数理解析研究所講究録
第 1991 巻 2016 年 154-157



# A VARIANT OF THE CHERN-SIMONS PERTURBATION THEORY

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## 1. Introduction

The Chern-Simons perturbation theory established by S. Axelrod and I. M. Singer in [1] and M. Kontsevich in [4] gives a topological invariant of a closed oriented 3-manifold with an acyclic local system. In the construction of this invariant, a chain map called a trace map plays an important role. A trace map is a chain map from the triple tensor product of the given local system to the trivial local system. It is, however, difficult to construct trace maps corresponding to the given local system. In this note, we give a variant of the degree 1 part of the Chern-Simons perturbation theory to construct new examples of trace maps.

**Acknowledgment.** The author would like to thank the organizers of the RIMS Seminar "Topology, Geometry and Algebra of low-dimensional manifolds" for inviting him. This work was supported by JSPS KAKENHI Grant Number 15K13437.

## 2. Review of the degree 1 part of the Chern-Simons perturbation theory

In this section, we review the degree 1 part (i.e., the 2$-$loop term) of the Chern-Simons perturbation theory based on the Kontsevich's construction in [4]. Let $M$ be a closed oriented 3-manifold. Let $E$ be a real local system on $M$. We consider $E$ as a covariant functor from the fundamental groupoid of $M$ to the category of finite dimensional real vector space. Let $\rho_E : \pi_1(M,x) \to \mathrm{Aut}(E_x)$ be the characteristic representation of the fundamental group corresponding to $E$, where $x \in M$ is a base point of $M$ and $E_x$ is the vector space corresponding to $x$. We assume the following conditions:

- The image of $\rho_E$ is in $SO(E_x)$, namely $\rho_E$ is an orthonormal representation.
- $E$ is acyclic, that is $H_k(M;E) = 0$ for any $k \in \mathbb{Z}$.

We will call such a local system an *acyclic orthonormal local system.*

Let $\mathbb{R}$ be the trivial local system on $M$. Take a chain map $Tr : E^{\otimes 3} \to \mathbb{R}$ (namely, for any $x, y \in M$ and a path $\gamma$ from $x$ to $y$, $Tr \circ (\gamma_* \otimes \gamma_* \otimes \gamma_*) = Tr$). We will call such a chain map a *trace map corresponding to E*.

Then for any cochain (resp. cocycle) $c \in C^*(M; E^{\otimes 3})$, we get a cochain (resp. cocycle) $Tr_* c \in C^*(M;\mathbb{R})$. In many cases, it is difficult to find a non trivial example of a trace map. For example, there is only the trivial trace map when the local system is corresponding to the surjective orthonormal representation $\pi_1(M) \to SO(2)$. We show two examples of trace maps.

**Example 2.1.** (1) (M. Kontsevich [4]) Let $\rho_G : \pi_1(M) \to G$ be a representation of the fundamental group in a semi-simple Lie group $G$ and we denote by $\rho : \pi_1(M) \to \mathrm{Aut}(\mathbf{g})$ the composition of the adjoint representation of $G$ and $\rho_G$ where $\mathbf{g}$ be the lie algebra of $G$. Let $E$ be the local system corresponding to $\rho$. In this setting we

Received December 31, 2015.

can take $Tr : E\otimes E\otimes E \to \mathbb{R}$ as $Tr(x\otimes y\otimes z) = \langle x, [y, z]\rangle$ where $\langle\ ,\ \rangle$ is the inner product of $\mathbf{g}$ and $[\ ,\ ]$ is the Lie bracket of $\mathbf{g}$.

(2) (G. Kuperberg and D. Thurston [5]) Let $M$ be a rational homology 3-sphere with a base point $\infty \in M$. Since the reduced homology groups $\tilde{H}_k(M\setminus\infty;\mathbb{R})$ are vanishing for all $k\in\mathbb{Z}$, we can take $E$ as a trivial local system $\mathbb{R}$. G. Kuperberg and D. Thurston developed the Chern-Simons perturbation theory for this situation. In this setting, $E\otimes E\otimes E = \mathbb{R}$. We can take $Tr : E\otimes E\otimes E = \mathbb{R}\to\mathbb{R}$ as the identity map.

