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HIGHER DIMENSIONAL THOMPSON GROUPS HAVE SERRE'S PROPERTY FA

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1. Higher dimensional Thompson groups nV

The Thompson group V is a subgroup of the homeomorphism group of the Cantor set C. Brin [3] defined higher dimensional Thompson groups nV as generalizations of V. For each n, nV is a subgroup of the homeomorphism group of C^n.

According to Brin's paper [3], we give the definition of higher dimensional Thompson groups. Hereafter, the symbol I denotes $[0,1)$. Symbols I_l and I_r denote $[0,1/2)\times I^{n-1}$ and $[1/2,1)\times I^{n-1}$, respectively.

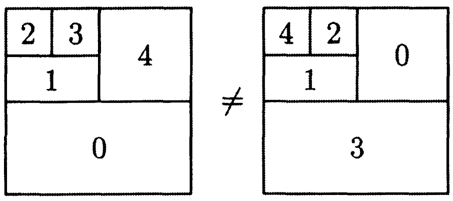
An *n-dimensional rectangle* is a subset of I^n, defined inductively as follows. The first rectangle is I^n itself. If $R=[a_1,b_1)\times\cdots\times[a_i,b_i)\times\cdots\times[a_n,b_n)$ is a rectangle, then for all $i\in\{1,\ldots,n\}$, the "i-th left half" and "the i-th right half" defined by

$$R_{l,i}=[a_1,b_1)\times\cdots\times[a_i,(a_i+b_i)/2)\times\cdots\times[a_n,b_n) \tag{1}$$

$$R_{r,i}=[a_1,b_1)\times\cdots\times[(a_i+b_i)/2,b_i)\times\cdots\times[a_n,b_n) \tag{2}$$

are again rectangles.

An *n-dimensional pattern* is a finite set of n-dimensional rectangles, with pairwise disjoint, non-empty interiors and whose union is I^n. A *numbered pattern* is a pattern with a one-to-one correspondence to $\{0,1,\ldots,r-1\}$ where r is the number of rectangles in the pattern. The following figure gives an example of a pair of 2-dimensional numbered patterns, which are different as numbered patterns although they are the same as patterns.



From now on, we will identify an n-dimensional rectangle with a subset of C^n and use the common symbol. We start with identifying I^n with C^n. Let R be a rectangle which is identified with a subset of C^n:

$$R'=C^n\cap[a'_1,b'_1]\times\cdots\times[a'_i,b'_i]\times\cdots\times[a'_n,b'_n]. \tag{3}$$

Define rectangles $R_{l,i}$ and $R_{r,i}$ in the same way as we obtained (1) and (2). These rectangles are identified respectively with the "i-th left third" and the "i-th right third" of R':

$$C^n\cap[a'_1,b'_1]\times\cdots\times[a'_i,(2a'+b'_i)/3]\times\cdots\times[a'_n,b'_n], \tag{4}$$

$$C^n\cap[a'_1,b'_1]\times\cdots\times[a'_i,(a'+2b'_i)/3]\times\cdots\times[a'_n,b'_n]. \tag{5}$$

We proceed by induction. In the same manner, every pattern describes a division of C^n. The following figure shows this correspondence, in the case of $n=2$.

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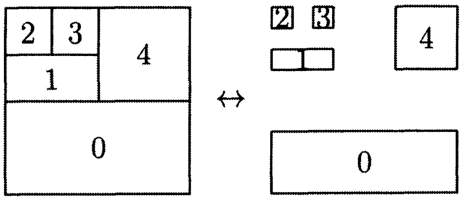
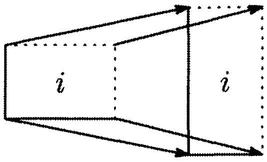
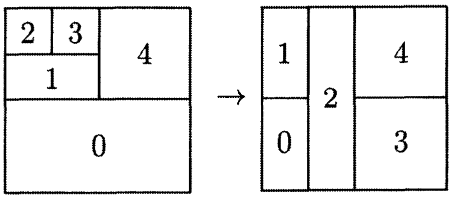


