ON EQUIVARIANT PERTURBATIVE INVARIANTS IN 3-DIMENSION
BY MORSE THEORY

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1. INTRODUCTION

Around 1992, Axelrod–Singer and Kontsevich independently developed the method to obtain (mathematical) topological invariants of 3-manifolds by perturbative expansion of Witten’s path integral (Chern–Simons perturbation theory, [1, 5]). The invariant is a series of terms corresponding to Feynman diagrams such that each term is given by integration over the configuration space of a 3-manifold. This is known to be very strong, for example, the expansion around the trivial connection dominates all \( \mathbb{Q} \)-valued Ohtsuki finite type invariants for integral homology 3-spheres ([7]). In this note, we explain about our attempt to construct ‘equivariant invariant’ of 3-manifolds with the first Betti number 1.

Around 2008, Ohtsuki constructed an equivariant refinement of the LMO invariant\(^1\) for 3-manifolds with the first Betti number 1 ([12, 13]), which pioneered a new direction of perturbative invariants of 3-manifolds. Inspired by Ohtsuki’s work, Lescop constructed an equivariant refinement of Chern–Simons perturbation theory for 3-manifolds with the first Betti number 1 for the 2-loop graphs by using a method similar to Marché ([9, 11]). Lescop’s construction is as follows.

Let \( M \) be a closed 3-manifold with \( H_1(M) = \mathbb{Z} \). The equivariant configuration space \( \text{Conf}_{K_2}(M) \) is defined as the set of tuples \( (x_1, x_2, \gamma) \), \( x_1, x_2 \in M \), satisfying the following conditions.

1. \( x_1 \neq x_2 \).
2. \( \gamma \) is the relative bordism class of paths \( c : [0, 1] \to M \) that go from \( x_1 \) to \( x_2 \).

The natural map \( \text{Conf}_{K_2}(M) \to \text{Conf}_2(M) = M \times M \setminus \Delta_M \) that forgets \( \gamma \) is an infinite cyclic covering. Instead of removing the diagonal \( \Delta_M \) in the definition of \( \text{Conf}_2(M) \), consider the blowing-up along \( \Delta_M \), namely replacing \( \Delta_M \) with its normal sphere bundle, to obtain a compactification \( \overline{\text{Conf}}_2(M) \) of \( \text{Conf}_2(M) \). Similarly, by blowing-up along the preimage of \( \Delta_M \) in the space of tuples \( (x_1, x_2, \gamma) \) satisfying only (2) above, we obtain the ‘closure’ \( \overline{\text{Conf}}_{K_2}(M) \) of \( \text{Conf}_{K_2}(M) \).

Lescop defined an invariant of 3-manifolds \( M \) with \( b_1(M) = 1 \) by an equivariant intersection theory in \( \text{Conf}_{K_2}(M) \). The principal term of it is given by the equivariant triple intersection \( \langle Q, Q, Q \rangle Z \) for a fundamental 4-chain

\[
Q \in C_4(\overline{\text{Conf}}_{K_2}(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t, t^{-1}]} \mathbb{Q}(t)
\]

which satisfies a certain boundary condition (equivariant propagator)\(^2\).

\(^1\)LMO invariant is defined combinatorially by using Kontsevich’s link invariant and is known to be universal among finite type invariants of homology 3-spheres.

\(^2\)The Poincaré dual of \( [Q] \in H^4(\text{Conf}_2(M), \partial \overline{\text{Conf}}_2(M); \mathbb{Q}(t)) \) generates \( H^2(\overline{\text{Conf}}_2(M); \mathbb{Q}(t)) \cong \mathbb{Q}(t) \). The meaning of the word ‘propagator’ here differs from the usual one.
Lescop proved the existence of an equivariant propagator by means of homology theoretic arguments. We developed a notion of ‘Z-paths’ (we previously called ‘AL-paths’) in a surface bundle $M$ over $S^1$ and gave an explicit equivariant propagator by the natural map from the moduli space of Z-paths to configuration space ([16]). By using the equivariant propagator, we construct an invariant of fiberwise Morse functions on $M$ ([17]). The construction of the invariant can be applied to a construction of a perturbative isotopy invariant of knots in $M$, which is useful for the study of finite type invariants of knots in $M$ ([18]).