**Remark 2.2.** M. Futaki, in his master thesis [3], investigated trace maps via the representation of the permutation group $\mathcal{S}_3$. He gave an example of a trace map different from Kontsevich's or Kuperberg-Thurston's trace map.

Let $p_i : M^2 \to M (i = 1, 2)$ be the projections. Let $\Delta = \{(x, x) \mid x \in M\} \subset M^2$. Let $C_2(M) = B\ell(M^2, \Delta)$ be the manifold with corners obtained by real blowing-up of $M^2$ along $\Delta$. We denote by $q : C_2(M) \to M^2$ the blow-down map. We remark that $C_2(M)$ is a compactification of the configuration space $M^2\setminus\Delta$. $q^*(p_1^*E\otimes p_2^*E)$ is a local system on $C_2(M)$. Let $\omega \in A^2(C_2(M); q^*(p_1^*E\otimes p_2^*E))$ be a closed 2-form on $C_2(M)$. Since $(q^*(p_1^*E\otimes p_2^*E))^{\otimes 3} = q^*(p_1^*(E^{\otimes 3})\otimes p_2^*(E^{\otimes 3}))$, $(Tr\otimes Tr)\omega^3$ is a closed 6-form on $C_2(M)$ with trivial coefficient: $(Tr\otimes Tr)\omega^3 \in A^6(C_2(M);\mathbb{R})$.

**Lemma 2.3.** *Let $\omega_0, \omega_1 \in A^2(C_2(M); q^*(p_1^*E\otimes p_2^*E))$ be closed 2-forms satisfying $\omega_0|_{\partial C_2(M)} = \omega_1|_{\partial C_2(M)}$. If $\omega_0|^2_{\partial C_2(M)} = 0$, then*

$$\int_{C_2(M)} (Tr\otimes Tr)\omega_0^3 = \int_{C_2(M)} (Tr\otimes Tr)\omega_1^3.$$

*Proof.* Because the assumption, $\omega_0-\omega_1$ represents the cohomology class of $H^2(M^2; p_1^*E\otimes p_2^*E)$. Since $H^2(M^2; p_1^*E\otimes p_2^*E) = 0$, there exists a 1-form $\eta \in A^1(M^2; p_1^*E\otimes p_2^*E)$ satisfying $d\eta = \omega_0 - \omega_1$. Because $\omega_0|^2_{\partial C_2(M)} = 0$, we can extend $\omega_0^2, \omega_0\omega_1, \omega_1^2 \in A^2(M^2\setminus\Delta; (p_1^*E\otimes p_2^*E)^{\otimes 2})$ to $M^2$ as closed 2-forms. (We may assume that $\omega_0 = \omega_1 = 0$ near $\partial C_2(M)$, by deforming $\omega_0, \omega_1$ by homotopy in near $\partial C_2(M)$ if necessary. ) Thanks to Stokes' theorem, we have

$$\begin{aligned}
&\int_{C_2(M)} (Tr\otimes Tr)\omega_0^3 - \int_{C_2(M)} (Tr\otimes Tr)\omega_1^3\\
&= \int_{C_2(M)} (Tr\otimes Tr)((\omega_0-\omega_1)(\omega_0^2+\omega_0\omega_1+\omega_1^2))\\
&= \int_{M^2\setminus\Delta} (Tr\otimes Tr)(d\eta|_{M^2\setminus\Delta}(\omega_0^2+\omega_0\omega_1+\omega_1^2))\\
&= \int_{M^2} (Tr\otimes Tr)(d(\eta(\omega_0^2+\omega_0\omega_1+\omega_1^2)))\\
&= \int_{M^2} d((Tr\otimes Tr)(\eta(\omega_0^2+\omega_0\omega_1+\omega_1^2)))\\
&= 0.
\end{aligned}$$

□

Let $\omega^\partial \in A^2(\partial C_2(M); q^*(p_1^*E\otimes p_2^*E))$ be a closed 2-form such that

$$[\omega^\partial] \in H^2(\partial C_2(M); q^*(p_1^*E\otimes p_2^*E))$$

is in the image of the restriction map

$$r^* : H^2(C_2(M); q^*(p_1^*E \otimes p_2^*E)) \to H^2(\partial C_2(M); q^*(p_1^*E \otimes p_2^*E))$$

and $(\omega^\partial)^2 = 0$. We take a closed 2-form $\omega \in A^2(C_2(M); q^*(p_1^*E \otimes p_2^*E))$ satisfying $\omega|_{\partial C_2(M)} = \omega^\partial$.

