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SYMMETRIC GROUPS, DIHEDRAL GROUPS, AND KNOT GROUPS

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ABSTRACT. The number of group homomorphisms of a knot group is a knot invariant. In this paper, we determine the numbers of group homomorphisms of knot groups to symmetric groups and dihedral groups in low degree.

1. INTRODUCTION

Let K be a knot and $G(K)$ the knot group, namely, the fundamental group of the exterior of the knot K in S^3. It is a useful method to investigate a given group that we construct a group homomorphism of the group to another well known group. For example, $SL(2;\mathbb{Z}/p\mathbb{Z})$-representations of knot groups are studied in [5]. In this paper, we consider group homomorphisms of knot groups to symmetric groups, and dihedral groups. To be precise, we calculate all the group homomorphisms of knot groups with up to 8 crossings to symmetric groups S_n of degree up to 6, and to dihedral groups D_{2n} of degree up to 18. Furthermore, they are classified by the order of the images. Throughout this paper, the numbers of homomorphisms are considered up to conjugation.

2. SYMMETRIC GROUP

First, we consider homomorphisms of knot groups to symmetric groups S_n:

$$S_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \,|\, {\sigma_i}^2 = 1,\ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } j \neq i \pm 1,\ (\sigma_i\sigma_{i+1})^3 = 1\rangle.$$

A representation onto symmetric group S_n corresponds to an n-fold covering of $S^3 - K$, see [2] for example. It is known that there exist subgroups of symmetric group S_3 and S_4 whose orders are divisors of 3! and 4! respectively. However, there does not exist a subgroup of S_5 whose order is $15, 30, 40$, though they are divisors of 5!. Similarly, there does not always exist a subgroup of symmetric subgroup S_n whose order is a divisor of $n!$. See [3], [4] in detail, for example.

Theorem 2.1. *All the prime knots with up to* 8 *crossings, except for two pairs* $(7_1, 8_{12})$ *and* $(7_3, 8_{13})$*, can be distinguished by the orders of the images of group homomorphisms to* S_n *up to* 6.

Theorem 2.1 is shown by Table 1 and Table 2. For example, Table 1 says that there exists a surjective homomorphism of $G(3_1)$ onto S_3. On the other hand, there does not exist a surjective homomorphism of $G(4_1)$ onto S_3. Then we conclude that these knots 3_1 and 4_1 are not equivalent. Moreover, though the numbers of group homomorphisms of $G(5_2)$ and $G(8_7)$ to S_n are same up to degree 6, there exists a homomorphism of $G(5_2)$ to S_6 such that the order of the image is 36 and there does not exist such a homomorphism of $G(8_7)$. Therefore we obtain that 5_2 and 8_7 are not equivalent.

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Remark 2.2. We can distinguish the pairs $(7_1, 8_{12})$ and $(7_3, 8_{13})$ by using homomorphisms to S_7.

We determine the numbers of homomorphisms to S_n in several cases as follows.

Proposition 2.3. *For any knot K,*
(1-a) $|\{f : G(K) \to S_3 \mid |\mathrm{im}\, f| = 2\}| = 1$, (1-b) $|\{f : G(K) \to S_3 \mid |\mathrm{im}\, f| = 3\}| = 1$,
(2-a) $|\{f : G(K) \to S_4 \mid |\mathrm{im}\, f| = 2\}| = 2$, (2-b) $|\{f : G(K) \to S_4 \mid |\mathrm{im}\, f| = 3\}| = 1$,
(2-c) $|\{f : G(K) \to S_4 \mid |\mathrm{im}\, f| = 4\}| = 1$, (2-d) $|\{f : G(K) \to S_4 \mid |\mathrm{im}\, f| = 8\}| = 0$
(3-a) $|\{f : G(K) \to S_5 \mid |\mathrm{im}\, f| = 2\}| = 2$, (3-b) $|\{f : G(K) \to S_5 \mid |\mathrm{im}\, f| = 3\}| = 1$,
(3-c) $|\{f : G(K) \to S_5 \mid |\mathrm{im}\, f| = 4\}| = 1$, (3-d) $|\{f : G(K) \to S_5 \mid |\mathrm{im}\, f| = 5\}| = 1$,
(3-e) $|\{f : G(K) \to S_5 \mid |\mathrm{im}\, f| = 8\}| = 0$.