2. MODULI SPACE OF Z-PATHS

We define the moduli space of Z-paths and its ‘closure’.3

2.1. Z-path. Let $M$ be an oriented closed 3-manifold. Assume that $M$ admits a structure of an oriented fiber bundle $\kappa : M \to S^1$. We say that a $C^\infty$ map $f : M \to \mathbb{R}$ is a fiberwise Morse function if the restriction $f_s = f|_{\kappa^{-1}(s)} : \kappa^{-1}(s) \to \mathbb{R}$ is Morse for each $s \in S^1$ (known to exist for every $\kappa$). The totality of the critical points of $f_s$, $s \in S^1$, forms a 1-submanifold of $M$ (closed braid) and we call each component of the 1-submanifold a critical locus. Let $\xi$ be the gradient of $f$ along the fibers, namely, the one whose restriction to each fiber over $s \in S^1$ is $\text{grad } f_s$. Let $\Sigma(\xi)$ denote the union of all critical loci of $\xi$. For a critical locus $p$ of a fiberwise Morse function $f$, the descending/ascending manifold are defined respectively by

$$\mathscr{D}_p(\xi) = \{x \in M | \lim_{t \to -\infty} \Phi_{-\xi}^t(x) \in p\}$$

$$\mathscr{A}_p(\xi) = \{x \in M | \lim_{t \to \infty} \Phi_{-\xi}^t(x) \in p\}$$

where $\Phi_{-\xi} : M \to M$ is the flow of $-\xi$.

Let $\kappa : \widetilde{M} \to \mathbb{R}$ be the pullback of $\kappa$ by the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$.

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\kappa} & S^1 \\
\downarrow{\pi} & & \downarrow \\
M & \xrightarrow{\kappa} & S^1
\end{array}$$

The induced map $\pi : \widetilde{M} \to M$ on the total space is an infinite cyclic covering. The function $\tilde{f} = f \circ \pi : \widetilde{M} \to \mathbb{R}$ is a fiberwise Morse function (for a fiber bundle over $\mathbb{R}$). Let $\tilde{\xi}$ denote the gradient for $\tilde{f}$ along the fibers. By replacing $\xi$ with $\tilde{\xi}$, the critical locus, its descending/ascending manifolds are defined similarly.

We say that an embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is horizontal if $\text{Im} \sigma$ is included in a single fiber of $\kappa$ and say that it is vertical if $\text{Im} \sigma$ is included in a single critical locus of $\tilde{f}$. A horizontal (resp. vertical) embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is descending if $\tilde{f}(\sigma(\mu)) \geq \tilde{f}(\sigma(\nu))$ (resp. $\kappa(\sigma(\mu)) \geq \kappa(\sigma(\nu))$). A horizontal embedding $\sigma : [\mu, \nu] \to \widetilde{M}$ is a flow-line of $\tilde{\xi}$ if for each $t \in (\mu, \nu)$, $d\sigma_t(\frac{\partial}{\partial t})$ is a positive multiple of $(-\tilde{\xi})_{\sigma(t)}$.

Definition 2.1. Let $x, y \in \widetilde{M}$ be such that $\kappa(x) \geq \kappa(y)$. A Z-path from $x$ to $y$ is a sequence $\gamma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ satisfying the following six conditions.

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3The definition in this note differs slightly from that of [16].
(1) For each i, σi is an embedding $[\mu_i, \nu_i] \rightarrow \hat{M}$ ($\mu_i \leq \nu_i$) and it is either horizontal or vertical.
(2) For each i, σi is descending.
(3) If σi is horizontal, then σi is a flow-line of $\tilde{\xi}$. If it is vertical, then $\mu_i < \nu_i$.
(4) $\sigma_i(\mu_i) = x$, $\sigma_n(\nu_n) = y$.
(5) $\sigma_i(\nu_i) = \sigma_{i+1}(\mu_{i+1})$ for $1 \leq i < n$.
(6) If σi is horizontal (resp. vertical) and if $i < n$, then σi+1 is vertical (resp. horizontal).

We say that two Z-paths are equivalent if they are related by piecewise reparametrizations. We call a sequence of paths of the form $\pi \circ \gamma = (\pi \circ \sigma_1, \ldots, \pi \circ \sigma_n)$ for a Z-path γ in $\hat{M}$ a Z-path in M.

Let $\mathcal{M}_2^z(\tilde{\xi})$ be the set of all equivalence classes of Z-paths in $\hat{M}$. This has a natural structure of a noncompact manifold with corners. Let t denote the covering translation of the covering $\pi : \hat{M} \rightarrow M$ that induces the translation $x \mapsto x - 1$ in $\mathbb{R}$. This induces diagonal $\mathbb{Z}$-actions $\gamma \mapsto t^n \gamma$, $(x, y) \mapsto (t^nx, t^ny)$ on $\mathcal{M}_2^z(\tilde{\xi})$ and $\hat{M} \times \hat{M}$. We denote the quotient spaces $\mathcal{M}_2^z(\tilde{\xi})/\mathbb{Z}$ and $(\hat{M} \times \hat{M})/\mathbb{Z}$ respectively by $\mathcal{M}_2^z(\tilde{\xi})_\mathbb{Z}$ and $\hat{M} \times_{\mathbb{Z}} \hat{M}$. We consider another $\mathbb{Z}$-action on the quotient spaces, denoted by $t^n$ by abuse of notation, as

$$t([x \times y]) = [x \times ty].$$

For a gradient $\xi$ along the fiber for a fiberwise Morse function f, let $\tilde{\xi}$ denote the nonsingular vector field $\xi + \text{grad} \kappa$ on $\hat{M}$. Let $\tilde{\xi} : M \rightarrow ST(M)$ ($ST$ denotes the unit tangent bundle) be the section given by $-\tilde{\xi}/\|\tilde{\xi}\|$.

