数理解析研究所講究録
第 1996 巻 2016 年 23-33

On m-complex symmetric operators

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Abstract

In this paper, we provide several spectral and local spectral properties of m-complex symmetric operators. Moreover, we study properties of nilpotent perturbations of m-complex symmetric operators. Finally, we discuss the structures of m-complex symmetric operators.

1 Introduction

The results in this paper will be appeared in other journals. Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$. A *conjugation* on $\mathcal{H}$ is an antilinear operator $C:\mathcal{H}\rightarrow\mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y\in\mathcal{H}$ and $C^{2}=I$. For any conjugation C, there is an orthonormal basis $\{e_{n}\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_{n} = e_{n}$ for all n (see [12] for more details). Note that $(CTC)^{k} = CT^{k}C$ and $(CTC)^{*} = CT^{*}C$ for every positive integer k, and $\|C\| = 1$. An operator $T\in\mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = CT^{*}C$. In this case, we say that T is complex symmetric with conjugation C. This terminology is due to the fact that T is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an l^{2}-space of the appropriate dimension (see [12]). The class of complex symmetric operators includes all normal operators, Hankel matrices, finite Toeplitz matrices, all truncated Toeplitz operators, and some Volterra integration operators. We refer the reader to [12]-[15] for more details.

[0]2010 Mathematics Subject Classification; Primary 47A11, Secondary 47B25.
Keywords; m-Complex symmetric operator; Decomposable; Single-valued extension property.
This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST)(2009-0093125). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012R1A1A3006841).

In 1970, J. W. Helton [16] initiated the study of operators $T \in \mathcal{L}(\mathcal{H})$ which satisfy an identity of the form;

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}T^{m-j}=0. \tag{1}$$

In light of complex symmetric operators, using the identity (1), we define m-complex symmetric operators as follows; an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C=0$$

for some positive integer m. In this case, we say that T is m-complex symmetric with conjugation C. Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*j}CT^{m-j}C$. Then T is an m-complex symmetric operator with conjugation C if and only if $\Delta_m(T) = 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be a *strict m-complex symmetric operator* if T is an m-complex symmetric operator but it is not an $(m-1)$-complex symmetric operator.

Note that

$$T^*\Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T). \tag{2}$$

Hence, if T is m-complex symmetric with conjugation C, then T is n-complex symmetric with conjugation C for all $n \geq m$. It is obvious that a 1-complex symmetric operator is complex symmetric.

2 Preliminaries

If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{su}(T)$, $\Gamma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, $\sigma_b(T)$, and $\sigma_w(T)$ for the spectrum, the surjective spectrum, the compression spectrum, the point spectrum, the approximate point spectrum, the essential spectrum, the left essential spectrum, the right essential spectrum, Browder spectrum, and Weyl spectrum of T, respectively.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset G of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function f on G such that $(T-\lambda)f(\lambda) \equiv 0$ on G, we have $f(\lambda) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for a vector $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of T at x is defined as the union of every open subset G of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-\lambda)f(\lambda) \equiv x$ on G. The *local spectrum* of T at x is given by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for a subset F of $\mathbb{C}$. An operator

$T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on G such that $(T-\lambda)f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G, we get that $f_n(\lambda)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are T-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that

$$\mathcal{H} = \mathcal{X} + \mathcal{Y}, \sigma(T|_{\mathcal{X}}) \subset \overline{U}, \text{ and } \sigma(T|_{\mathcal{Y}}) \subset \overline{V}.$$

It is well-known that

$$\text{Decomposable} \Rightarrow \text{Bishop's property } (\beta) \Rightarrow \text{SVEP}.$$

In general, the converse implications do not hold (see [20] for more details).

3 Examples

In this section, we consider several examples of m-complex symmetric operators with conjugation C. It is well-known that if T is nilpotent of order 2, then T is complex symmetric by [11, Theorem 5]. But if T is nilpotent of order k with $k > 2$, then T may not be complex symmetric.

