFINITE COMPLETIONS AND 3-MANIFOLD GROUPS

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ABSTRACT. The profinite completion of a group gives a way to encode all finite quotients of the group. In this note we consider 3-manifold groups and discuss some properties or invariants of a compact 3-manifold that can be detected by the profinite completion of its fundamental group. In particular we study the case of knot complements in the 3-sphere. The material of this note is largely based on [6].

INTRODUCTION

In this note we give a summary of recent results about profinite properties of 3-manifold groups and the relations with the geometry and topology of 3-manifolds. The goal is not to give details of the proofs, but to present a brief overview of the on-going developments and to point out some interesting questions and problems. The material is largely based on [6] where the details can be found.

In the first section we briefly review some basic background on profinite completions of residually finite groups, basic references are [30] and [35]. In the second section we discuss the notion of profinite rigidity for the class of finitely generated and residually finite groups and present examples of such groups which cannot be distinguished by their profinite completions. The third section deals with the class of 3-manifolds groups: we overview the main results known about profinite properties of 3-manifold groups. More material about topics of these two sections can be found in [29]. The fourth section presents the main results obtained in [6] concerning the fiberedness and Thurston norm of 3-manifolds with respect to the profinite completion of their fundamental groups. The last section is devoted to knot groups and their profinite properties, according to [6].

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1. Profinite completion

In this note π will be a finitely generated and residually finite group. Let $Q(\pi)$ be the set of isomorphism classes of finite quotients of π. A general question is:

Question 1.1. *What properties of π can be deduced from the set $Q(\pi)$?*

For example if all finite quotient of π are abelian, then π is abelian.

Finite quotients of π correspond to finite index normal subgroups of π. So properties related to finite quotients of π are encoded in the profinite completion of π.

Let $\mathcal{N}(\pi)$ be the collection of all finite index subgroups $\Gamma \subset \pi$. The set $\mathcal{N}(\pi)$ is a directed set with pre-order $\Gamma' \geq \Gamma$ if $\Gamma' \subset \Gamma$.

If $\Gamma' \geq \Gamma$ then there is an induced epimorphism $h_{\Gamma',\Gamma}\colon \pi/\Gamma' \to \pi/\Gamma$. So to a group π one can associate the inverse system $\{\pi/\Gamma, h_{\Gamma',\Gamma}\}_\Gamma$ with $\Gamma \in \mathcal{N}(\pi)$.

The profinite completion of π is defined as the inverse limit of this system:

$$\widehat{\pi} = \varprojlim \pi/\Gamma.$$

Here is a more direct way, to define the profinite completion $\widehat{\pi}$. Let each finite quotient π/Γ for $\Gamma \in \mathcal{N}(\pi)$ be equipped with the discrete topology. Then the product

$$\prod_{\Gamma \in \mathcal{N}(\pi)} \pi/\Gamma$$

is a compact group. The diagonal map $g \in \pi \to \{g\Gamma\}_{\Gamma\in\mathcal{N}(\pi)}$ defines a homomorphism:

$$i_\pi\colon \pi \to \prod_{\Gamma \in \mathcal{N}(\pi)} \pi/\Gamma.$$

This homomorphism $i_\pi\colon \pi \to \widehat{\pi}$ is injective since π is residually finite. The profinite completion of π can be defined as the closure :

$$\widehat{\pi} = \overline{i_\pi(\pi)} \subset \prod_{\Gamma \in \mathcal{N}(\pi)} \pi/\Gamma.$$

By construction $\widehat{\pi}$ is a compact topological group. A subgroup $U < \widehat{\pi}$ is open if and only if it is closed and of finite index. A subgroup $H < \widehat{\pi}$ is closed if and only if it is the intersection of all open subgroups of $\widehat{\pi}$ containing it.

The following result of N. Nikolov and D. Segal [25] is crucial for the study of profinite completions of finitely generated groups. Its proof uses the classification of finite simple groups.

Theorem 1.2. [25] *Let π be a finitely generated group. Then every finite index subgroup of $\widehat{\pi}$ is open. In particular $\widehat{\widehat{\pi}} = \widehat{\pi}$.*

In particular, there is a one-to-one correspondence between the normal subgroups with the same finite index in π and $\widehat{\pi}$: $\Gamma \in \mathcal{N}(\pi) \to \overline{\Gamma} \in \mathcal{N}(\widehat{\pi})$, and $\overline{\Gamma} = \widehat{\Gamma}$. The inverse map is given by $H \in \mathcal{N}(\widehat{\pi}) \to H \cap \pi \in \mathcal{N}(\pi)$.

An important consequence of the result of Nikolov and Segal is the following:

Corollary 1.3. *Let π be a finitely generated group. For any finite group G the map $i_\pi \colon \pi \to \widehat{\pi}$ induces a bijection $i_\pi^* \colon \mathrm{Hom}(\widehat{\pi}, G) \to \mathrm{Hom}(\pi, G)$.*

Given two groups A and B, a group homomorphism $\varphi \colon A \to B$ induces a continuous homomorphism $\widehat{\varphi} \colon \widehat{A} \to \widehat{B}$. Moreover if φ is an isomorphism, so is $\widehat{\varphi}$.