**Theorem 2.2** ([16]). Let $\Sigma$ be an oriented connected closed surface and let $M$ be the mapping torus of an orientation preserving diffeomorphism $\varphi : \Sigma \rightarrow \Sigma$. Let $\tilde{\Delta}_M \subset \hat{M} \times_{\mathbb{Z}} \hat{M}$ be the preimage of the diagonal $\Delta_M$ of $M \times M$.

1. There is a natural 'closure' $\mathcal{H}_2^\mathbb{Z}(\tilde{\xi})_Z$ of $\mathcal{M}_2^z(\tilde{\xi})_Z$ that is a countable union of compact manifolds with corners.
2. Suppose that $\kappa$ induces an isomorphism $H_1(M)/\text{Torsion} \cong H_1(S^1)$. Let $\tilde{b} : \mathcal{H}_2^\mathbb{Z}(\tilde{\xi})_Z \rightarrow \hat{M} \times_{\mathbb{Z}} \hat{M}$ be the map that assigns the endpoints. Let $\tilde{b} : \mathcal{H}_2^\mathbb{Z}(\tilde{\xi})_Z \rightarrow \hat{M} \times_{\mathbb{Z}} \hat{M}$ be the blow-up of $\mathcal{H}_2^\mathbb{Z}(\tilde{\xi})_Z$ along $\tilde{b}^{-1}(\Delta_M)$. Then $\tilde{b}$ induces a map

$$\tilde{b} : \mathcal{H}_2^\mathbb{Z}(\tilde{\xi})_Z \rightarrow \text{Conf}_{K_2}(M).$$
that represents a 4-dimensional $\mathbb{Q}(t)$-chain $Q(\xi)$ of $\overline{\text{Conf}}_{K_{2}}(M)$. Moreover, the following identity in $H_{3}(\partial\overline{\text{Conf}}_{K_{2}}(M); \mathbb{Q}) \otimes_{\mathbb{Q}[t, t^{-1}]} \mathbb{Q}(t)$ holds.

$$[\partial Q(\xi)] = [s_{\xi}(M)] + \frac{t\zeta_{\varphi}}{\zeta_{\varphi}}[ST(M)|_{K}],$$

where $\zeta_{\varphi}$ is the Lefschetz zeta function for $\varphi$ and $K$ is a knot such that $\kappa_{*}(K)$ is the positive generator of $H_{1}(S^{1})$. Furthermore, there is a product $P(t)$ of cyclotomic polynomials such that $P(t)\Delta(M)Q(\xi)$ is a $\mathbb{Q}[t, t^{-1}]$-chain ($\Delta(M)$ is the Alexander polynomial of $M$).

2.2. Closure of the moduli space of Z-paths. We define the space $\mathcal{M}_{2}(\xi)$ of horizontal paths in $\tilde{M}$ by

$$\mathcal{M}_{2}(\xi) = \{(x, y) \in \tilde{M} \times \tilde{M}; \kappa(x) = \kappa(y), y = \Phi_{-\xi}^{t}(x) \text{ for some } t > 0\}.$$  

Let $b : \mathcal{M}_{2}(\xi) \to \tilde{M} \times \tilde{M}$ denote the inclusion map. For a continuous parameter $s \in S$ such as real numbers, we denote the sum $\bigcup_{s \in S} V_{s}$ by $\int_{s \in S} V_{s}$ and if the parameter is at most countable, then we denote it by $\sum_{s \in S} V_{s}$ or $V_{s_{1}} + V_{s_{2}} + \cdots$ etc.

For a generic $\xi$, the intersection $\mathcal{D}_{p}(\xi) \cap \mathcal{A}_{q}(\xi)$ is transversal and hence is a smooth manifold. There is a free $\mathbb{R}$-action on $\mathcal{D}_{p}(\xi) \cap \mathcal{A}_{q}(\xi)$ by $x \mapsto \Phi_{-\xi}^{T}(x)$ ($T \in \mathbb{R}$). We put $\mathcal{M}_{pq}(\xi) = (\mathcal{D}_{p}(\xi) \cap \mathcal{A}_{q}(\xi))/\mathbb{R}$.