**Definition 2.4.**

$$I(M, E, Tr, \omega^\partial) = \int_{C_2(M)} (Tr \otimes Tr)\omega^3.$$

**Example 2.5.** Bott and Cattaneo proved in [2] that when $M$ is a homology 3-sphere we can choose $\omega|_{\partial C_2(M)} = p(\tau)^*\iota_*\omega_{S^2}$. Here $p(\tau) : \partial C_2(M) \cong M \times S^2 \to S^2$ is the projection map induced by a framing $\tau : TM \cong M \times \mathbb{R}^3$ and $\omega_{S^2} \in A^2(S^2; \mathbb{R})$ is a closed 2-form on $S^2$ such that $\int_{S^2} \omega_{S^2} = 1$ and $\iota : \mathbb{R} \to E \otimes E$ is a chain map defined by $\iota(1) = 1_E$, where $1_E \in E \otimes E^* = E \otimes E$ is the evaluation map.

## 3. A variant of the Chern-Simons perturbation theory

In this section we extend the invariant $I(M, E, Tr, \omega^\partial)$. The extended invariant is more flexible than the original one. Let $E_1, E_2$ and $E_3$ be complex acyclic local systems on $M$. We denote by $\overline{E_1}, \overline{E_2}$ and $\overline{E_3}$ the conjugate local systems of $E_1, E_2$ and $E_3$ respectively. Let $Tr : E_1 \otimes E_2 \otimes E_3 \to \mathbb{C}$ be a chain map. Take a closed 2-form $\omega_i^\partial \in A^2(\partial C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ such that $[\omega^\partial] \in H^2(\partial C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ is in the image of the restriction map $r^* : H^2(C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i})) \to H^2(\partial C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ and $\omega_i^\partial \wedge \omega_j^\partial = 0$ for any $i = 1, 2, 3$ and $j = 1, 2, 3$. We take a closed 2-form $\omega_i \in A^2(C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ satisfying $\omega_i|_{\partial C_2(M)} = \omega_i^\partial$.

**Definition 3.1.**

$$I(M, (E_i)_{i=1,2,3}, Tr, (\omega_i^\partial)_{i=1,2,3}) = \int_{C_2(M)} (Tr \otimes Tr)(\omega_1 \wedge \omega_2 \wedge \omega_3).$$

**Remark 3.2.**
- By the same reason as in Lemma 2.3, $I(M, (E_i)_{i=1,2,3}, Tr, (\omega_i^\partial)_{i=1,2,3})$ is independent of the choices of $\omega_1, \omega_2$ and $\omega_3$.
- We can not define $I(M, E, Tr, \omega^\partial)$ for a non-acyclic local system $E$. We expect that it may be obtained as an appropriate limit of $I(M, (E_i)_{i=1,2,3}, Tr, (\omega_i^\partial)_{1,2,3})$.

## 4. An Example

Let $M = S^1 \times S^1 \times S^1$. We denote by $[S_1^1] \in H_1(M; \mathbb{Z})$ the homology class represented by the first $S^1$ factor. Let $\alpha_1, \alpha_2, \alpha_3 \in U(1) \setminus \{1\}$ be any complex numbers satisfying $\alpha_1\alpha_2\alpha_3 = 1$. For $i = 1, 2, 3$, $E_i$ is the complex local system corresponding to the abelian representation $\rho_i : H_1(M; \mathbb{Z}) \to \mathbb{Z}[S_1^1] \to U(1), n[S_1^1] \mapsto \alpha_i^n$, $n \in \mathbb{Z}$. Here $H_1(M; \mathbb{Z}) \to \mathbb{Z}[S_1^1]$ is the projection. In this situation, $H_k(M; E_i) = 0$ for any $k \in \mathbb{Z}$ and $i = 1, 2, 3$.