If the groups A and B are finitely generated, any homomorphism $\widehat{A} \to \widehat{B}$ is continuous, by [25]. On the other hand, a homomorphism $\phi \colon \widehat{A} \to \widehat{B}$ is not necessarily induced by a homomorphism $\varphi \colon A \to B$.

The following result holds:

Lemma 1.4. *Let A and B be two finitely generated groups and $f \colon \widehat{A} \to \widehat{B}$ be an isomorphism. Then for any finite group G the isomorphism $f \colon \widehat{A} \to \widehat{B}$ induces a bijection $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ given by:*

$$i_A^* \circ f^* \circ {i_B^*}^{-1} \colon \mathrm{Hom}(B, G) \xrightarrow{{i_B^*}^{-1}} \mathrm{Hom}(\widehat{B}, G) \xrightarrow{f^*} \mathrm{Hom}(\widehat{A}, G) \xrightarrow{i_A^*} \mathrm{Hom}(A, G).$$

For $\beta \in \mathrm{Hom}(B, G)$ we denote by $\beta \circ f = i_A^* \circ f^* \circ {i_B^*}^{-1}(\beta)$ the resulting homomorphism in $\mathrm{Hom}(A, G)$.

It is clear from the definition that two groups A and B with isomorphic profinite completions have the same finite quotients: $\mathcal{Q}(A) = \mathcal{Q}(B)$. The converse also holds when A and B are finitely generated, see [11], [30]

Lemma 1.5. *Two finitely generated groups A and B have isomorphic profinite completions if and only if they have the same set of finite quotients.*

The proof of Lemma 1.5 follows from the fact that for a finitely generated group π the system of characteristic finite index subgroups $C(n) := \cap_{[\pi:\Gamma] \leq n} \Gamma$ is cofinal for the system of all finite index subgroups. So this system suffices to define the profinite completion, i.e. $\widehat{\pi} = \varprojlim \pi / C(n)$.

We call the finite quotients $\pi / C(n)$ the *characteristic quotients* of π.

2. Profinite rigidity

Following Grunewald and Zalesskii [17] we define the *genus* of a finitely generated and residually finite group π as the set $\mathcal{G}(\pi)$ of isomorphism classes of finitely generated, residually finite groups Γ such that $\widehat{\Gamma} \cong \widehat{\pi}$.

Definition 2.1. A residually finite and finitely generated group π is *profinitely rigid* if $\mathcal{G}(\pi) = \{\pi\}$.

Question 2.2. *Which groups are profinitely rigid? Can $\mathcal{G}(\pi)$ be infinite?*

In general these questions are wide open. One may ask a weaker question:

Question 2.3. *What group theoretic properties are shared by groups in $\mathcal{G}(\pi)$?*

Such properties are called *profinite properties* of a group. For example, being abelian is a profinite property.

The next lemma says that the abelianizations are the same.

Lemma 2.4. *Let A and B be two finitely generated and residually finite groups. If $\widehat{A} \cong \widehat{B}$, then $A^{ab} \cong B^{ab}$*

Corollary 2.5. *A finitely generated abelian group is profinitely rigid.*

Surprisingly, the analogous result is not known for free group:

Question 2.6. *Is a finitely generated free group profinitely rigid?*

The following result of G. Baumslag [4] and R. Hirshon [21]] shows that in general the profinite completion $\widehat{\pi}$ does not determine the group π.

Theorem 2.7. [4, 21] *Let Let Γ and π two finitely generated groups. If $\Gamma \times \mathbb{Z} \cong \pi \times \mathbb{Z}$ then $\widehat{\Gamma} \cong \widehat{\pi}$.*

Given a group A and a class $\psi \in \mathrm{Aut}(A)$, one can build the corresponding semidirect product $A_\psi := A \rtimes_\psi \mathbb{Z}$. It corresponds to the split exact sequence

$$1 \to A \to A_\psi \to \mathbb{Z} \to 1,$$

where the action of $\mathbb{Z}$ on A is given by ψ. The isomorphism type of A_ψ depends only on the class of ψ in $\mathrm{Out}(A)$.

As a consequence of Theorem 2.7, one gets examples of finitely generated and residually finite groups which are not profinitely rigid:

Corollary 2.8. *Let A be a finitely presented and residually finite group and $\psi \in \mathrm{Aut}(A)$ such that ψ^n is an inner automorphism for some $n \in \mathbb{Z}$. Then for any $k \in \mathbb{Z}$ relatively prime to n, $\widehat{A_{\psi^k}} \cong \widehat{A_\psi}$.*

Example 2.9. [4] Let $\pi_1 = \mathbb{Z}/25\mathbb{Z} \rtimes_\psi \mathbb{Z}$ and $\pi_2 = \mathbb{Z}/25\mathbb{Z} \rtimes_{\psi^2} \mathbb{Z}$, $\psi \in \mathrm{Aut}(\mathbb{Z}/25\mathbb{Z})$ be given by $\psi(x) = x^6$ for a generator $x \in \mathbb{Z}/25\mathbb{Z}$. Then $\widehat{\pi_1} \cong \widehat{\pi_2}$. In this example ψ is of order 5 in $\mathrm{Out}(\mathbb{Z}/25\mathbb{Z})$.