**Proposition 2.3.** There is a natural closure $\overline{\mathcal{M}}_{2}(\xi)$ of $\mathcal{M}_{2}(\xi)$ and the extension $\overline{b} : \overline{\mathcal{M}}_{2}(\xi) \to \tilde{M} \times \tilde{M}$ of $b$ such that for a generic $\xi$ the following hold ($\Delta_{S} \subseteq S \times S$ denotes the diagonal for any set $S$).

1. $\overline{\mathcal{M}}_{2}(\xi) - \overline{b}^{-1}(\Delta_{\tilde{M}})$ is a manifold with corners.
2. $\overline{b}$ induces a diffeomorphism $\text{Int} \overline{\mathcal{M}}_{2}(\xi) \to \mathcal{M}_{2}(\xi)$.
3. The codimension $r$ stratum of $\overline{\mathcal{M}}_{2}(\xi) - \overline{b}^{-1}(\Delta_{\tilde{M}})$ corresponds to broken flow-lines that are broken $r$ times at critical points. The codimension $r$ stratum of $\overline{\mathcal{M}}_{2}(\xi) - \overline{b}^{-1}(\Delta_{\tilde{M}})$ for $r \geq 1$ is canonically diffeomorphic to

$$\int_{s \in \mathbb{R}} \sum_{q_{1} \in \Sigma(\xi)} \mathcal{A}_{q_{1}}(\xi_{s}) \times \mathcal{D}_{q_{1}}(\xi_{s}) - \sum_{q_{1} \in \Sigma(\xi)} \Delta_{q_{1}} \quad (r = 1)$$

$$\int_{s \in \mathbb{R}} \sum_{q_{1}, \ldots, q_{r} \in \Sigma(\xi)} \mathcal{A}_{q_{1}}(\xi_{s}) \times \mathcal{M}_{q_{1}q_{2}}(\xi_{s}) \times \cdots \times \mathcal{M}_{q_{r-1}q_{r}}(\xi_{s}) \times \mathcal{D}_{q_{r}}(\xi_{s}) \quad (r \geq 2)$$

The formula for the codimension $r$ stratum ($r \geq 2$) in Proposition 2.3 can be rewritten as follows.

$$\int_{s \in \mathbb{R}} X(s) \times \prod_{r-1} \Omega(s) \times \cdots \times \Omega(s) \times Y(s).$$
Here, if $\Sigma(\tilde{\xi}) = \{p_1, p_2, \ldots, p_N\}$, then
\[
X(s) = (\mathcal{A}_{p_1}(\tilde{\xi}) \mathcal{A}_{p_2}(\tilde{\xi}) \cdots \mathcal{A}_{p_N}(\tilde{\xi}))
\]
\[
Y(s) = (\mathcal{D}_{p_1}(\tilde{\xi}) \mathcal{D}_{p_2}(\tilde{\xi}) \cdots \mathcal{D}_{p_N}(\tilde{\xi}))
\]
\[
\Omega(s) = (\mathcal{M}_{p_Np_1}(\tilde{\xi}) \mathcal{M}_{p_3p_1}(\tilde{\xi}) \cdots \mathcal{M}_{p_3p_N}(\tilde{\xi}) \cdots \mathcal{M}_{p_3p_N}(\tilde{\xi}) \cdots \mathcal{M}_{p_1pN}(\tilde{\xi}))
\]
and the direct product of matrices is defined by replacing multiplications and sums with direct products and disjoint unions, respectively.

**Proposition 2.4.** Let $p$ be a critical locus of $\tilde{\xi}$ and let $\overline{\mathcal{D}}_{p}(\tilde{\xi}) = \overline{b}^{-1}(p \times M)$, $\overline{\mathcal{A}}_{p}(\tilde{\xi}) = \overline{b}^{-1}(M \times p)$. For a generic $\tilde{\xi}$, the following are satisfied.

1. $\overline{\mathcal{D}}_{p}(\tilde{\xi})$ (resp. $\overline{\mathcal{A}}_{p}(\tilde{\xi})$) is a manifold with corners.
2. $\overline{b}$ induces a diffeomorphism $\text{Int} \overline{\mathcal{D}}_{p}(\tilde{\xi}) \to \mathcal{D}_{p}(\tilde{\xi})$ (resp. $\text{Int} \overline{\mathcal{A}}_{p}(\tilde{\xi}) \to \mathcal{A}_{p}(\tilde{\xi})$).
3. The codimension $r$ stratum of $\overline{\Omega} = ((1-\delta_{ij})\overline{\mathcal{M}}_{p_ip_j}(\tilde{\xi}))$ for $r \geq 1$ is canonically diffeomorphic to
\[
\int_{s \in \mathbb{R}} \Omega(s) \times \cdots \times \Omega(s) \times Y(s) \quad (\text{resp. } \int_{s \in \mathbb{R}} X(s) \times \Omega(s) \times \cdots \times \Omega(s))
\]

**Proposition 2.5.** Let $p, q$ be critical loci of $\tilde{\xi}$ and let $\overline{\mathcal{M}}_{pq}(\tilde{\xi}) = \overline{b}^{-1}(p \times q)$. For a generic $\tilde{\xi}$, the following hold.

