THE CANONICAL DECOMPOSITION OF THE TWO-BRIDGE KNOT WITH SLOPE 12/29 AND THE CANNON-THURSTON MAP ASSOCIATED WITH THE KNOT

NAOKI SAKATA

1. INTRODUCTION

This note is a summary of the author's talk at the workshop "Topology, Geometry and Algebra of low-dimensional manifolds".

Let $M$ be a hyperbolic once-punctured torus bundle over $S^1$. Then $M$ admits a canonical decomposition into ideal polyhedra by Epstein-Penner [6] and Weeks [18]. In fact, Jørgensen [12] has constructed the canonical decomposition of $M$. The canonical decomposition induces a triangulation of the peripheral torus. Thus we have a triangulation of the complex plane $\mathbb{C}$ as the lift of the triangulation of the torus to the universal cover. On the other hand, we have another tessellation of $\mathbb{C}$ which nicely reflects the nature of the Cannon-Thurston map associated with $M$. The tessellation is called the "Cannon-Dicks-Thurston fractal tessellation" (see Definition 2.4).

Dicks and Sakuma [5] have proved that there exists a nice relation between the two tessellations. In particular, the two tessellations share the same vertex set.

It is natural to expect that similar results hold in a more general setting, and the author has been trying to realize this expectation for hyperbolic fibered two-bridge links. The first task is to generalize the fractal tessellation to more general hyperbolic punctured surface bundles, and the author announced such a generalization for hyperbolic punctured surface bundles in [14], provided that all singularities of the stable and unstable foliations of the monodromy are at punctures of the fiber surface. The condition is satisfied for the hyperbolic fibered two-bridge knots with slope $r$ such that $r$ has a continued fraction expansion $\pm[2,2,\ldots,2]$. The main purpose of this note is to present an idea of the proof of the following theorem.

**Theorem 1.1.** For the hyperbolic fibered two-bridge knot $K$ with slope $12/29 = [2,2,2,2]$, the triangulation induced by the canonical decomposition and the fractal tessellation share the same vertex set.

After the announcement of this result at the workshop, the author learned the work of Guéritaud [10, 11], which generalizes the fractal tessellation and establishes beautiful relation between "veering" ideal triangulations (cf. [1]) of hyperbolic punctured surface bundles and the fractal tessellations, generalizing the result of Dicks and Sakuma. Motivated by this work, the author proved the following theorem.

**Theorem 1.2** ([15, Theorem 1.1]). The canonical decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0 < |r| < 1/2)$ is veering if and only if the slope $r$ has the continued fraction expansion $\pm[2,2,\ldots,2]$.

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As a consequence of Guéritaud’s result and the above theorem, the canonical decomposition of such a two-bridge knot and the fractal tessellation are intimately related. In particular, Theorem 1.1 follows immediately.

2. THE CANNON-DICKS-THURSTON FRACTAL TESSELLATION

In this section, we will recall some properties of the Cannon-Dicks-Thurston fractal tessellations. Let \( F \) be a complete hyperbolic punctured surface of finite area, and let \( \rho_0 : \pi_1(F) \to \text{PSL}(2, \mathbb{R}) \) be the holonomy representation of \( F \). For a pseudo-Anosov homeomorphism \( h \) of \( F \), the mapping torus \( M_h := F \times [0,1] / \langle x,0 \rangle \sim (h(x),1) \) admits a complete hyperbolic structure, by Thurston’s work. Let \( \rho : \pi_1(F) \to \text{PSL}(2, \mathbb{C}) \) be the restriction of the holonomy representation of \( M \) to \( \pi_1(F) \). Generalizing the work of Cannon and Thurston [4], Bowditch proved the following theorem.

**Theorem 2.1.** There exists a unique \( \pi_1(F) \)-equivariant continuous surjection \( \omega \) from the limit set of \( \Gamma_0 = \rho_0(\pi_1(F)) \) to that of \( \Gamma = \rho(\pi_1(F)) \).

This surjection \( \omega \) is called a Cannon-Thurston map. Note that the limit sets of \( \Gamma_0 \) and \( \Gamma \) are equal to the boundary of the hyperbolic plane \( \partial \mathbb{H}^2 \) and the boundary of the 3-dimensional hyperbolic space \( \partial \mathbb{H}^3 = \mathbb{C} \), respectively. Thus the surjection \( \omega \) is a sphere filling curve.

Moreover, Bowditch has given a complete description of the combinatorics of the Cannon-Thurston map.

**Theorem 2.2** ([3, Theorem 9.1]). Let \( \mathcal{F}_{\pm} \) be the stable and unstable foliations of \( h \), and let \( \tilde{\mathcal{F}}_{\pm} \) be the lift of \( \mathcal{F}_{\pm} \) to the universal cover \( \hat{\mathbb{H}}^2 \). For any leaf of \( \tilde{\mathcal{F}}_{\pm} \), the images of its endpoints under the Cannon-Thurston map \( \omega \) are identical. Furthermore, this generates all the identifications occurring under \( \omega \).