Example 3.1 Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. If T is nilpotent of order $k > 2$ and $T^* \neq CTC$, then T is a $(2k-1)$-complex symmetric operator with conjugation C.

Example 3.2 Let C be a conjugation on $\mathcal{H}$. If $R \in \mathcal{L}(\mathcal{H})$ is a self-adjoint operator and $R = CRC$ where $RQ = QR$, $Q^* \neq CQC$ and $Q^k = 0$ for some $k > 2$, then an operator $R + Q$ is $(2k-1)$-complex symmetric with conjugation C.

Example 3.3 Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$. If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $T^3 = 0$ and T is a not complex symmetric operator by [15]. Hence T is a 5-complex symmetric operator with conjugation C. However, since $T^3 = 0$ and $T^2 \neq 0$, it follows that

$$\sum_{j=0}^{4} (-1)^{4-j} \binom{4}{j} T^{*j} C T^{4-j} C = 6T^{*2}CT^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 24 \end{pmatrix} \neq 0.$$

So it is not a 4-complex symmetric operator.

4 m-complex symmetric operators

In this section, we provide spectral properties of m-complex symmetric operators. Recall that two vectors x and y are said to be C-orthogonal if $\langle Cx, y\rangle = 0$.

Theorem 4.1 *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. Then the following statements hold.*
(i) *If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*.*
(ii) *Eigenvectors of T corresponding to distinct eigenvalues are C-orthogonal.*
(iii) *If $\lambda \in \sigma_{ap}(T)$, then $\overline{\lambda} \in \sigma_{ap}(T^*)$.*
(iv) *Let $\lambda \neq \mu$. If $\{x_n\}$, $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$, then $\lim_{n\to\infty}\langle Cx_n, y_n\rangle = 0$.*

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *isoloid* if for any $\lambda \in \operatorname{iso}\sigma(T)$, $\lambda \in \mathbb{C}$ is an eigenvalue of T, where $\operatorname{iso}\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. For $D \subset \mathbb{C}$, we denote $D^* = \{\bar{z} : z \in D\}$.

Corollary 4.2 *Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric with conjugation C. If T is isoloid, then T^* is also isoloid.*

Theorem 4.3 *If $\{T_k\}$ is a sequence of m-complex symmetric operators with conjugation C such that $\lim_{k\to\infty}\|T_k - T\| = 0$, then T is also m-complex symmetric with conjugation C.*

We provide equivalent statements for an m-complex symmetric operator.

Proposition 4.4 *Let $T \in \mathcal{L}(\mathcal{H})$ be invertible and let C be a conjugation on $\mathcal{H}$. Then the following assertions hold.*
(i) *If $T^{*j}CT^{m-j}C = CT^{m-j}CT^{*j}$ for $j = 0, 1, \cdots, m$, then T is m-complex symmetric with conjugation C if and only if $CT^{*-1}C$ is m-complex symmetric with conjugation C.*
(ii) *T is m-complex symmetric with conjugation C if and only if T^{-1} is m-complex symmetric with conjugation C.*

Theorem 4.5 *If $T \in \mathcal{L}(\mathcal{H})$ is an m-complex symmetric operator with conjugation C, then T^n is also m-complex symmetric with conjugation C for some positive integer n.*

Corollary 4.6 *Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric with conjugation C. If $\lim_{n\to\infty}\|T^n x\|^{\frac{1}{n}} = 0$, then $\lim_{n\to\infty}\|T^{*mn}Cx\|^{\frac{1}{n}} = 0$.*

Example 4.7 Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_3}, \overline{z_2}, \overline{z_1})$ on $\mathbb{C}^3$. If $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$, then $T^* \neq CTC = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $T^{*2} = CT^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$. Hence T^2 is a 1-complex symmetric operator but T is not a 1-complex symmetric operator with conjugation C.

We next study the local spectral properties of m-complex symmetric operators.