1. $\overline{\mathcal{M}}_{pq}(\tilde{\xi})$ is a manifold with corners.
2. There is a natural diffeomorphism $\text{Int} \overline{\mathcal{M}}_{pq}(\tilde{\xi}) \to \mathcal{M}_{pq}(\tilde{\xi})$.
3. The codimension $r$ stratum of $\overline{\Omega} = ((1-\delta_{ij})\overline{\mathcal{M}}_{p_ip_j}(\tilde{\xi}))$ for $r \geq 1$ is canonically diffeomorphic to
\[
\int_{s \in \mathbb{R}} \Omega(s) \times \cdots \times \Omega(s) \times \Omega(s)
\]

A fiberwise space over a space $B$ is a pair of a space $E$ and a continuous map $\phi : E \to B$. A fiber over a point $s \in B$ is $E(s) = \phi^{-1}(s)$ ([2]). For two fiberwise spaces $E_1 = (E_1, \phi_1)$ and $E_2 = (E_2, \phi_2)$ over $B$, a fiberwise product $E_1 \times_B E_2$ is defined as the following subspace of $E_1 \times E_2$:
\[
E_1 \times_B E_2 = \int_{s \in B} E_1(s) \times E_2(s).
\]
Namely, $E_1 \times_B E_2$ is the pullback of $E_1 \rightarrow B \rightarrow E_2$. 
For a sequence \( A_i = (A_i, \phi_i) \), \( \phi_i : A_i \to \mathbb{R} \) \((i = 1, 2, \ldots, n)\) of fiberwise spaces over \( \mathbb{R} \), we define its iterated integrals as
\[
\int_{\mathbb{R}} A_1 A_2 \cdots A_n = \int_{s_1 > s_2 > \cdots > s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n)
\]
\[
= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 > \cdots > s_n\}),
\]
\[
\int_{\mathbb{R}} A_1 A_2 \cdots A_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n} A_1(s_1) \times A_2(s_2) \times \cdots \times A_n(s_n)
\]
\[
= (\phi_1 \times \cdots \times \phi_n)^{-1}(\{(s_1, \ldots, s_n) \in \mathbb{R}^n \mid s_1 \geq \cdots \geq s_n\}).
\]

For a matrix \( P = (A_{ij}) \) of fiberwise spaces over \( \mathbb{R} \), we define a fiber of \( s \in \mathbb{R} \) by \( P(s) = (A_{ij}(s)) \). Then iterated integrals for matrices of fiberwise spaces over \( \mathbb{R} \) can be defined by similar formulas as above.

We define matrices \( X, Y, \Omega \) of fiberwise spaces over \( \mathbb{R} \) by
\[
X = (\mathcal{A}_{p_1}(\tilde{\xi}) \mathcal{A}_{p_2}(\tilde{\xi}) \cdots \mathcal{A}_{p_n}(\tilde{\xi})), \quad Y = (D_{p_1}(\tilde{\xi}) D_{p_2}(\tilde{\xi}) \cdots D_{p_n}(\tilde{\xi})),
\]
\[
\Omega = ((1-\delta_{ij})\mathcal{M}_{p_i p_j}(\tilde{\xi}))_{1 \leq i, j \leq N}.
\]

Then the space of \( Z \)-paths in \( \tilde{M} \) is rewritten by means of the iterated integrals as follows.
\[
\mathcal{M}_2(\tilde{\xi}) = \mathcal{M}_2(\tilde{\xi}) + \int_{\mathbb{R}} X^t Y + \int_{\mathbb{R}} X \Omega^t Y + \int_{\mathbb{R}} X \Omega \Omega^t Y + \cdots
\]

We would like to define the 'closure' of this space.

**Lemma 2.6.** For a generic \( \tilde{\xi} \), the space \( \int_{\mathbb{R}} X \Omega \cdots \Omega^t Y \) is the disjoint union of finitely many manifolds with corners, and the closure of its codimension 1 stratum is given by the following formula.
\[
\int_{\mathbb{R}} (\partial X) \Omega \cdots \Omega^t Y + \sum_{i=1}^{\infty} \int_{\mathbb{R}} X \Omega \cdots \Omega (\partial \Omega) \Omega \cdots \Omega^t Y + \int_{\mathbb{R}} X \Omega \cdots \Omega^t Y
\]
\[
+ \int_{\mathbb{R}} (X \times R \Omega) \Omega \cdots \Omega^t Y + \sum_{i=1}^{\infty} \int_{\mathbb{R}} X \Omega \cdots \Omega (\Omega \times R \Omega) \Omega \cdots \Omega^t Y + \int_{\mathbb{R}} X \Omega \cdots \Omega (\Omega \times R \Omega)^t Y.
\]

For \( n \geq 0 \), let \( S_n \) (resp. \( T_n \)) denote the first line (resp. the second line) of the formula in Lemma 2.6.

