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THE CANONICAL DECOMPOSITION OF THE TWO-BRIDGE KNOT WITH SLOPE 12/29 AND THE CANNON-THURSTON MAP ASSOCIATED WITH THE KNOT

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1. INTRODUCTION

This note is a summary of the author's talk at the workshop "Topology, Geometry and Algebra of low-dimensional manifolds".

Let M be a hyperbolic once-punctured torus bundle over S^1. Then M admits a *canonical decomposition* into ideal polyhedra by Epstein-Penner [6] and Weeks [18]. In fact, Jørgensen [12] has constructed the canonical decomposition of M. The canonical decomposition induces a triangulation of the peripheral torus. Thus we have a triangulation of the complex plane $\mathbb{C}$ as the lift of the triangulation of the torus to the universal cover. On the other hand, we have another tessellation of $\mathbb{C}$ which nicely reflects the nature of the Cannon-Thurston map associated with M. The tessellation is called the "Cannon-Dicks-Thurston fractal tessellation" (see Definition 2.4).

Dicks and Sakuma [5] have proved that there exists a nice relation between the two tessellations. In particular, the two tessellations share the same vertex set.

It is natural to expect that similar results hold in a more general setting, and the author has been trying to realize this expectation for hyperbolic fibered two-bridge links. The first task is to generalize the fractal tessellation to more general hyperbolic punctured surface bundles, and the author announced such a generalization for hyperbolic punctured surface bundles in [14], provided that all singularities of the stable and unstable foliations of the monodromy are at punctures of the fiber surface. The condition is satisfied for the hyperbolic fibered two-bridge knots with slope r such that r has a continued fraction expansion $\pm[2,2,\ldots,2]$. The main purpose of this note is to present an idea of the proof of the following theorem.

Theorem 1.1. *For the hyperbolic fibered two-bridge knot K with slope $12/29=[2,2,2,2]$, the triangulation induced by the canonical decomposition and the fractal tessellation share the same vertex set.*

After the announcement of this result at the workshop, the author learned the work of Guéritaud [10, 11], which generalizes the fractal tessellation and establishes beautiful relation between "veering" ideal triangulations (cf. [1]) of hyperbolic punctured surface bundles and the fractal tessellations, generalizing the result of Dicks and Sakuma. Motivated by this work, the author proved the following theorem.

Theorem 1.2 ([15, Theorem 1.1]). *The canonical decomposition of a hyperbolic fibered two-bridge link $K(r)$ $(0<|r|<1/2)$ is veering if and only if the slope r has the continued fraction expansion $\pm[2,2,\ldots,2]$.*

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As a consequence of Guéritaud's result and the above theorem, the canonical decomposition of such a two-bridge knot and the fractal tessellation are intimately related. In particular, Theorem 1.1 follows immediately.

2. The Cannon-Dicks-Thurston fractal tessellation

In this section, we will recall some properties of the Cannon-Dicks-Thurston fractal tessellations. Let F be a complete hyperbolic punctured surface of finite area, and let $\rho_0 : \pi_1(F) \to \mathrm{PSL}(2,\mathbb{R})$ be the holonomy representation of F. For a pseudo-Anosov homeomorphism h of F, the mapping torus $M_h := F \times [0,1]/(x,0) \sim (h(x),1)$ admits a complete hyperbolic structure, by Thurston's work. Let $\rho : \pi_1(F) \to \mathrm{PSL}(2,\mathbb{C})$ be the restriction of the holonomy representation of M to $\pi_1(F)$. Generalizing the work of Cannon and Thurston [4], Bowditch proved the following theorem.

Theorem 2.1. *There exists a unique $\pi_1(F)$-equivariant continuous surjection ω from the limit set of $\Gamma_0 = \rho_0(\pi_1(F))$ to that of $\Gamma = \rho(\pi_1(F))$.*

This surjection ω is called a *Cannon-Thurston map.* Note that the limit sets of Γ_0 and Γ are equal to the boundary of the hyperbolic plane $\partial\mathbb{H}^2$ and the boundary of the 3-dimensional hyperbolic space $\partial\mathbb{H}^3 = \widehat{\mathbb{C}}$, respectively. Thus the surjection ω is a sphere filling curve.

Moreover, Bowditch has given a complete description of the combinatorics of the Cannon-Thurston map.