We will construct a self-homeomorphism of C^n from a pair of numbered patterns with the same number of rectangles. Let $P = \{P_i\}_{0\leq i\leq r-1}$ and $Q = \{Q_i\}_{0\leq i\leq r-1}$ be numbered patterns. We define $g(P,Q) : I^n \to I^n$ which takes each P_i onto Q_i affinely so as to preserve the orientation, as drawn in the following figure. Namely, the restriction of $g(P,Q)$ to each P_i has the form $(x_1, \dots, x_n) \mapsto (a_1 + 2^{j_1}x_1, \dots, a_n + 2^{j_n}x_n)$ for some integers $j_1, \dots, j_n$.



With the former identification of rectangles with subsets of C^n, the above construction defines a self-homeomorphism of C^n. We again write $g(P,Q)$ for this homeomorphism.

When $n = 2$, we illustrate $g(P,Q)$ as a triplet of P, Q and an arrow which indicates the domain and the range. Below we show an example.



The *n-dimensional Thompson group* nV is a subgroup of self-homeomorphisms of C^n which consists of the maps with the form $g(P,Q)$. Every element of nV is identified with a partially affine, partially orientation preserving bijection from I^n to itself.

Theorem 1.1 (Bleak and Lanoue [2]). *n_1V and n_2V are isomorphic if and only if $n_1 = n_2$.*

Higher dimensional Thompson groups have some important properties in common with Thompson groups.

Theorem 1.2 (Brin [4]). *For all $n \in \mathbb{N}$, nV is simple.*

2. The number of ends and actions on trees

Let Γ be a connected locally finite graph. We equip Γ with graph metric. For a finite subtree K, $\|\Gamma - K\|$ denotes the number of unbounded connected components of $\Gamma - K$. The *number of ends* of Γ, $e(\Gamma)$, is defined to be the supremum of $\|\Gamma - K\|$ taken over all the finite subtrees.

Throughout this section, G denotes a finitely generated group and S denotes a finite generating set of G. The *Cayley graph* $\Gamma_{G,S}$ is a graph whose vertex set is G, and there is an oriented edge from $g \in G$ to $h \in G$ if some $s \in S$ satisfies $g \cdot s = h$. G acts freely on $\Gamma_{G,S}$ from the left.

The *number of ends* of G, $e(G)$, is the number of ends of $\Gamma_{G,S}$.

Proposition 2.1. *$e(G)$ satisfies the following.*

(i) *$e(G)$ does not depend on the choice of S.*
(ii) *(The Freudenthal-Hopf Theorem) $e(G)$ is 0, 1, 2 or ∞.*
(iii) *$e(G) = 0$ if and only if G is finite.*
(iv) *$e(G) = 2$ if and only if G has an infinite cyclic subgroup of finite index.*

The following result, Stallings' theorem, provides a group-theoretical characterization of the case where $e(G) \geq 2$.

Theorem 2.2 (Stallings [9], Bergman [1]). *$e(G) \geq 2$ if and only if G has a structure of an amalgamated product or an HNN-extension on some finite subgroup.*

In the light of this theorem, we can characterize the case of $e(G) = 1$ in terms of group actions on trees. From now on, we consider only simplicial trees and simplicial actions without edge-inversions. We say that G has *property FA* if every action of G on a tree T has a fixed point. Here, a fixed point means $x \in T$ such that $g(x) = x$ for every $g \in G$.

Theorem 2.3 (Serre [8]). *If an infinite group G has propery FA, then $e(G) = 1$.*

The following proposition is a basic fact about group actions on trees. Let G be a group acting on a tree. Let $g \in G$. If some $x \in T$ satisfies $g(x) = x$, then g is said to be *elliptic*. Otherwise, we say g is *hyperbolic*.