**Lemma 2.7.** There is a natural stratification preserving diffeomorphisms
\[
\partial X \cong X \times \mathbb{R} \Omega, \quad \partial^t Y \cong \Omega \times \mathbb{R}^t Y,
\]
\[
\partial \Omega \cong \Omega \times \mathbb{R} \Omega, \quad \partial \mathcal{M}_2(\tilde{\xi}) \cong \Delta_{\tilde{M}} + X \times \mathbb{R}^t Y.
\]

These induce, for \( n \geq 0 \), a stratification preserving diffeomorphism
\[
S_n \cong T_{n+1}.
\]

Let \( S_{-1} \subset \partial \mathcal{M}_2(\tilde{\xi}) \) be the face that corresponds to \( X \times \mathbb{R}^t Y \) by the diffeomorphism of Lemma 2.7.
Definition 2.8.

\[ \overline{\mathcal{M}}_2^z(\tilde{\xi}) = \left[ \mathcal{M}_2(\tilde{\xi}) + \int_{\mathbb{R}} X^t Y + \int_{\mathbb{R}} X \Omega Y + \cdots \right] / \sim \]

Here, for each \( n \geq 0 \), we identify \( S_{n-1} \) with \( T_n \) by the diffeomorphism of Lemma 2.7. \( \mathbb{Z} \) acts on \( \overline{\mathcal{M}}_2^z(\tilde{\xi}) \) by \((x_1, x_2, \ldots, x_n) \mapsto (tx_1, tx_2, \ldots, tx_n)\). We put \( \overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}} = \overline{\mathcal{M}}_2^z(\tilde{\xi}) / \mathbb{Z} \).

Outline of the proof of Theorem 2.2. By fixing orientations on the manifold pieces in the stratified space \( \overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}} \), the map \( \overline{b} : \overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}} \to \tilde{M} \times_{\mathbb{Z}} \tilde{M} \) represents a \( \mathbb{Q}(t) \)-chain of \( \tilde{M} \times_{\mathbb{Z}} \tilde{M} \). (The proof that the coefficients are rational functions is an analogue of the proof of the rationality of Novikov complexes by Pajitnov ([14, 15]).) By Lemmas 2.6, 2.7 and by checking the orientations on the gluing parts, it turns out that the boundary of \( \overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}} \) concentrates on the lift \( \tilde{\Delta}_M \) of the diagonal \( \Delta_M \). Hence the boundary of \( B\ell_{\overline{b}^{-1}(\tilde{\Delta}_M)}(\overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}}) \) consists of \( \mathbb{Z} \)-paths with endpoints agree (in \( \tilde{M} \)) and of closed \( \mathbb{Z} \)-paths (in \( M \)). One sees that in the homology class of the boundary of \( B\ell_{\overline{b}^{-1}(\tilde{\Delta}_M)}(\overline{\mathcal{M}}_2^z(\tilde{\xi})_{\mathbb{Z}}) \), the part for closed \( \mathbb{Z} \)-paths corresponds to the logarithmic derivative of the Lefschetz zeta function.

3. Perturbation theory for \( \hat{\Lambda} \)-coefficients

Put \( \Lambda = \mathbb{Q}[t, t^{-1}] \), \( \hat{\Lambda} = \mathbb{Q}(t) \). Since a recipe for the perturbation theory for Lie algebra local coefficient systems is given by Axelrod-Singer, Kontsevich ([1, 5]), it is expected that one can obtain a perturbative invariant with \( \hat{\Lambda} \)-coefficients if there is an appropriate propagator with \( \hat{\Lambda} \)-coefficients. The \( \mathbb{Q}(t) \)-chain given by the moduli space of \( \mathbb{Z} \)-paths can be considered as an appropriate equivariant propagator and we use it.

3.1. \( \hat{\Lambda} \)-colored graph. We call a finite connected graph with edges oriented a graph. A vertex-orientation of a graph is an assignment of cyclic order of edges incident to each vertex. For a vertex-oriented graph \( \Gamma \), a \( \hat{\Lambda} \)-coloring of \( \Gamma \) is a mapping \( \phi : \text{Edges}(\Gamma) \to \hat{\Lambda} \).