For the case when \( F \) is a once-punctured torus, Cannon and Dicks [4] have introduced a certain fractal tessellation of the complex plane \( \mathbb{C} = \mathbb{C} \setminus \{\infty\} \), where \( \infty \) is a parabolic fixed point of \( \Gamma \). The fractal tessellation nicely reflects the behavior of the Cannon-Thurston map. This result was extended by Guéritaud [11] to a more general setting as follows.

**Proposition 2.3** (cf. [11, Theorem 1.2]). Suppose that all singularities of the invariant foliations \( \mathcal{F}_{\pm} \) of the pseudo-Anosov homeomorphism \( h \) are at punctures of \( F \). Then there exists a \( \mathbb{Z} \)-family of Jordan curves \( J_i \) of \( \partial \mathbb{H}^3 \), bounding domains \( D_i \), with the following properties:

1. For each \( i \in \mathbb{Z} \), the Jordan curve \( J_i \) passes through \( \infty \), a parabolic fixed point of \( \Gamma \).
2. For each \( i \in \mathbb{Z} \), \( D_i \supset D_{i+1} \).
3. \( \bigcap_{i \in \mathbb{Z}} D_i = \emptyset \) and \( \bigcup_{i \in \mathbb{Z}} D_i = \mathbb{C} \).
4. For \( i, i' \in \mathbb{Z} \), if \( |i - i'| > 1 \), then \( J_i \cap J_{i'} = \{\infty\} \).
5. For \( i, i' \in \mathbb{Z} \), if \( |i - i'| = 1 \), then \( J_i \cap J_{i'} \) is a discrete set which accumulates at \( \infty \) from both directions.
6. The closure of each component of \( \partial \mathbb{H}^3 \setminus \bigcup_{i \in \mathbb{Z}} J_i \) is homeomorphic to the disk.

Moreover, Guéritaud describes the way in which the Cannon-Thurston map \( \omega \) fills \( \hat{\mathbb{C}} = \partial \mathbb{H}^3 \) (see [11, Section 1.2], and see also [4]). By the above proposition, we have a CW-decomposition of the complex plane \( \mathbb{C} = \partial \mathbb{H}^3 \setminus \{\infty\} \).
Definition 2.4. The symbol $C$ denotes the CW-decomposition of $C$ defined as follows:

$$
C^{(0)} = \bigcup_{i \in \mathbb{Z}} (J_i \cap J_{i+1}) \setminus \{\infty\},
$$

$$
C^{(1)} = \{\text{cl}(\gamma) \mid \gamma \text{ is a component of } J_i \setminus (J_{i-1} \cup J_{i+1}), i \in \mathbb{Z}\},
$$

$$
C^{(2)} = \{\text{cl}(\delta) \mid \delta \text{ is a component of } \partial \mathbb{H}^3 \setminus \bigcup_{i \in \mathbb{Z}} J_i, i \in \mathbb{Z}\}.
$$

We call $C$ the Cannon-Dicks-Thurston fractal tessellation (a fractal tessellation, in brief) associated with the Cannon-Thurston map $\omega$.

In order to state the main result, we introduce the notion of a "spider".

Definition 2.5. Suppose that all singularities of the invariant foliations $F_{\pm}$ of the pseudo-Anosov homeomorphism $h$ are at punctures of $F$.

(1) Let $p$ be a parabolic fixed point of $\Gamma_0$. We denote by $s(p)$ the union of the leaves of $F_{\pm}$ which have $p$ as an endpoint, and we call it the spider with head $p$. We call an endpoint of a leaf of $s(p)$ a foot of $s(p)$.

(2) A parabolic fixed point $p$ of $\Gamma_0$ with $p \neq \infty$ is called a neighbor of $\infty$ if the number of points of $s(\infty) \cap s(p)$ is equal to 2, where $\infty$ is the parabolic fixed point of $\Gamma_0$ such that $\omega(\infty) = \infty$.

Note that, by Theorem 2.2, the preimage of $\omega(p) \in \partial \mathbb{H}^3$, under the Cannon-Thurston map $\omega$, is equal to the set of all feet of $s(p)$. By the construction of $C$, we have the following lemma.

Lemma 2.6. Suppose that $v$ is a vertex of the fractal tessellation. Then there exists a unique parabolic fixed point $p$ of $\Gamma_0$ such that $v = \omega(p)$. Furthermore, $p$ is a neighbor of $\infty$.

3. Singular Euclidean structure on the fiber surface of the two-bridge knot with slope $12/29$

In this section, we describe the stable and unstable foliations of the monodromy, $h$, of the hyperbolic fibered two-bridge knot $K$ with slope $12/29$. To this end, we give a singular Euclidean structure on the fiber surface, $F$, with respect to which $h$ acts by affine transformation, following the construction of Thurston [17] (cf. [7]).