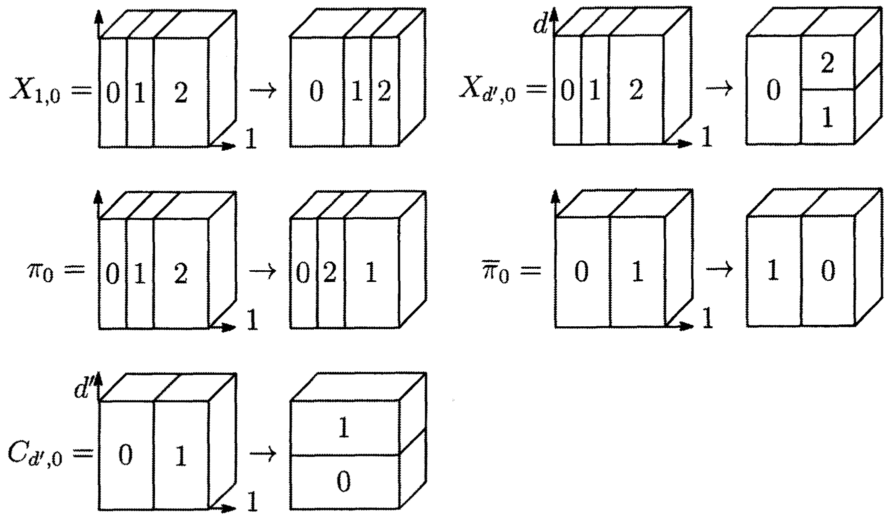
Proposition 2.4 (Serre [8]). *Let G be a group acting on a tree T. Let $g \in G$.*

(i) $\mathrm{Fix}(g) = \{ x \in T \mid g(x) = x \}$ *is either empty or a subtree of T.*
(ii) *If g is hyperbolic, g acts on a unique simplicial line in T by translation. This line is called the axis of g.*
(iii) *(Serre's lemma) Assume that G is generated by a finite set of elements $\{s_j\}_{1\leq i\leq m}$ such that every element is elliptic, and the products of every two elements are elliptic, equivalently, every two elements have a common fixed point. Then there is $x \in T$ which is fixed by every element of G.*

3. nV HAS PROPERTY FA

We would like to explain the idea to show that each nV has property FA. First, we take a finite generating set of nV. Next, we modify the generating set as to satisfy the requirements of Serre's lemma.

For every n, nV is known to have a useful presentation, described in the following. We define $X_{1,0}$, $X_{d',0}$, $C_{d',0}$, π_0, $\overline{\pi}_0 \in nV$ $(2 \leq d' \leq n)$ as shown in the following figure. For $i \geq 1$, $X_{d,i}$ $(1 \leq d \leq n)$ is defined inductively. On I_r, $X_{d,i}$ restricts to the identity. For $x \in I_l$, we write $x = (x_1, x_2)$ where $x_1 \in [0, 1/2)$ and $x_2 \in I^{n-1}$. We define $\phi : I_l \to I^n$ by $\phi(x_1, x_2) = (2x_1, x_2)$. On I_l, $X_{d,i} = X_{d,i-1}\phi$. Similarly, $C_{d',i}$, π_i and $\overline{\pi}_i$ restrict to the identity on I_r and $C_{d',i-1}\phi$, $\pi_{i-1}\phi$ and $\overline{\pi}_{i-1}\phi$ on I_l, respectively.



Theorem 3.1 (Hennig and Matucci [7, Theorem 23]). *Let*

$$\Sigma = \{X_{d,i}, C_{d',i}, \pi_i, \overline{\pi}_i\}_{1\leq d\leq n,\ 2\leq d'\leq n,\ i\geq 0}. \tag{6}$$

(i) Σ *is a generating set of* nV.
(ii) *The elements of* Σ *satisfy the following relations:*