Proof. There exists only one subgroup of S_3 of order 2 (up to conjugation), which is generated by one element and a cyclic group. A non-trivial homomorphism of $G(K)$ to this group maps all elements to its generator. Then the number of such homomorphisms is 1 and we get (1-a). By similar arguments, we obtain (1-b), (2-a), (2-b), (3-a), (3-b), and (3-d). Note that there are two subgroups of S_4 and S_5 of order 2 respectively.

There are three conjugacy classes of subgroups of S_4 (and S_5) of order 4. One of them is a cyclic group $\mathbb{Z}/4\mathbb{Z}$ and $G(K)$ admits one surjective homomorphism onto this subgroup. It is easy to see that $G(K)$ does not admit a surjective homomorphism onto the other subgroups. Then the number of homomorphisms to subgroups of S_4 (and S_5) of order 4 is one.

The subgroup of S_4 (and S_5) of order 8, which is the 2-Sylow subgroup, is the dihedral group D_8. As we see later in Theorem 3.1, there does not exist a surjective homomorphism of $G(K)$ onto D_8. Therefore the order of the image of homomorphism to S_4 (and S_5) is not 8.

This completes the proof. □

3. Dihedral group

Next, we will see homomorphisms of knot groups to dihedral groups D_{2n}:

$$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ srs = r^{-1} \rangle.$$

It is well known that D_6 is isomorphic to S_3. In general, D_{2n} can be regarded as a subgroup of S_n. The subgroups of D_{2n} are determined in [1], namely, they are generated by $\{r^d\}$ or $\{r^d, r^k s\}$, where d is a divisor of n and $0 \leq k < d$.

Theorem 3.1. *Let K be a knot and $f : G(K) \to D_8$ a group homomorphism. Then the image of f is a cyclic group $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$. In particular, f is not surjective. Moreover, $|\{f : G(K) \to D_8 \mid \mathrm{im}\, f = \mathbb{Z}/2\mathbb{Z}\}| = 3$ and $|\{f : G(K) \to D_8 \mid \mathrm{im}\, f = \mathbb{Z}/4\mathbb{Z}\}| = 1$.*

Proof. It it known that the conjugacy decomposition of D_8 is the following:

$$D_8 = \{e\} \cup \{r, r^3\} \cup \{r^2\} \cup \{s, r^2 s\} \cup \{rs, r^3 s\}.$$

Note that $s \cdot r \cdot s^{-1} = r^{-1} = r^3$, $r \cdot s \cdot r^{-1} = r^2 s$, and $r \cdot rs \cdot r^{-1} = r^3 s$. We fix the Wirtinger presentation of knot group:

$$G(K) = \langle x_1, x_2, \ldots, x_k \mid x_{i_1} x_1 x_{i_1}^{-1} x_2^{-1} = 1, x_{i_2} x_2 x_{i_2}^{-1} x_3^{-1} = 1, \ldots, x_{i_k} x_k x_{i_k}^{-1} x_1^{-1} = 1 \rangle.$$

Remark that $x_1, x_2, \ldots, x_k$ are conjugate to one another. Then all the $f(x_1), f(x_2), \ldots, f(x_k)$ are also conjugate. If $f(x_i)$ is r, then the image of f is a cyclic group $\mathbb{Z}/4\mathbb{Z}$. Similarly, if $f(x_i)$ is r^2, then the image of f is $\mathbb{Z}/2\mathbb{Z}$.

Next, we assume $f(x_i) = s$. Since $f(x_1)$ and $f(x_{i_1})$ are contained in the same conjugacy class, $f(x_{i_1})$ is s or r^2s. We see that

$$f(x_{i_1}x_1x_{i_1}^{-1}) = \begin{cases} s \cdot s \cdot s^{-1} = s \\ r^2s \cdot s \cdot (r^2s)^{-1} = r^2sr^{-2} = r^4s = s \end{cases}.$$

In either case, $f(x_2) = s$, by $f(x_{i_1}x_1x_{i_1}^{-1}x_2^{-1}) = 1$. Inductively, all the x_i are sent to s. Therefore the image of f is a cyclic group $\mathbb{Z}/2\mathbb{Z}$.

Finally, we assume $f(x_1) = rs$. In this case, all the x_i are sent to rs by similar argument. Since $(rs)^2 = 1$, the image of f is also a cyclic group $\mathbb{Z}/2\mathbb{Z}$.