Since A is residually finite and finitely generated, the profinite completion $\widehat{A_\psi}$ can be computed from $\widehat{A}$ and $\widehat{\mathbb{Z}}$, see [17], [26].

The system of characteristic finite index subgroups $C(n) := \cap_{[A:\Gamma]\leq n}\Gamma$ is cofinal in A. For each $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that ψ^m induces the identity on the characteristic quotient $A/C(n)$. It follows that $C(n)_{\psi^m} := C(n) \rtimes_{\psi^m} \mathbb{Z}$ is a cofinal system of normal finite index subgroups of A_ψ, since $A \cap C(n)_{\psi^m} = C(n)$. In particular A_ψ is residually finite and its profinite topology induces that of A, so the closure $\overline{A} \in \widehat{A_\psi}$ can be identified with $\widehat{A}$.

By using the automorphisms induced by the elements of $\mathrm{Aut}(A)$ on the finite quotients $A/C(n)$ and the equality $\widehat{A} = \varprojlim A/C(n)$, one can define an injective homomorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(\widehat{A})$. Since $\mathrm{Aut}(A)$ is itself residually finite, the above homomorphism extends to a homomorphism $\widehat{\mathrm{Aut}(A)} \to \mathrm{Aut}(\widehat{A})$. Therefore any homomorphism ψ : $\mathbb{Z} \to \mathrm{Aut}(A)$ extends to a homomorphism $\hat{\psi}\colon \widehat{\mathbb{Z}} \to \widehat{\mathrm{Aut}(A)} \to \mathrm{Aut}(\widehat{A})$. These are key observations for the proof of the following results:

Proposition 2.10. [17, 26] *Let A be a finitely generated and residually finite group and $\psi \in \mathrm{Aut}(A)$, then:*

(1) $\widehat{A_\psi} = \widehat{A \rtimes_\psi \mathbb{Z}} = \widehat{A} \rtimes_{\hat{\psi}} \widehat{\mathbb{Z}}$.

(2) $\widehat{A_\psi} = \widehat{A} \times \widehat{\mathbb{Z}}$ *if and only if ψ induces an inner automorphisms on the finite characteristic quotients of A*

In [26] is given an example of a finitely generated and residually finite group A with an automorphism $\psi \in \mathrm{Aut}(A)$ such that no positive power of ψ is an inner automorphism, but $\widehat{A_\psi} = \widehat{A} \times \widehat{\mathbb{Z}}$.

3. 3-MANIFOLD GROUPS

In the remainder of this paper M will be a compact orientable aspherical 3-manifold with empty or toroidal boundary. A typical example is the exterior $E(K)$ of a knot K in S^3. By Perelman's Geometrization Theorem $\pi_1(M)$ is residually finite, see [19].

3.1. **Rigidity.**

Definition 3.1. An orientable compact 3-manifold M is called *profinitely rigid* if $\widehat{\pi_1(M)}$ distinguishes $\pi_1(M)$ from all other 3-manifold groups.

There are closed 3-manifolds which are not profinitely rigid. At the moment the examples known are *Sol manifolds*, see [32], [16], or *surface bundle with periodic monodromy, i.e. Seifert fibered manifolds*, see [20]. There are no hyperbolic examples known, so the following question makes sense:

Question 3.2 (Rigidity). *Which compact, orientable, irreducible 3-manifolds are profinitely rigid? In particular what about hyperbolic 3-manifolds?*

The answer is positive for the figure-eight knot group by the work of M. Bridson and A. Reid [8]:

Theorem 3.3. [8] *The figure-eight knot group is detected by its profinite completion, among 3-manifold groups.*

We describe now the Seifert fibered examples given by J. Hempel. Let F be a closed orientable surface, $h \in \mathrm{Homeo}^+(F)$ and $M = F \rtimes_h S^1$ be the surface bundle over S^1 with monodromy h. Let $h_\star \in \mathrm{Aut}(\pi_1(F))$ be the automorphism induced by h, then $\pi_1(F)_{h_\star} = \pi_1(F) \rtimes_{h_\star} \mathbb{Z} \cong \pi_1(M)$.

By recent results of I. Agol [2] and D. Wise [41] virtually surface bundles are generic in dimension 3. A surface bundle over S^1 is hyperbolic if and only if its monodromy is pseudo-Anosov by Thurston's hyperbolisation theorem, see [27]. It is Seifert fibered if and only if its monodromy is periodic, see [18].