Theorem 2.2 ([3, Theorem 9.1]). *Let $\mathcal{F}_\pm$ be the stable and unstable foliations of h, and let $\widetilde{\mathcal{F}}_\pm$ be the lift of $\mathcal{F}_\pm$ to the universal cover $\mathbb{H}^2$. For any leaf of $\widetilde{\mathcal{F}}_\pm$, the images of its endpoints under the Cannon-Thurston map ω are identical. Furthermore, this generates all the identifications occurring under ω.*

For the case when F is a once-punctured torus, Cannon and Dicks [4] have introduced a certain fractal tessellation of the complex plane $\mathbb{C} = \widehat{\mathbb{C}} \setminus \{\infty\}$, where ∞ is a parabolic fixed point of Γ. The fractal tessellation nicely reflects the behavior of the Cannon-Thurston map. This result was extended by Guéritaud [11] to a more general setting as follows.

Proposition 2.3 (cf. [11, Theorem 1.2]). *Suppose that all singularities of the invariant foliations $\mathcal{F}_\pm$ of the pseudo-Anosov homeomorphism h are at punctures of F. Then there exists a $\mathbb{Z}$-family of Jordan curves J_i of $\partial\mathbb{H}^3$, bounding domains D_i, with the following properties:*

1. *For each $i \in \mathbb{Z}$, the Jordan curve J_i passes through ∞, a parabolic fixed point of Γ.*
2. *For each $i \in \mathbb{Z}$, $D_i \supset D_{i+1}$.*
3. *$\bigcap_{i\in\mathbb{Z}} D_i = \emptyset$ and $\bigcup_{i\in\mathbb{Z}} D_i = \mathbb{C}$.*
4. *For $i, i' \in \mathbb{Z}$, if $|i - i'| > 1$, then $J_i \cap J_{i'} = \{\infty\}$.*
5. *For $i, i' \in \mathbb{Z}$, if $|i - i'| = 1$, then $J_i \cap J_{i'}$ is a discrete set which accumulates at ∞ from both directions.*
6. *The closure of each component of $\partial\mathbb{H}^3 \setminus \bigcup_{i\in\mathbb{Z}} J_i$ is homeomorphic to the disk.*

Moreover, Guéritaud describes the way in which the Cannon-Thurston map ω fills $\widehat{\mathbb{C}} = \partial\mathbb{H}^3$ (see [11, Section 1.2], and see also [4]). By the above proposition, we have a CW-decomposition of the complex plane $\mathbb{C} = \partial\mathbb{H}^3 \setminus \{\infty\}$.

Definition 2.4. The symbol $\mathcal{C}$ denotes the CW-decomposition of $\mathbb{C}$ defined as follows:

$$\mathcal{C}^{(0)} = \bigcup_{i\in\mathbb{Z}}(J_i \cap J_{i+1}) \setminus \{\infty\},$$
$$\mathcal{C}^{(1)} = \{\mathrm{cl}(\gamma) \mid \gamma \text{ is a component of } J_i \setminus (J_{i-1} \cup J_{i+1}),\ i \in \mathbb{Z}\},$$
$$\mathcal{C}^{(2)} = \left\{\mathrm{cl}(\delta) \mid \delta \text{ is a component of } \partial\mathbb{H}^3 \setminus \textstyle\bigcup_{i\in\mathbb{Z}} J_i,\ i \in \mathbb{Z}\right\}.$$

We call $\mathcal{C}$ the *Cannon-Dicks-Thurston fractal tessellation* (a *fractal tessellation*, in brief) associated with the Cannon-Thurston map ω.

In order to state the main result, we introduce the notion of a "spider".

Definition 2.5. Suppose that all singularities of the invariant foliations $\mathcal{F}_\pm$ of the pseudo-Anosov homeomorphism h are at punctures of F.

(1) Let p be a parabolic fixed point of Γ_0. We denote by $\mathfrak{s}(p)$ the union of the leaves of $\widetilde{\mathcal{F}}_\pm$ which have p as an endpoint, and we call it the *spider* with head p. We call an endpoint of a leaf of $\mathfrak{s}(p)$ a *foot* of $\mathfrak{s}(p)$.

(2) A parabolic fixed point p of Γ_0 with $p \neq \infty$ is called a *neighbor* of ∞ if the number of points of $\mathfrak{s}(\infty) \cap \mathfrak{s}(p)$ is equal to 2, where ∞ is the parabolic fixed point of Γ_0 such that $\omega(\infty) = \infty$.

Note that, by Theorem 2.2, the preimage of $\omega(p) \in \partial\mathbb{H}^3$, under the Cannon-Thurston map ω, is equal to the set of all feet of $\mathfrak{s}(p)$. By the construction of $\mathcal{C}$, we have the following lemma.