Definition 3.1 (Garoufalidis-Rozansky [4]).

\[
\mathcal{A}_n(\hat{\Lambda}) = \frac{\text{span}_{\mathbb{Q}}\{ \Gamma : 3\text{-valent, } 2n \text{ vertices, } \hat{\Lambda}\text{-colored vertex-oriented graphs} \}}{\text{AS, IHX, Orientation reversal, Linearity, Holonomy}}
\]

\[
\begin{align*}
p(t) &= p(t^{-1}) & p + \alpha q &= p + \alpha(q) \\
\text{Orientation} & \quad \text{Reversal} & \quad \text{Linearity} & \quad \text{Holonomy}
\end{align*}
\]
3.2. Equivariant configuration space and equivariant intersection. Let $\kappa : M \to S^1$ be a fiber bundle and let $\Theta = 1 \xrightarrow{(1)} 2 \xrightarrow{(2)} 3$. We define $M^\Theta$ as

$$M^\Theta := \{(x_1, x_2; \gamma_1, \gamma_2, \gamma_3) \mid x_1, x_2 \in M, $$

$$\gamma_i : \text{homotopy class of } c_i : [0, 1] \to S^1 \text{ such that } c_i(0) = \kappa(x_1), c_i(1) = \kappa(x_2)\}.$$  

When $H_1(M) = \mathbb{Z}_n$, the homotopy class $\gamma_i$ of $c_i$ is the same thing as the relative bordism class of the lift $\tilde{c}_i : [0, 1] \to M$ of $c_i$. The equivariant configuration space $\overline{Conf}_\Theta(M)$ for $\Theta$ is defined by

$$\overline{Conf}_\Theta(M) := Bl(M^\Theta, \text{preimage of } \Delta_M),$$

where $Bl(X, A)$ is the blow-up of a (real) manifold $X$ along a submanifold $A$. The projection $\overline{Conf}_\Theta(M) \to \overline{Conf}_2(M)$ is a $\mathbb{Z}^3$-covering and we have $\pi_0(\overline{Conf}_\Theta(M)) \approx H^1(\Theta; \mathbb{Z}) = [\Theta, S^1]$.

By extending the intersections of chains by $\hat{\Lambda}$-linearity, we define the multilinear form

$$Q_1 \otimes Q_2 \otimes Q_3 \mapsto \langle Q_1, Q_2, Q_3 \rangle_{\Theta} \in C_0(\overline{Conf}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda \otimes^3} \hat{\Lambda}^{S^3}$$

($\Lambda^{S^3} = \mathbb{Q}[t_1^\pm 1, t_2^\pm 1, t_3^\pm 1]$, $\hat{\Lambda}^{S^3} = \mathbb{Q}(t_1) \otimes_{\mathbb{Q}} \mathbb{Q}(t_2) \otimes_{\mathbb{Q}} \mathbb{Q}(t_3)$) for ‘generic’ 4-dimensional $\hat{\Lambda}$-chains $Q_1, Q_2, Q_3$ in $\overline{Conf}_K(M)$.

We define the trace $\text{Tr}_\Theta : \hat{\Lambda}^{S^3} \to \mathfrak{A}_1(\hat{\Lambda})$ for $\hat{\Lambda}$-colored graphs by

$$\text{Tr}_\Theta(F_1(t_1) \otimes F_2(t_2) \otimes F_3(t_3)) = \left[ \begin{array}{c} \tilde{F}_1(t) \\ \tilde{F}_2(t) \\ \tilde{F}_3(t) \end{array} \right].$$

This induces the following map.

$$\text{Tr}_\Theta : H_0(\overline{Conf}_\Theta(M); \mathbb{Q}) \otimes_{\Lambda \otimes^3} \hat{\Lambda}^{S^3} \to H_0(\overline{Conf}_2(M); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathfrak{A}_1(\hat{\Lambda}) = \mathfrak{A}_1(\hat{\Lambda}).$$

Similarly, for a 3-valent graph $\Gamma$ with $2n$ vertices (3n edges), we obtain

$$Q_1 \otimes Q_2 \otimes \cdots \otimes Q_{3n} \mapsto \langle Q_1, Q_2, \ldots, Q_{3n} \rangle_{\Gamma} \in C_0(\overline{Conf}_\Gamma(M); \mathbb{Q}) \otimes_{\Lambda \otimes^{3n}} \hat{\Lambda}^{S^{3n}}$$

$$\Rightarrow \text{Tr}_\Gamma \langle Q_1, Q_2, \ldots, Q_{3n} \rangle_{\Gamma} \in H_0(\overline{Conf}_{2n}(M); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathfrak{A}_n(\hat{\Lambda}) = \mathfrak{A}_n(\hat{\Lambda}).$$

**Definition 3.2.** Let $\kappa_i : M \to S^1$ be an oriented surface bundle such that $\kappa_i \simeq \kappa$. Let $f : M \to \mathbb{R}$ be an oriented fiberwise Morse function (w.r.t. $\kappa_i$), let $\xi_i$ be the gradient for $\kappa_i$ along the fibers ($i = 1, 2, \ldots, 3n$). We define $Z_n$ as follows.