The above shows us the numbers of homomorphisms to D_8 too. □

4. Tables

The following are tables of the numbers of homomorphisms to S_n and D_{2n}. The first columns of these tables line up prime knots with up to 8 crossings. The numbers of knots follow the Rolfsen's book [6]. The other columns give us the numbers of homomorphisms (up to conjugation) to S_n and D_{2n} such that the order of the image is k. For example, the second column of Table 1 shows the numbers of homomorphisms to subgroups of S_3 of order 2. We omit the columns for the number of trivial homomorphisms, since the number is always 1.

Table 1: S_3, S_4, and S_5

K	S_3			S_4							S_5											
	2	3	6	2	3	4	6	8	12	24	2	3	4	5	6	8	10	12	20	24	60	120
3_1	1	1	1	2	1	1	1	0	1	1	2	1	1	1	3	0	0	1	0	1	1	0
4_1	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	1	1	0	1	1	2
5_1	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	1	0	0	0	2	2
5_2	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	1	1
6_1	1	1	1	2	1	1	1	0	0	1	2	1	1	1	3	0	0	0	2	1	0	0
6_2	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	1
6_3	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	1	0
7_1	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	0
7_2	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	0	1	2	0	0	0
7_3	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	0	1	0	0	0	1
7_4	1	1	1	2	1	1	1	0	0	1	2	1	1	1	3	0	1	0	0	1	1	0
7_5	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	0
7_6	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	2	0	0	1
7_7	1	1	1	2	1	1	1	0	0	1	2	1	1	1	3	0	0	0	0	1	0	2
8_1	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	0	1	0	0	1	0
8_2	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	1
8_3	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	2	0
8_4	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	0	1	0	0	1	1
8_5	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	0	1	0	3	2	1
8_6	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	2	1
8_7	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	1	1
8_8	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	1	0	2	0	1	1

K	S_3			S_4							S_5											
	2	3	6	2	3	4	6	8	12	24	2	3	4	5	6	8	10	12	20	24	60	120
8_9	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	1	0	0	0	0	0
8_{10}	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	0	1	0	3	2	1
8_{11}	1	1	1	2	1	1	1	0	1	1	2	1	1	1	3	0	0	1	2	1	1	1
8_{12}	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	0
8_{13}	1	1	0	2	1	1	0	0	1	0	2	1	1	1	1	0	0	1	0	0	0	1
8_{14}	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	0	0
8_{15}	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	0	1	2	3	2	1
8_{16}	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	1	0	0	0	1	1
8_{17}	1	1	0	2	1	1	0	0	0	0	2	1	1	1	1	0	0	0	0	0	1	2
8_{18}	1	1	4	2	1	1	4	0	5	4	2	1	1	1	9	0	1	5	0	4	4	4
8_{19}	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	0	1	0	3	1	3
8_{20}	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	0	1	0	3	2	0
8_{21}	1	1	1	2	1	1	1	0	1	3	2	1	1	1	3	0	1	1	0	3	3	3

Table 2: S_6

K	S_6																			
	2	3	4	5	6	8	9	10	12	16	18	20	24	36	48	60	72	120	360	720
3_1	3	2	2	1	6	0	0	0	2	0	2	0	6	0	0	2	0	0	0	0
4_1	3	2	2	1	2	0	0	1	2	0	0	0	2	2	0	0	0	4	4	0
5_1	3	2	2	1	2	0	0	1	0	0	0	0	0	0	0	4	0	4	4	2
5_2	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	2	0	2	2	0
6_1	3	2	2	1	6	0	0	0	0	0	2	2	4	0	0	0	0	0	2	0
6_2	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0
6_3	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	2	0	0	2	0
7_1	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7_2	3	2	2	1	2	0	0	0	2	0	0	2	2	0	0	0	0	0	4	0
7_3	3	2	2	1	2	0	0	0	2	0	0	0	2	0	0	0	0	2	2	0
7_4	3	2	2	1	6	0	0	1	0	0	2	0	4	0	0	2	0	0	4	4
7_5	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	4	0
7_6	3	2	2	1	2	0	0	0	0	0	0	2	0	0	0	0	0	2	0	0
7_7	3	2	2	1	6	0	0	0	0	0	2	0	4	0	0	0	0	4	6	0
8_1	3	2	2	1	2	0	0	0	2	0	0	0	2	0	0	2	0	0	2	0
8_2	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	0	0	2	4	0
8_3	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	4	0	0	4	4
8_4	3	2	2	1	2	0	0	0	2	0	0	0	2	0	0	2	0	2	0	2
8_5	3	2	2	1	6	0	0	0	2	0	2	0	14	0	0	4	0	2	10	0
8_6	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	4	0	2	4	0
8_7	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	2	0	2	4	0
8_8	3	2	2	1	2	0	0	1	0	0	0	2	0	0	0	2	0	2	2	0
8_9	3	2	2	1	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
8_{10}	3	2	2	1	6	0	0	0	2	0	2	0	14	0	0	4	0	2	6	2
8_{11}	3	2	2	1	6	0	0	0	2	0	2	2	6	0	0	2	0	2	0	0