Lemma 2.6. *Suppose that v is a vertex of the fractal tessellation. Then there exists a unique parabolic fixed point p of Γ_0 such that $v = \omega(p)$. Furthermore, p is a neighbor of ∞.*

3. Singular Euclidean structure on the fiber surface of the two-bridge knot with slope 12/29

In this section, we describe the stable and unstable foliations of the monodromy, h, of the hyperbolic fibered two-bridge knot K with slope $12/29$. To this end, we give a singular Euclidean structure on the fiber surface, F, with respect to which h acts by affine transformation, following the construction of Thurston [17] (cf. [7]).

First of all, we recall the fiber surface F of K and the monodromy h. The rational number $12/29$ has the following continued fraction expansion:

$$12/29 = [2,2,2,2] = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}}.$$

Thus the two-bridge knot K is the boundary of the fiber surface F obtained by successively plumbing the unknotted four positive and negative Hopf-bands (see Figure 1). The symbols α_1, α_2, β_1 and β_2 denote the simple loops in F as shown in Figure 1. Each of them is a core curve of a Hopf-band. For a simple loop γ, T_γ denotes the Dehn twist along γ. Let T_A (resp. T_B) be the product of T_{α_1} and T_{α_2} (resp. T_{β_1} and T_{β_2}). Then the monodromy h is a pseudo-Anosov homeomorphism $T_A \circ (T_B)^{-1}$.

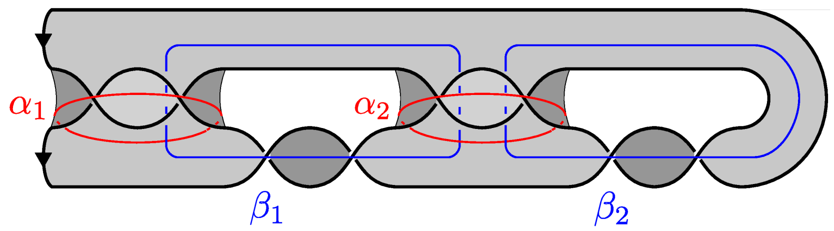


FIGURE 1. The fiber surface of the hyperbolic fibered two-bridge knot K with slope 12/29.

Next, we give a singular Euclidean structure on the fiber surface F. We cut F into three octagons, F_1, F_2 and F_3, as illustrated in Figure 2(a). Let E_i be the rectangle

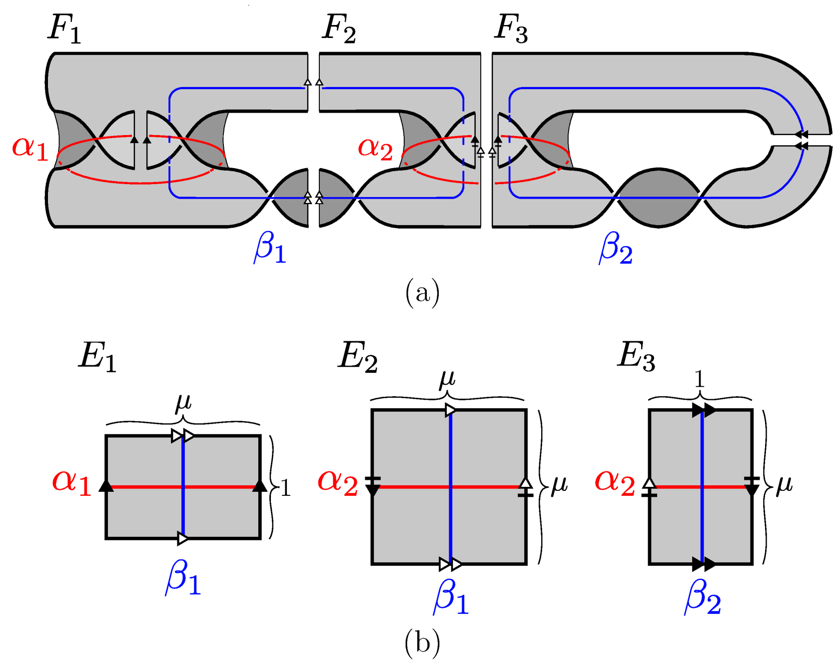


FIGURE 2. (a) The octagons F_1, F_2 and F_3. (b) The Euclidean rectangles E_1, E_2 and E_3.

obtained from F_i by collapsing each of the four components of $F_i \cap \partial F$ into a point, for each $i \in \{1,2,3\}$. Set $\mu = (1+\sqrt{5})/2$. Identify each rectangle E_i with the Euclidean rectangle as illustrated in Figure 2(b). For any Euclidean rectangles E_i, the length of the edges crossed α_1 or β_2 (resp. α_2 or β_1) is equal to 1 (resp. μ). This gives a singular Euclidean structure on F.