The following proposition follows from Corollary 2.8 by taking $A = \pi_1(F)$:

Proposition 3.4. [18] *There are surface bundles with periodic monodromies whose fundamental groups have the same profinite completion, but are not isomorphic.*

It has been shown by G. Wilkes [38] that these are the only possible examples for closed Seifert fibered 3-manifolds:

Theorem 3.5. [38] *Let M be a closed orientable irreducible Seifert fibered space. Let N be a compact orientable 3-manifold with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$. Then either:*

(1) *is profinitely rigid, i.e. $\pi_1(N) \cong \pi(M)$, or*
(2) *M and N are surface bundles with periodic monodromies h and h^k, for k coprime to the order of h (Hempel examples).*

A consequence of Wilkes' result and Proposition 2.10 is:

Corollary 3.6. *Let F be a closed orientable surface. A homeomorphism h of F is homotopic to the identity if and only if it induces an inner automorphisms on every finite characteristic quotient of $\pi_1(F)$.*

One could ask whether the actions induced by h on all the finite characteristic quotients of $\pi_1(F)$ suffice to determine h, up to conjugacy and isotopy, when h is not periodic.

The following examples of torus bundles with Anosov monodromies show that it is not true, see P. Stebe [32], L. Funar [16]: these 3-manifolds have solvable fundamental groups:

Proposition 3.7. [16, 32] *There exist infinitely many pairs of torus bundles with Anosov monodromies whose fundamental groups have the same profinite completion, but are not isomorphic.*

In these examples $\pi_1(F) = A \cong \mathbb{Z} \times \mathbb{Z}$ and the monodromies induce linear automorphisms $\psi, \varphi \in \mathrm{GL}(2, \mathbb{Z})$ which are represented by non conjugate Anosov matrices Ψ and Φ, whose images in $\mathrm{GL}(2, \mathbb{Z}/n\mathbb{Z})$ are conjugate for every integer $n > 1$. Here is an example due to P. Stebe [32]:

$$\Psi = \begin{pmatrix} 188 & 275 \\ 121 & 177 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 188 & 11 \\ 3025 & 177 \end{pmatrix}.$$

More examples can be found in L. Funar's work [16].

A continuous map $f\colon M \to N$ induces an homomorphism $f_\star\colon \pi_1(M) \to \pi_1(N)$ and thus an homomorphism $\hat{f}_\star\colon \widehat{\pi_1(N)} \to \widehat{\pi_1(M)}$. The following result (see[29, Thm 8.3]) follows from the residual finiteness of compact 3-manifold groups together with the fact that these groups are good (cf. section 4.1 and also [3, H 26], [10]).

Proposition 3.8. *Let $f\colon M \to N$ a continuous map between two closed orientable aspherical 3-manifolds. Then $\hat{f}_\star\colon \widehat{\pi_1(N)} \to \widehat{\pi_1(M)}$ is an isomorphism if and only if f is homotopic to a homeomorphism.*

In particular for the examples given in Propositions 3.4 and 3.7 the isomorphism between the profinite completions is not induced by a continuous map between the manifolds.

The following finiteness problem is of interest:

Question 3.9 (Finiteness). *Given a 3-manifold M, are there only finitely many 3-manifolds N with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$?*

By analogy with surface bundles over the circle, the question for surface homeomorphisms can be stated as:

Question 3.10. *Let F be a closed orientable surface. Are there only finitely many homeomorphisms h of F, up to isotopy, which induce the same outer automorphism on every finite characteristic quotient of $\pi_1(F)$?*

An important invariant of pseudo-Anosov homeomorphism $h \in \mathrm{Homeo}^+(F)$ is the dilatation factor $\lambda(h)$. An affirmative answer to Question 3.10 for pseudo-Anosov homeomorphisms would follow from a proof that $\lambda(h)$ is a profinite invariant, namely:

Question 3.11. *Let F be a closed orientable surface and h a pseudo-Anosov homeomorphism on F. Do the actions induced by h on all finite characteristic quotients of $\pi_1(F)$ determine its dilatation factor $\lambda(h)$?*

The main question addressed in the remaining of this note is:

Question 3.12. *Which invariants or properties of M are detected by $\widehat{\pi_1(M)}$?*

An invariant σ (or a property P) is a profinite invariant (or a profinite property) if, given two compact, aspherical, orientable 3-manifold M and N with $\widehat{\pi_1(N)} \cong \widehat{\pi_1(M)}$, M and N have the same invariant σ (or M has the property P if and only if N does).