First of all, we recall the fiber surface $F$ of $K$ and the monodromy $h$. The rational number $12/29$ has the following continued fraction expansion:

$$
12/29 = \left[2, 2, 2, 2\right] = \cfrac{1}{2 + \frac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2}}}}.
$$

Thus the two-bridge knot $K$ is the boundary of the fiber surface $F$ obtained by successively plumbing the unknotted four positive and negative Hopf-bands (see Figure 1). The symbols $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ denote the simple loops in $F$ as shown in Figure 1. Each of them is a core curve of a Hopf-band. For a simple loop $\gamma$, $T_\gamma$ denotes the Dehn twist along $\gamma$. Let $T_A$ (resp. $T_B$) be the product of $T_{\alpha_1}$ and $T_{\alpha_2}$ (resp. $T_{\beta_1}$ and $T_{\beta_2}$). Then the monodromy $h$ is a pseudo-Anosov homeomorphism $T_A \circ (T_B)^{-1}$. 
FIGURE 1. The fiber surface of the hyperbolic fibered two-bridge knot $K$ with slope $12/29$.

Next, we give a singular Euclidean structure on the fiber surface $F$. We cut $F$ into three octagons, $F_1$, $F_2$ and $F_3$, as illustrated in Figure 2(a). Let $E_i$ be the rectangle obtained from $F_i$ by collapsing each of the four components of $F_i \cap \partial F$ into a point, for each $i \in \{1, 2, 3\}$. Set $\mu = (1 + \sqrt{5})/2$. Identify each rectangle $E_i$ with the Euclidean rectangle as illustrated in Figure 2(b). For any Euclidean rectangles $E_i$, the length of the edges crossed $\alpha_1$ or $\beta_2$ (resp. $\alpha_2$ or $\beta_1$) is equal to 1 (resp. $\mu$). This gives a singular Euclidean structure on $F$.

We can see that the homeomorphisms $T_A$ and $T_B$, respectively, act by affine transformations on $F$, and their derivative $dT_A$ and $dT_B$ can be described by the following matrices $D_A$ and $D_B$:

$$D_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

$$D_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}.$$
Since the monodromy $h$ is equal to $T_A \circ (T_B)^{-1}$, it acts on $F$ by affine transformation with respect to the singular Euclidean structure, such that its derivative $dh = dT_A \circ (dT_B)^{-1}$ is described by the following matrix $D_h$:

$$D_h = D_A(D_B)^{-1} = \begin{pmatrix} 1 + \mu^2 & \mu \\ \mu & 1 \end{pmatrix}. $$

The stable and unstable foliations of the monodromy $h$ are described as follows. Set $\lambda = (\mu + 3 + \sqrt{\mu+6})/2$. Let $v_+$ be the vector $(1, (-\mu + \sqrt{\mu+5})/2)^T$. Then the vector $v_+$ is an eigenvector of $D_h$, and $\lambda$ is the eigenvalue associated with $v_+$. Hence the set consisting of the straight lines on $F$ with the same slope $(-\mu + \sqrt{\mu+5})/2$ forms a foliation, $\mathcal{F}_+$, which is invariant under $h$. In fact, the invariant foliation $\mathcal{F}_+$ is the stable foliation of $h$ with dilatation $\lambda$. Similarly, the vector $v_- := (1, (-\mu - \sqrt{\mu+5})/2)^T$ is the other eigenvector of $D_h$, and it gives the unstable foliation, $\mathcal{F}_-$, of $h$.

4. Idea of the proof of Theorem 1.1

First of all, we recall the edges of the canonical decomposition, $\mathcal{D}$, of $S^3 \setminus K$, where $K$ is the hyperbolic two-bridge knot with slope $12/29$. By the work of Guéritaud [8] (cf. [2, 9, 16]), we can see that the edges of $\mathcal{D}$ are as shown in Figure 3(a). Moreover, by the author’s previous work [13], the canonical decomposition $\mathcal{D}$ is “layered” with respect to the fiber structure of $S^3 \setminus K$. In particular, each edge of $\mathcal{D}$ can be isotoped into the fiber surface $F$ (see Figure 3(b)).

![Figure 3](image)

**Figure 3.** The edges of the canonical decomposition $\mathcal{D}$ of $S^3 \setminus K$.

By Lemma 2.6, each vertex of the fractal tessellation $C$ corresponds a unique neighbor $p$ of $\infty$. On the other hand, each vertex of the cusp triangulation induced by the canonical decomposition $\mathcal{D}$ is contained in a unique edge of $\mathcal{D}$. Since the edges of $\mathcal{D}$ are isotoped into the fiber surface $F$, what we need to show is the following identity:

$$\{ \pi(g_p) \mid p \text{ is a neighbor of } \infty \} = \mathcal{D}^{(1)} ,$$

where $g_p$ is the vertical geodesic, in $\mathbb{H}^2$, above $p \in \partial \mathbb{H}^2$, and $\pi : \mathbb{H}^2 \to F = \mathbb{H}^2/T_0$ is the natural projection. By using the singular Euclidean structure introduced in Section 3, we can describe each neighbor of $\infty$. Hence we can prove the above identity.
REFERENCES


DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, 1-3-1 KAGAMIYAMA, HIGASHI-HIROSHIMA, 739-8526, JAPAN (RESEARCH FELLOW OF JAPAN SOCIETY FOR THE PROMOTION OF SCIENCE)

E-mail address: sakata202988e-05@hiroshima-u.ac.jp