We next give a closed 2-form $\omega_i^\partial \in A^2(\partial C_2(M); q^*(p_1^*E \otimes p_2^*\overline{E}))$ explicitly. We consider $S^1$ as $\mathbb{R}/\mathbb{Z}$ and let $(x, y, z)$ be the coordinate of $M = \mathbb{R}^3/\mathbb{Z}^3$. Then we have a (global) coordinate $(x_1, y_1, z_1, x_2, y_2, z_2)$ of $M \times M$.

Let

$$N(\Delta) = \{(x_1, y_1, z_1, x_2, y_2, z_2) \mid |x_1 - x_2| < \varepsilon_1, |y_1 - y_2| < \varepsilon_1, |z_1 - z_2| < \varepsilon_1\}$$

be a tubular neighborhood of $\Delta$ in $M^2$ for an enough small positive number $\varepsilon_1 > 0$. We identify $C_2(M)$ with $M^2 \setminus N(\Delta)$.

The normal bundle of $\Delta$ is canonically isomorphic to the tangent bundle $TM$. Then $\partial C_2(M) = \partial N(\Delta)$ is identified with $\Delta \times S^2$ via the standard trivialization of $TM = TS^1 \times TS^1 \times TS^1$. Let $\iota : \mathbb{C} \to E_i \otimes \overline{E_i}|_\Delta = \mathbb{C}$ be the identity chain map.

Take a smooth function $\varphi : \mathbb{R} \to [0,1]$ satisfying the following conditions:

- There is an enough small real number $\varepsilon_1 >> \varepsilon_2 > 0$, $\mathrm{supp}(\varphi) \subset (-\varepsilon_2, \varepsilon_2)$,
- $\varphi(0) = 1$.

For $i = 1, 2, 3$ we set

$$\omega_i^\partial = (1 - \alpha_i)(\iota_*(\varphi(y_2 - y_1)\varphi(z_2 - z_1)(dy_2 - dy_1) \wedge (dz_2 - dz_1)).$$

**Proposition 4.1.** *$\omega_i^\partial$ is in the image of the restriction map $r^* : H^2(C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i})) \to H^2(\partial C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ and $\omega_i^\partial \wedge \omega_j^\partial = 0$ for any $i = 1, 2, 3$ and $j = 1, 2, 3$. Furthermore*

$$I(M, (E_i)_{i=1,2,3}, Tr, (\omega_i^\partial)_{i=1,2,3}) = 0.$$

*Proof.* We give an extended 2-form $\omega_i \in A^2(C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ of $\omega_i^\partial$ for $i = 1, 2, 3$ explicitly. Let

$$\omega_{\mathbb{R}} = \varphi(y_2 - y_1)\varphi(z_2 - z_1)(dy_2 - dy_1) \wedge (dz_2 - dz_1) \in A^2(M^2 \setminus \Delta; \mathbb{R}).$$

Obviously, we can extend $\omega_{\mathbb{R}}$ to $C_2(M)$.

Let $s : \Delta \to \partial C_2(M^2)$ be the section defined by

$$s(x, y, z, x, y, z) = \left(x, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, x, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Since the image $s(\Delta)$ of the section $s$ is a deformation retract of the support of $\omega_{\mathbb{R}}$, we can extend the chain map $\iota$ (See Example 2.5.) to $\mathrm{supp}(\omega_{\mathbb{R}})$. We denote by $\iota_i$ such an extended chain map. Therefore we have the closed 2-form $\omega_i = (\iota_i)_*\omega_{\mathbb{R}} \in A^2(C_2(M); q^*(p_1^*E_i \otimes p_2^*\overline{E_i}))$ for $i = 1, 2, 3$. By the construction, $\omega_i|_{\partial C_2(M)} = \omega_i^\partial$ and $\omega_1 \wedge \omega_2 \wedge \omega_3 = 0$. $\square$

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