3.2. **Geometries.**

It is natural to ask whether the profinite completion detects Thurston's geometric structures. For closed aspherical orientable 3-manifolds this has been settled by H. Wilton and P. Zalesskii [40]:

Theorem 3.13. [40] *Let M be a closed aspherical orientable 3-manifold, then $\widehat{\pi_1(M)}$ detects:*

(1) *whether M is hyperbolic.*
(2) *whether M is Seifert fibered.*

From the fact that profinite completions distinguish Fuchsian groups [29], they deduce the following corollary:

Corollary 3.14. *Let M and N two closed orientable aspherical 3-manifolds such that $\widehat{\pi_1(M)} \cong \widehat{\pi_1(N)}$. If M admits a geometric structure then N admits the same geometric structure.*

Case (2) of Theorem 3.13 is used by Wilkes in the proof of Theorem 3.5

The non-empty boundary case is still open. The Seifert fibered case is settled in [6] for knot exteriors. Coming back to the case of surface bundles over the circle, one gets the following corollary:

Corollary 3.15. *Let F be a closed orientable surface and h a homeomorphism on F. Whether h is pseudo-Anosov or periodic is detected by the actions induced by h on all the finite characteristic quotients of $\pi_1(F)$.*

One may also remark that the profinite completion distinguishes hyperbolic geometry among Thurston's eight geometries because hyperbolic manifold groups are residually non-abelian simple, see [24].

3.3. **Volume conjecture.**

The volume $\mathrm{Vol}(M)$ of a compact orientable aspherical 3-manifold M with empty or toroidal boundary is defined as the sum of the volumes of the hyperbolic pieces in the geometric decomposition of M.

A strong conjecture, see [23], asserts that the logarithmic growth of the torsion part of the homology of the finite covers of M determines $\mathrm{Vol}(M)$.

Let $\mathcal{N}(\pi_1(M))$ be the collection of all finite index subgroups Γ of $\pi_1(M)$.

Conjecture 3.16 (Asymptotic volume conjecture).

$$\limsup_{\Gamma \in \mathcal{N}(\pi_1(M))} \log(\mathrm{Tor}(\Gamma^{ab})) = \mathrm{Vol}(M)/6\pi.$$

The lower bound has been established by T. Le [23]:

Theorem 3.17. [23]

$$\limsup_{\Gamma\in\mathcal{N}(\pi_1(M))} \log(\mathrm{Tor}(\Gamma^{ab})) \leq \mathrm{Vol}(M)/6\pi.$$

The volume conjecture justifies the following question:

Question 3.18. *Is* $\mathrm{Vol}(M)$ *a profinite invariant?*

A positive answer to this question would answer the finiteness question 3.9 for the case of hyperbolic 3-manifolds.

A much weaker question is still open:

Question 3.19. *Does the profinite completion* $\widehat{\pi_1(M)}$ *detect whether* $\mathrm{Vol}(M)$ *vanishes or not?*

Because of Perelman's geometrization theorem, this is equivalent to decide whether M is a graph manifold or not. This question is addressed in [7] using the notion of pro-virtually abelian completion of $\pi_1(M)$.

4. Thurston norm

We study now the relation between the Thurston norm of a 3-manifold and the profinite completion of its fundamental group. We recall that M is a compact, orientable, aspherical 3-manifold, with ∂M empty or an union of tori.

We define the *complexity* of a compact orientable surface F with connected components $F_1, \ldots, F_k$ to be:

$$\chi_-(F) := \sum_{i=1}^{d} \max\{-\chi(F_i), 0\}.$$

Then the *Thurston norm* of a cohomology class $\phi \in H^1(M;\mathbb{Z})$ is defined as

$$\|\phi\|_M := \min\{\chi_-(F) \mid F \subset M \text{ properly embedded and dual to } \phi\}.$$

By homogeneity $\|.\|_M$ extends to a seminorm on $H^1(M;\mathbb{R})$, see [34]. It is a true norm if M is hyperbolic.

In the following let M_1 and M_2 be two 3-manifolds such that there exists an isomorphism $f\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$. Such an isomorphism induces in particular an isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$ and therefore $H_1(M_1;\mathbb{Z})$ and $H_1(M_2;\mathbb{Z})$ are abstractly isomorphic.

In general this abstract isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$ is not induced by an isomorphism $H_1(M_1;\mathbb{Z}) \to H_1(M_2;\mathbb{Z})$.

In order to compare the Thurston norms of M_1 and M_2, the following definition is introduced in [6]:

Definition 4.1. (1) An isomorphism $f\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is called *regular* if the induced isomorphism $\widehat{H_1(M_1;\mathbb{Z})} \to \widehat{H_1(M_2;\mathbb{Z})}$ is induced by an isomorphism $f_*\colon H_1(M_1;\mathbb{Z}) \to H_1(M_2;\mathbb{Z})$.

(2) A class $\phi \in H^1(N;\mathbb{R})$ is called *fibered* if there is a fibration $p\colon M \to S^1$ such that $\phi = p_*\colon \pi_1(M) \to \mathbb{Z}$.

The following result, obtained in [6], shows that for a regular isomorphism $f\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ the corresponding isomorphism $f_*\colon H_1(M_1;\mathbb{Z}) \to H_1(M_2;\mathbb{Z})$ preserves the Thurston norm and the fibred classes. So it sends the unit ball to the unit ball and preserves the fibered faces.