K	S_6																			
	2	3	4	5	6	8	9	10	12	16	18	20	24	36	48	60	72	120	360	720
8_{12}	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8_{13}	3	2	2	1	2	0	0	0	2	0	0	0	2	0	0	0	0	2	2	0
8_{14}	3	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0	0	0	2	4
8_{15}	3	2	2	1	6	0	0	0	2	0	2	2	14	0	0	4	0	2	2	6
8_{16}	3	2	2	1	2	0	0	1	0	0	0	0	0	0	0	2	0	2	16	4
8_{17}	3	2	2	1	2	0	0	0	0	0	0	0	0	2	0	2	0	4	4	0
8_{18}	3	2	2	1	18	0	0	1	10	0	14	0	26	2	0	8	0	8	10	8
8_{19}	3	2	2	1	6	0	0	0	2	0	2	0	14	2	0	2	0	6	6	2
8_{20}	3	2	2	1	6	0	0	0	2	0	2	0	14	0	0	4	0	0	4	0
8_{21}	3	2	2	1	6	0	0	1	2	0	2	0	14	2	0	6	0	6	6	2

Table 3: $D_8, D_{10}, D_{12}, D_{14}, D_{16}$, and D_{18}

K	D_8			D_{10}			D_{12}					D_{14}			D_{16}				D_{18}				
	2	4	8	2	5	10	2	3	4	6	12	2	7	14	2	4	8	16	2	3	6	9	18
3_1	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	0
4_1	3	1	0	1	2	2	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
5_1	3	1	0	1	2	2	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
5_2	3	1	0	1	2	0	3	1	0	1	0	1	3	3	3	1	2	0	1	1	0	3	0
6_1	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	3
6_2	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
6_3	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
7_1	3	1	0	1	2	0	3	1	0	1	0	1	3	3	3	1	2	0	1	1	0	3	0
7_2	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
7_3	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
7_4	3	1	0	1	2	2	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	0
7_5	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
7_6	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
7_7	3	1	0	1	2	0	3	1	0	3	0	1	3	3	3	1	2	0	1	1	1	3	0
8_1	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_2	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_3	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_4	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_5	3	1	0	1	2	0	3	1	0	3	0	1	3	3	3	1	2	0	1	1	1	3	0
8_6	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_7	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_8	3	1	0	1	2	2	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_9	3	1	0	1	2	2	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_{10}	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	3
8_{11}	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	3
8_{12}	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_{13}	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_{14}	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0

K	D_8			D_{10}			D_{12}					D_{14}			D_{16}				D_{18}				
	2	4	8	2	5	10	2	3	4	6	12	2	7	14	2	4	8	16	2	3	6	9	18
8_{15}	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	0
8_{16}	3	1	0	1	2	2	3	1	0	1	0	1	3	3	3	1	2	0	1	1	0	3	0
8_{17}	3	1	0	1	2	0	3	1	0	1	0	1	3	0	3	1	2	0	1	1	0	3	0
8_{18}	3	1	0	1	2	2	3	1	0	9	0	1	3	0	3	1	2	0	1	1	4	3	0
8_{19}	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	0
8_{20}	3	1	0	1	2	0	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	3
8_{21}	3	1	0	1	2	2	3	1	0	3	0	1	3	0	3	1	2	0	1	1	1	3	0

References

[1] S. Cavior, *The subgroups of the dihedral group*, Math. Mag., **48** (1975), 107.

[2] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, 2002.

[3] D. Holt, *Enumerating subgroups of the symmetric group*, Computational group theory and the theory of groups, II, Contemp. Math., **511** (2010), 33-37,

[4] G. Pfeiffer, available at http://schmidt.nuigalway.ie/subgroups/.

[5] T. Kitano and M. Suzuki, *On the number of $SL(2;\mathbb{Z}/p\mathbb{Z})$-representations of knot groups*, J. Knot Theory Ramifications, **21** (2012), 18 pages.

[6] D. Rolfsen, Knots and Links, AMS Chelsea Publishing, 1976.

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