Theorem 4.8 *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. Then T^* has the property (β) if and only if T is decomposable.*

An operator $X \in \mathcal{L}(\mathcal{H})$ is *quasiaffinity* if it has trivial kernel and dense range and $S \in \mathcal{L}(\mathcal{H})$ is *quasiaffine transform* of T if there is a quasiaffinity X such that $XS = TX$. Two operators S and T are *quasisimilar* if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$. A closed subspace $\mathcal{M} \subset \mathcal{H}$ is *invariant* for T if $T\mathcal{M} \subset \mathcal{M}$, and *hyperinvariant* for T if it is invariant for every operator in $\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : TS = ST\}$ of T.

Corollary 4.9 *Let $T \in \mathcal{L}(\mathcal{H})$ be m-complex symmetric operators. Then the following statements hold.*
(i) *If T^* is hyponormal, i.e., $TT^* \geq T^*T$, then T is decomposable.*
(ii) *If T^* has the property (β) and $\sigma(T)$ has nonempty interior, then T has a nontrivial invariant subspace.*
(iii) *If $\sigma(T)$ is not singleton and $S \in \mathcal{L}(\mathcal{H})$ is quasisimilar to T, then S has a nontrivial hyperinvariant subspace.*

Theorem 4.10 *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. If T^* has the single-valued extension property, then T has the single-valued extension property. Moreover, in this case, $\sigma_{T^*}(x) \subset \sigma_T(Cx)^*$ for all $x \in \mathcal{H}$. Furthermore, $CH_T(F) \subset H_{T^*}(F^*)$ where $F^* := \{\overline{z} : z \in F\}$ for any set F in $\mathbb{C}$.*

Assume that T has the single-valued extension property. If there exists a constant k such that for every $x, y \in \mathcal{H}$ with $\sigma_T(x) \cap \sigma_T(y) = \emptyset$ we have

$$\| x \| \leq k \| x + y \|$$

where k is independent of x and y, we say that an operator T satisfies *Dunford's boundedness condition* (B).

Corollary 4.11 *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-complex symmetric operator with conjugation C. If T^* has the single-valued extension property and the Dunford's boundedness condition (B), then T also has the Dunford's boundedness condition (B).*

Next, we investigate properties of nilpotent perturbations of m-complex symmetric operators.

Proposition 4.12 *Let $T \in \mathcal{L}(\mathcal{H})$, let C be a conjugation on $\mathcal{H}$, and let N be nilpotent of order $n (n > 2)$ with $NT = TN$. Then the following statements hold.*
(i) *Assume that T is strict m-complex symmetric with the conjugation C for $m > 1$. If T commutes with CT^*C and CN^*C, then $T + N$ is a $(2n + m - 2)$-complex symmetric operator.*
(ii) *Suppose that T is a complex symmetric operator with a conjugation C. If T commutes with CN^*C, then an operator $T + N$ is a $(2n - 1)$-complex symmetric operator.*

Theorem 4.13 *If $T \in \mathcal{L}(\mathcal{H})$ is m-complex symmetric with the conjugation C and N is a nilpotent operator of order n with $TN = NT$, then the following statements are equivalent:*
(i) *T is decomposable.*
(ii) *T^* has the property (β).*
(iii) *$T + N$ is decomposable.*
(iv) *$T^* + N^*$ has the property (β).*

Corollary 4.14 *If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric with the conjugation C and N is a nilpotent operator of order n with $TN = NT$, then the following statements are equivalent:*
(i) *T is decomposable.*
(ii) *T^* has the property (β).*
(iii) *T has the property (β).*
(iv) *$T + N$ is decomposable.*
(v) *$T^* + N^*$ has the property (β).*
(vi) *$T + N$ has the property (β).*

Let us recall that we say that *Weyl's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso\sigma(T) : 0 < dim\ ker(T - \lambda) < \infty\}$ and $iso\Delta$ denotes the set of all isolated points of Δ. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$. As some applications of Theorem 4.13, we get the following corollary.