Theorem 4.2. [6] *Let M_1 and M_2 be two aspherical 3-manifolds with empty or toroidal boundary. If $f\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is a regular isomorphism, then:*

(1) *For any class $\phi \in H^1(M_2;\mathbb{R})$, $\|\phi\|_{M_2} = \|f^*\phi\|_{M_1}$.*

(2) *$\phi \in H^1(M_2;\mathbb{R})$ is fibered if and only if $f^*\phi \in H^1(M_1;\mathbb{R})$ is fibered.*

When $\partial M_1 \neq \emptyset$ and ϕ is a fibered class, this result has also been obtained by A. Reid and M. Bridson [8], by a different method.

We now briefly describe the main steps of the proof of Theorem 4.2

4.1. Cohomological properties: Goodness.

Following J.P. Serre [31] a group π is called *good* if the following holds: for any finite abelian group A and any representation $\alpha\colon \pi \to \mathrm{Aut}_{\mathbb{Z}}(A)$ the inclusion $\iota\colon \pi \to \widehat{\pi}$ induces for any j an isomorphism $\iota^*\colon H^j_\alpha(\widehat{\pi};A) \to H^j_\alpha(\pi;A)$ of twisted cohomology groups.

If π is good of finite cohomological dimension then $\widehat{\pi}$ is torsion free.

The following theorem of W. Cavendish [10] is crucial for the proofs of the results in [6] to transfer cohomological informations via profinite completion. Its proof uses Agol's virtual fibration theorem:

Theorem 4.3. [10] *The fundamental group of any compact aspherical* 3*-manifold is good.*

Corollary 4.4. *For a compact aspherical* 3*-manifold the property of being closed is a profinite property.*

4.2. Twisted Alexander polynomials.

Let X be a CW-complex, $\phi \in H^1(X;\mathbb{Z})$ and $\alpha\colon \pi_1(X) \to \mathrm{GL}(k,\mathbb{F})$ be a representation, $\mathbb{F}$ being a field. Set $\mathbb{F}[t^{\pm1}]^k := \mathbb{F}^k \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm1}]$ and consider the tensor representation:

$$\alpha \otimes \phi\colon \pi_1(X) \to \mathrm{Aut}_{\mathbb{F}[t^{\pm1}]}(\mathbb{F}[t^{\pm1}]^k),$$

given by:

$$g \mapsto \Big(\sum_i v_i \otimes p_i(t) \mapsto \sum_i \alpha(g)(v_i) \otimes t^{\phi(g)} p_i(t)\Big).$$

That makes $\mathbb{F}[t^{\pm 1}]^k$ a left $\mathbb{Z}[\pi_1(X)]$-module and the corresponding twisted homology groups $H_i^{\alpha\otimes\phi}(X;\mathbb{F}[t^{\pm 1}]^k)$ are naturally $\mathbb{F}[t^{\pm 1}]$-modules.

Definition 4.5. The *i-th twisted Alexander polynomial* $\Delta^{\alpha}_{X,\phi,i} \in \mathbb{F}[t^{\pm 1}]$ is the order of the $\mathbb{F}[t^{\pm 1}]$-module $H_i^{\alpha\otimes\phi}(X;\mathbb{F}[t^{\pm 1}]^k)$.

The twisted Alexander polynomials are well-defined up to multiplication by some at^k where $a \in \mathbb{F}\setminus\{0\}$ and $k \in \mathbb{Z}$ (i.e. a unit in $\mathbb{F}[t^{\pm 1}]$).

For a polynomial $f(t) = \sum_{k=r}^{s} a_k t^k \in \mathbb{F}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we now define $\deg(f(t)) = s - r$. For the zero polynomial set $\deg(0) := +\infty$.

The following results are crucial for the proof of Theorem 4.2. The first statement, see [13], gives a non-vanishing criterion for a non-zero class $\phi \in H^1(M;\mathbb{Z})$ to be fibered, in terms of twisted Alexander polynomials.

The second statement, see [14], [15], computes the Thurston norm of a non-zero class $\phi \neq 0 \in H^1(M;\mathbb{Z})$ in term of the degrees of some twisted Alexander polynomials.

Theorem 4.6. [13, 14, 15] *Let M be a compact, aspherical, orientable 3-manifold with empty or toroidal boundary and $\phi \neq 0 \in H^1(M;\mathbb{Z})$:*

(1) *The class ϕ is fibered if and only if $\Delta^{\alpha}_{M,\phi,1} \neq 0$ for all primes p and all representations $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{F}_p)$.*
(2) *There exists a prime p and a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{F}_p)$ such that*

$$\|\phi\|_M = \max\left\{0, \frac{1}{k}\left(-\deg\left(\Delta^{\alpha}_{M,\phi,0}\right) + \deg\left(\Delta^{\alpha}_{M,\phi,1}\right) - \deg\left(\Delta^{\alpha}_{M,\phi,2}\right)\right)\right\}.$$

The proof of theorem 4.6 relies heavily on the work of Agol [1, 2], Przytycki-Wise [28] and Wise [41].