$$X_{d'',j}X_{d,i} = X_{d,i}X_{d'',j+1} \qquad (i < j, 1 \leq d, d'' \leq n), \tag{7}$$
$$C_{d',j}X_{d,i} = X_{d,i}C_{d',j+1} \qquad (i < j, 1 \leq d \leq n, 2 \leq d' \leq n), \tag{8}$$
$$Y_jX_{d,i} = X_{d,i}Y_{j+1} \qquad (i < j, Y \in \{\pi, \overline{\pi}\}, 1 \leq d \leq n), \tag{9}$$
$$\pi_jX_{d,i} = X_{d,i}\pi_j \qquad (i > j+1, 1 \leq d \leq n), \tag{10}$$
$$\pi_jC_{d',i} = C_{d',i}\pi_j \qquad (i > j+1, 2 \leq d' \leq n), \tag{11}$$
$$\pi_j\pi_i = \pi_i\pi_j \qquad (|i-j| > 2), \tag{12}$$
$$\overline{\pi}_j\pi_i = \pi_i\overline{\pi}_j \qquad (j > i+1), \tag{13}$$
$$\overline{\pi}_iX_{1,i} = \pi_i\overline{\pi}_{i+1} \qquad (i \geq 0), \tag{14}$$
$$C_{d',i}X_{1,i} = X_{d',i}C_{d',i+2}\pi_{i+1} \qquad (i \geq 0, 2 \leq d' \leq n), \tag{15}$$
$$\pi_iX_{d,i} = X_{d,i+1}\pi_i\pi_{i+1} \qquad (i \geq 0, 1 \leq d \leq n). \tag{16}$$

Relations (7), (8) and (9) are similar to the "almost commutative" relation of Thompson's group F. According to those relations, we can see that $\{X_{d,i}, C_{d',i}, \pi_i, \overline{\pi}_i\}_{i\leq m}$ generates nV for every $m \geq 1$.

We would like to modify Σ to consist of elliptic elements. For this purpose, we use the following characterization for an element of nV to be elliptic.

Lemma 3.2. *Let* $g \in nV$ *act identically on some rectangle. If* nV *acts on a tree,* g *is elliptic.*

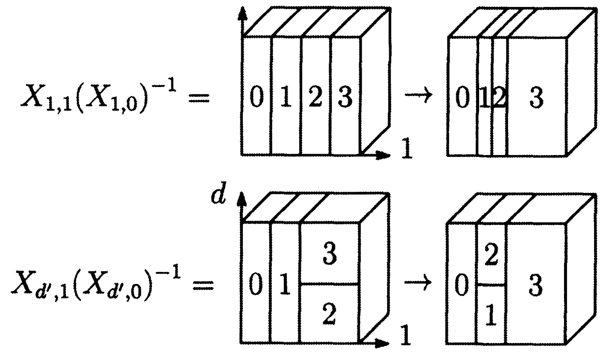
The above lemma was shown in [6] in the case of $n = 1$. For each rectangle $R \subsetneq I^n$, we consider a subgroup which consists of elements whose supports are included in R. We may observe that such subgroups are conjugate to each other, and that they are isomorphic to V, which is simple. The proof depends on these facts, which are also true in the case of nV for general n.

Lemma 3.3. *The set*

$$S = \{ X_{d,1}, X_{d,1}(X_{d,0})^{-1}, C_{d',2}, \pi_0, \pi_3, \overline{\pi}_3 \}_{1 \le d \le n, 2 \le d' \le n} \tag{17}$$

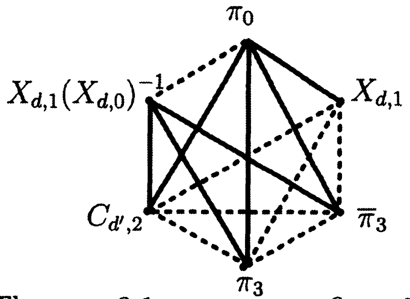
generates nV.

We show the newly appeared elements $X_{d,1}(X_{d,0})^{-1}$ in the figure below.



Each element of S restricts to the identity on some rectangle. If two elements restrict to the identity on a rectangle, then their product again restricts to the identity on the rectangle and is elliptic. If two elliptic elements commute, then they have a common fixed point.

The following figure shows that almost all the pairs of elements of S satisfy one of those two conditions. Solid segments represent the commutativity and dotted ones indicate that two endpoints restrict to the identity on the same rectangle.



According to the relations in Theorem 3.1, we may confirm that the exceptional pairs also have common fixed points.

Theorem 3.4. *nV has property FA. Especially,* $e(nV) = 1$.

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