$$Z_n := \sum_{\Gamma} \text{Tr}_\Gamma \langle Q(\xi_1), Q(\xi_2), \ldots, Q(\xi_{3n}) \rangle_{\Gamma} \in \mathfrak{A}_n(\hat{\Lambda}).$$

The sum is over all (labeled) 3-valent graphs with 2n vertices.
$Z$-graph

**Theorem 3.3** ([17]).

\[ \hat{Z}_n = Z_n - Z_n^{\text{anomaly}}(\vec{\rho}_W) \in \mathcal{A}_n(\hat{\Lambda}) \]

is an invariant of \((M, s, [\kappa], [f])\). \((Z_n^{\text{anomaly}}(\vec{\rho}_W)\) is a term obtained by counting affine graphs in a rank 3 vector bundle over some compact 4-manifold \(W\) such that \(\partial W = M\). Here,

1. \(s\) is a spin structure on \(M\).
2. \([\kappa] \in H^1(M)\) is the homotopy class of \(\kappa\).
3. \([f]\) is the 'concordance class' of an oriented fiberwise Morse function \(f : M \to \mathbb{R}\). (Oriented fiberwise Morse functions \(f_0\) and \(f_1\) are concordant if there is a generic homotopy \(F : M \times [0, 1] \to \mathbb{R}\) between \(f_0\) and \(f_1\) such that for each birth-death locus, its projection to \(S^1 \times [0, 1]\) is a simple closed curve and is not nullhomotopic.)

To get an invariant of \((M, [\kappa])\), one must show that \(\hat{Z}_n\) does not depend on the choice of concordance class of oriented fiberwise Morse functions, namely, that \(\hat{Z}_n\) is invariant under a generic homotopy of oriented fiberwise Morse functions. However, as suggested by the definition of concordance, the topology of the moduli space of Z-paths may change if there is a birth-death locus whose projection on \(S^1 \times [0, 1]\) is nullhomotopic. We guess that the restriction of the homotopy to concordances might be too strong.

Though, this is sufficient to study finite type isotopy invariants of knots in a 3-manifold ([18]). Thanks to the definition of \(\hat{Z}_n\) by Z-paths, Theorem 3.3 can be proved by a standard argument (constructing a cobordism between moduli spaces on the endpoints) without difficulty.

**3.3. Z-graph.** Now we explain that \(Z_n\) can be defined by counting certain graphs. In the following, we only consider the graph \(\Gamma = \Theta\) for simplicity.

**Definition 3.4.** Put \(\Sigma = \kappa^{-1}(0)\). For \((a_1, a_2, a_3) \in \mathbb{Z}^3\), we define

\[ \mathcal{M}_{\Theta(a_1, a_2, a_3)}^\mathbb{Z}(\Sigma; \xi_1, \xi_2, \xi_3) \]

as the set of maps \(I : \Theta \to M\) such that

1. \(i\)-th edge is a Z-path for \(\xi_i\).
2. \(#(i\text{-th edge of } I) \cap \Sigma = a_i\) (count with signs)

We call such a map \(I : \Theta \to M\) a Z-graph.

This definition is an analogue of the flow-graphs considered in Fukaya's Morse homotopy theory [3]. The following lemma can be proved by a transversality argument as in [3].
Lemma 3.5. For a generic $\kappa_i$, $\xi_i$ ($i = 1, 2, 3$), the moduli space $\mathcal{M}_{\Theta(a_1,a_2,a_3)}^{\mathbb{Z}}(\Sigma; \xi_1, \xi_2, \xi_3)$ is a compact oriented 0-dimensional manifold $(\forall (a_1,a_2,a_3) \in \mathbb{Z}^3)$.

Proposition 3.6. Choose $\kappa_i$, $\xi_i$ ($i = 1, 2, 3$) generically as in the Lemma. Put

$$F_\Theta := \sum_{(a_1,a_2,a_3) \in \mathbb{Z}^3} \# \mathcal{M}_{\Theta(a_1,a_2,a_3)}^{\mathbb{Z}}(\Sigma; \xi_1, \xi_2, \xi_3) t_1^{a_1} t_2^{a_2} t_3^{a_3}.$$

Then there exist a polynomial $P(t_1,t_2,t_3) \in \mathbb{Q}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ and a product $C(t) \in \Lambda$ of cyclotomic polynomials such that

$$F_\Theta = \frac{P(t_1,t_2,t_3)}{C(t_1)C(t_2)C(t_3)\Delta(t_1)\Delta(t_2)\Delta(t_3)} = \langle Q(\tilde{\xi}_1), Q(\tilde{\xi}_2), Q(\tilde{\xi}_3) \rangle_\Theta$$

holds. ($\Delta(t)$ is the Alexander polynomial of $M$)

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