Given $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$ and $n \in \mathbb{N}$, set $\phi_n\colon \pi_1(M) \xrightarrow{\phi} \mathbb{Z} \to \mathbb{Z}_n$. For a representation $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{F}_p)$ and $n \in \mathbb{N}$, let $\mathbb{F}_p[\mathbb{Z}_n]^k = \mathbb{F}_p^k \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}_n]$ and $\alpha\otimes\phi_n\colon \pi_1(M) \to \mathrm{Aut}(\mathbb{F}_p[\mathbb{Z}_n]^k)$ the induced representation.

The following proposition shows that the degrees of twisted Alexander polynomials can be computed from the dimension of some twisted homology groups, namely:

Proposition 4.7. [6] *Let $\phi \in H^1(M;\mathbb{Z})\setminus 0$ and $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{F}_p)$, then:*

(1) $\deg \Delta^{\alpha}_{M,\phi,0} = \max\Big\{ \dim_{\mathbb{F}_p}\left(H_0^{\alpha\otimes\phi_n}(M;\mathbb{F}_p[\mathbb{Z}_n]^k)\right) \Big| n \in \mathbb{N}\Big\}$
(2) $\deg \Delta^{\alpha}_{M,\phi,1} = \max\Big\{\dim_{\mathbb{F}_p}\big(H_1^{\alpha\otimes\phi_n}(M;\mathbb{F}_p[\mathbb{Z}_n]^k)\big) - \dim_{\mathbb{F}_p}\big(H_0^{\alpha\otimes\phi_n}(M;\mathbb{F}_p[\mathbb{Z}_n]^k)\big)\Big| n \in \mathbb{N}\Big\}$.

The next proposition and the goodness of aspherical compact 3-manifold groups will conclude the proof of Theorem 4.2.

Proposition 4.8. [6] *Let π_1 and π_2 be good groups and $f\colon \widehat{\pi_1} \xrightarrow{\cong} \widehat{\pi_2}$ an isomorphism. Let $\beta\colon \pi_2 \to \mathrm{GL}(k, \mathbb{F}_p)$ be a representation. Then for any i there is an isomorphism*

$$H_i^{\beta\circ f}(\pi_1; \mathbb{F}_p^k) \cong H_i^{\beta}(\pi_2; \mathbb{F}_p^k).$$

Since 3-manifold groups are good, one gets:

Corollary 4.9. *Let M_1 and M_2 be two 3-manifolds. Suppose $f\colon \widehat{\pi_1(M_1)} \to \widehat{\pi_1(M_2)}$ is a regular isomorphism. Then for any $\phi \neq 0 \in H^1(M_2, \mathbb{Z})$ and any representation $\alpha\colon \pi_1(M_2) \to \mathrm{GL}(k, \mathbb{F}_p)$ one has:*

$$\deg\left(\Delta^{\alpha\circ f}_{M_1,\phi\circ f,i}\right) = \deg\left(\Delta^{\alpha}_{M_2,\phi,i}\right), \quad i = 0, 1, 2.$$

When the first Betti number $b_1(M_1) = 1$, then $b_1(M_2) = 1$ and the regular assumption is not needed anymore because of the following lemma :

Lemma 4.10. [6] *Let M be a 3-manifold with $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ and $\beta\colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F}_p)$ a representation. Let $\phi_n\colon \pi_1(M) \to \mathbb{Z}_n$ and $\psi_n\colon \pi_1(M) \to \mathbb{Z}_n$ be two epimorphisms. Then given any i there exists an isomorphism $H_i^{\beta\otimes\phi_n}(M; \mathbb{F}_p[\mathbb{Z}_n]^k) \cong H_i^{\beta\otimes\psi_n}(M; \mathbb{F}_p[\mathbb{Z}_n]^k)$.*

Knot exteriors in S^3 are typical examples of manifolds with first Betti number 1 and are considered in the next section.

5. Knot groups

The exterior $E(K) = S^3 \setminus \mathcal{N}(K)$ of a knot $K \subset S^3$ is a compact orientable 3-manifold with $b_1 = 1$. The fundamental group $\pi_1(E(K))$ is called the group of the knot K.

There is a canonical epimorphism $\pi_1(E(K)) \to H_1(E(K); \mathbb{Z}) \cong \mathbb{Z}$. Let $\phi_K \in H^1(E(K); \mathbb{Z})$ be the corresponding class. If K is non-trivial, then the Thurston norm of ϕ_K equals $2g(K) - 1$, where $g(K)$ is the Seifert genus of K. The knot K is called fibered if ϕ_K is a fibered class.

The following theorem summarizes the results obtained in [6] about profinite completions of knot groups.