Corollary 4.15 *Let $R = T + N \in \mathcal{L}(\mathcal{H})$ where T is m-complex symmetric and $N^n = 0$ with $TN = NT$. If T^* has the property (β), then the following assertions hold.*
(i) *R and R^* have the property (β) and the single-valued extension property.*
(ii) *If $\sigma(R)$ has nonempty interior, then R has a nontrivial invariant subspace.*
(iii) *$H_R(F)$ is a hyperinvariant subspace for R.*
(iv) *If f is any function analytic on a neighborhood of $\sigma(R)$, then both Weyl's and Browder's theorems hold for $f(R)$ and $\sigma_w(f(R)) = \sigma_b(f(R)) = f(\sigma_w(R)) = f(\sigma_b(R))$.*

Lemma 4.16 *If T is an m-complex symmetric operator, then the following relations hold.*
(i) *$\sigma_p(T) \subseteq \sigma_p(T^*)^*$, $\sigma_{ap}(T) \subset \sigma_{ap}(T^*)^*$, $\Gamma(T^*)^* \subseteq \Gamma(T)$, $\sigma_{su}(T^*)^* \subseteq \sigma_{su}(T)$, and*

$$\sigma(T) = \sigma_{ap}(T^*)^* = \sigma_{su}(T).$$

(ii) *$\sigma_{le}(T) \subseteq \sigma_{le}(T^*)^*$, $\sigma_{re}(T^*)^* \subseteq \sigma_{re}(T)$, and $\sigma_e(T) \subseteq \sigma_{re}(T)$.*
(iii) *If T^* has the single-valued extension property, then*

$$\sigma(T) = \sigma_{ap}(T) = \sigma_{ap}(T^*)^* = \sigma(T^*)^*.$$

Proposition 4.17 *Let $R = T + N$ be an operator in $\mathcal{L}(\mathcal{H})$ with the same hypotheses as in Theorem 4.13. Then the following properties hold.*
(i) *$\sigma_p(R) \subset \sigma_p(T^*)^* \cup \{0\}$, $\Gamma(R^*)^* \subset \Gamma(T) \cup \{0\}$, and $\sigma_{ap}(R) \subseteq \sigma_{ap}(T^*)^* \cup \{0\}$.*
(ii) *$\sigma_{le}(R) \subset \sigma_{le}(T)$ and $\sigma_{re}(R^*)^* \subset \sigma_{re}(T^*)^*$.*

5 Structure of m-complex symmetric operators

In this section, we focus on structures of m-complex symmetric operators. For $0 < p \leq 1$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$. We call an operator $T \in \mathcal{L}(\mathcal{H})$ *skew complex symmetric* if there exists a conjugation C on $\mathcal{H}$ such that $T = -CT^*C$.

Theorem 5.1 *Let T be an operator on $\mathcal{H}$ and C be a conjugation on $\mathcal{H}$. Then the following statements hold.*
(i) *If m is even, then $\Delta_m(T)$ is complex symmetric with the conjugation C. In this case, if $\Delta_m(T)$ is p-hyponormal, then it is normal.*
(ii) *If m is odd, then $\Delta_m(T)$ is skew complex symmetric with the conjugation C. In this case, if $\Delta_m(T) = 0$ and $\Delta_{m-1}(T)$ is p-hyponormal, then $T^*\Delta_{m-1}(T) =$*

$\Delta_{m-1}(T)CTC$ *and* $\Delta_{m-1}(T)$ *is normal.*
(iii) *Let*

$$K_m(T) := \bigcap_{n\geq m} ker(\Delta_n(T)).$$

If $K_1(T) \neq \{0\}$ *and* $K_m(T) \neq \mathcal{H}$, *then the subspace* $C(K_m(T))$ *is a nontrivial invariant subspace for* T.