We can see that the homeomorphisms T_A and T_B, respectively, act by affine transformations on F, and their derivative dT_A and dT_B can be described by the following matrices D_A and D_B:

$$D_A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix},$$

$$D_B = \begin{pmatrix} 1 & 0 \\ -\mu & 1 \end{pmatrix}.$$

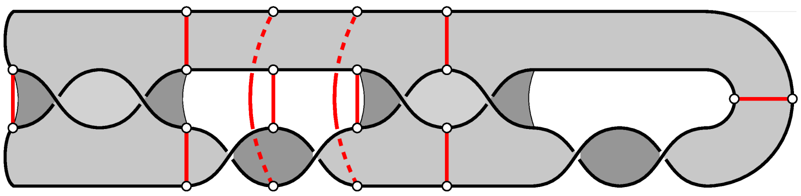
Since the monodromy h is equal to $T_A \circ (T_B)^{-1}$, it acts on F by affine transformation with respect to the singular Euclidean structure, such that its derivative $dh = dT_A \circ (dT_B)^{-1}$ is described by the following matrix D_h:

$$D_h = D_A(D_B)^{-1} = \begin{pmatrix} 1+\mu^2 & \mu \\ \mu & 1 \end{pmatrix}.$$

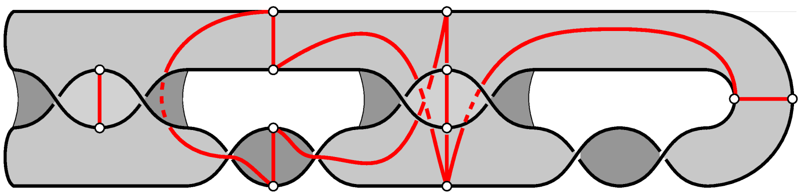
The stable and unstable foliations of the monodromy h are described as follows. Set $\lambda = (\mu + 3 + \sqrt{7\mu + 6})/2$. Let $\boldsymbol{v}_+$ be the vector $\left(1, (-\mu + \sqrt{\mu+5})/2\right)^{\mathsf{T}}$. Then the vector $\boldsymbol{v}_+$ is an eigenvector of D_h, and λ is the eigenvalue associated with $\boldsymbol{v}_+$. Hence the set consisting of the straight lines on F with the same slope $(-\mu+\sqrt{\mu+5})/2$ forms a foliation, $\mathcal{F}_+$, which is invariant under h. In fact, the invariant foliation $\mathcal{F}_+$ is the stable foliation of h with dilatation λ. Similarly, the vector $\boldsymbol{v}_- := \left(1, (-\mu - \sqrt{\mu+5})/2\right)^{\mathsf{T}}$ is the other eigenvector of D_h, and it gives the unstable foliation, $\mathcal{F}_-$, of h.

4. Idea of the proof of Theorem 1.1

First of all, we recall the edges of the canonical decomposition, $\mathcal{D}$, of $S^3 \setminus K$, where K is the hyperbolic two-bridge knot with slope 12/29. By the work of Guéritaud [8] (cf. [2, 9, 16]), we can see that the edges of $\mathcal{D}$ are as shown in Figure 3(a). Moreover, by the author's previous work [13], the canonical decomposition $\mathcal{D}$ is "layered" with respect to the fiber structure of $S^3 \setminus K$. In particular, each edge of $\mathcal{D}$ can be isotoped into the fiber surface F (see Figure 3(b)).



(a)



(b)

Figure 3. The edges of the canonical decomposition $\mathcal{D}$ of $S^3 \setminus K$.

By Lemma 2.6, each vertex of the fractal tessellation $\mathcal{C}$ corresponds a unique neighbor p of ∞. On the other hand, each vertex of the cusp triangulation induced by the canonical decomposition $\mathcal{D}$ is contained in a unique edge of $\mathcal{D}$. Since the edges of $\mathcal{D}$ are isotoped into the fiber surface F, what we need to show is the following identity:

$$\{\pi(g_p) \mid p \text{ is a neighbor of } \infty\} = \mathcal{D}^{(1)},$$

where g_p is the vertical geodesic, in $\mathbb{H}^2$, above $p \in \partial\mathbb{H}^2$, and $\pi : \mathbb{H}^2 \to F = \mathbb{H}^2/\Gamma_0$ is the natural projection. By using the singular Euclidean structure introduced in Section 3, we can describe each neighbor of ∞. Hence we can prove the above identity.

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