Theorem 5.1. [6] *Let K_1 and K_2 be two knots in S^3 such that $\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))}$. Then the following hold:*

1. *K_1 and K_2 have the same Seifert genus: $g(K_1) = g(K_2)$;*
2. *K_1 is fibered if and only if K_2 is fibered;*
3. *If no zero of Δ_{K_1} is a root of unity, then $\Delta_{K_1} = \pm\Delta_{K_2}$;*
4. *If K_1 is a torus knot, then $K_1 = K_2$;*
5. *If K_1 is the figure-eight knot, then $K_1 = K_2$;*
6. *If $E(K_1)$ and $E(K_2)$ have a homeomorphic finite cyclic cover, either $K_1 = K_2$ or Δ_{K_1} and Δ_{K_2} are product of cyclotomic polynomials.*

The statements (1) and (2) are direct consequences of Theorem 4.2 and Lemma 4.10. Statement (3) follows from Proposition 5.2 below and D. Fried's result that the Alexander polynomial of a knot can be recovered from the torsion parts of the first homology groups of the n-fold cyclic covers of its exterior, provided that no zero is a root of unity, see [12].

Given a knot K let $E_n(K)$ denote the n-fold cyclic cover of $E(K)$. By construction $\pi_1(E_n(K)) = \ker(\pi(K) \to H_1(E(K);\mathbb{Z}) \to \mathbb{Z}/n\mathbb{Z})$.

Lemma 5.2. *Let K_1 and K_2 be two knots such that $\widehat{\pi(K_1)} \cong \widehat{\pi(K_2)}$. Then the following hold:*

(1) *For each $n \geq 1$ we have $H_1(E_n(K_1;\mathbb{Z}) \cong H_1(E_n(K_2);\mathbb{Z})$.*
(2) *The Alexander polynomial Δ_{K_1} has a zero that is an n-th root of unity if and only if Δ_{K_2} has a zero that is an n-th root of unity.*

The proof of this lemma follows from the following facts, see [6] for the details.

The isomorphism $\pi_1\widehat{(E(K_1))} \cong \pi_1\widehat{(E(K_2))}$ implies that $\pi_1\widehat{(E_n(K_1))} \cong \pi_1\widehat{(E_n(K_2))}$, since a knot group admits a unique homomorphism onto $\mathbb{Z}/n\mathbb{Z}$ for each n. Therefore we see that $H_1(E_n(K_1);\mathbb{Z}) \cong H_1((E_n(K_2);\mathbb{Z})$.

By the Fox formula $H_1(E_n(K);\mathbb{Z}) \cong \mathbb{Z} \oplus A$, with $|A| = \left|\prod_{k=1}^{n} \Delta_K\left(e^{2\pi ik/n}\right)\right|$, see [36]. In particular $b_1(E_n(K)) = 1$ if and only if no n-th root of unity is a zero of Δ_K.

The next corollary follows now easily from statements (1) to (3) of Theorem 5.1, Lemma 5.2 and the fact that the trefoil knot and the figure-eight knot are the only fibered knots of genus 1.

Corollary 5.3. *Let J be the trefoil knot or the figure-eight knot. If K is a knot with $\pi_1\widehat{(E(J))} \cong \pi_1\widehat{(E(K))}$, then J and K are equivalent.*

In fact $\pi_1\widehat{(E(J))}$ detects the trefoil or the figure-eight complement among all compact connected 3-manifolds, see [8].

Let $T_{p,q}$ be a torus knot of type (p,q) with $0 < p < q$, statements (1) to (3) of Theorem 5.1 and Lemma 5.2 imply the following claim:

Claim 5.4. $\pi_1\widehat{(E(T_{p,q}))} \cong \pi_1\widehat{(E(T_{r,s}))} \Leftrightarrow (p,q) = (r,s)$

Hence each torus knot is distinguished, among knots, by the profinite completion of its group because of the following result:

Proposition 5.5. [6] *Let J be a torus knot. If K is a knot with $\pi_1\widehat{(E(J))} \cong \pi_1\widehat{(E(K)}$, then K is a torus knot.*

The proof of the last statement (6) uses the fact that the logarithmic Mahler measure of the Alexander polynomial is a profinite invariant by [33] and the study of knots with cyclically commensurable exteriors developed in [5]

Since prime knots with isomorphic groups have homeomorphic complements by W. Whitten [37], the following question makes sense:

Question 5.6. *Let K_1 and K_2 be two prime knots in S^3. If $\widehat{\pi_1(E(K_1))} \cong \widehat{\pi_1(E(K_2))}$, does it follow that $K_1 = K_2$?*

The group of a prime knot K does not necessarily determine the topological type of the knot exterior $E(k)$, if it contains a properly embedded essential annulus. This means that K is a torus knot or a cable knot and that the essential annulus cobounds with some annulus in $\partial E(K)$ a solid torus V in $E(K)$. Then by [22, Chapter X] some Dehn flip along V may produce a Haken manifold M that is homotopically equivalent but not homeomorphic to $E(K)$ and thus does not imbed in S^3. However one may ask whether the profinite completion can detect knot groups among 3-manifold groups.

Question 5.7. *Let M be a compact orientable aspherical 3-manifold and let $K \subset S^3$ be a knot. Does $\widehat{\pi_1(M)} \cong \widehat{\pi_1(E(K))}$ imply that $\pi_1(M)$ is isomorphic to a knot group?*

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