Corollary 5.2 *Let* T *be an operator on* $\mathcal{H}$ *and* C *be a conjugation on* $\mathcal{H}$. *Then the following statements hold.*
(i) *If* m *is even, then* $\sigma(\Delta_m(T)) = \sigma_{ap}(\Delta_m(T))$.
(ii) *If* m *is odd, then* $\sigma(\Delta_m(T)) = \sigma_{ap}(\Delta_m(T)) \cup [-\sigma_{ap}(\Delta_m(T))]$.
(iii) *If* m *is odd and* $\Delta_m(T)$ *has finite rank* k, *then the rank of* $\Delta_m(T)$ *is even.*
(iv) *If* $K_1(T) \neq \{0\}$ *and* $1 \notin \sigma_p(CTC)$, *then* $C(K_1(T))$ *has at least two distinct elements of* $\mathcal{H}$.
(v) *Put* $F_n(T) := \bigcap_{n\leq j\leq m-1} ker(\Delta_j(T))$ *for* $n = 1, 2, \cdots, m-1$. *If* T *is a strict* m*-complex symmetric operator and* $F_1(T) \neq \{0\}$, *then* $CF_n(T)$ *is a nontrivial invariant subspace for* T *for* $n = 1, 2, \cdots, m-1$.

Example 5.3 *Let* C *be a conjugation operator given by* $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ *on* $\mathbb{C}^3$. *If* $R = I + N$ *where* $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ *on* $\mathbb{C}^3$, *then* $N^3 = 0$. *Hence*

$$\Delta_4(R) = \Delta_4(N) = 6N^{*2}CN^2C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 54 \end{pmatrix} \neq 0.$$

Thus R *is not a* 4*-complex symmetric operator. Hence* $R = I + N$ *is* 5*-complex symmetric with* C.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *normaloid* if $\|T\| = r(T)$ where $r(T)$ is the spectral radius of T. A vector $x \in \mathcal{H}$ is said to be *isotropic* if $\langle x, Cx\rangle = 0$ ([14]). We next state some conditions for $(m+1)$-complex symmetric operators to be m-complex symmetric operators.

Theorem 5.4 *Let* $T \in \mathcal{L}(\mathcal{H})$ *and* C *be a conjugation on* $\mathcal{H}$. *Suppose* $\Delta_{m+1}(T) = 0$, $\Delta_m(T)$ *is normaloid, and an eigenvector corresponding to every eigenvalue in* $\sigma_p(\Delta_m(T))$ *is not isotropic. Assume that one of the following statements holds;*
(i) *When* m *is even, for every* $\mu \in \sigma_{ap}(\Delta_m(T))$ *there exist* $\lambda \in \sigma(\Delta_1(T))$ *and a sequence* $\{x_n\}$ *of unit vectors such that* $|\lambda|^m = |\mu|$ *and*

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1(T) - \lambda)x_n\| = 0.$$

(ii) *When m is odd, for every $\mu \in \sigma_{ap}(\Delta_m(T))$ there exist $\lambda \in \sigma(T^* + CTC)$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and*

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|((T^* + CTC) - \lambda)x_n\| = 0.$$

Then $\Delta_m(T) = 0$.

Corollary 5.5 *Let C be a conjugation on $\mathcal{H}$ and let $T \in \mathcal{L}(\mathcal{H})$ be a strict $(m+1)$-complex symmetric operator, and an eigenvector corresponding to every eigenvalue in $\sigma_p(\Delta_m(T))$ be not isotropic. Assume that one of the following statements holds;*
(i) *When m is even, for every $\mu \in \sigma_{ap}(\Delta_m(T))$, there exist $\lambda \in \sigma(\Delta_1(T))$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and*

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|(\Delta_1(T) - \lambda)x_n\| = 0.$$

(ii) *When m is odd, for every $\mu \in \sigma_{ap}(\Delta_m(T))$, there exist $\lambda \in \sigma(T^* + CTC)$ and a sequence $\{x_n\}$ of unit vectors such that $|\lambda|^m = |\mu|$ and*

$$\lim_{n\to\infty} \|(\Delta_m(T) - \mu)x_n\| = \lim_{n\to\infty} \|((T^* + CTC) - \lambda)x_n\| = 0.$$

Then $\Delta_m(T)$ is not normaloid.

Theorem 5.6 *For an operator $T \in \mathcal{L}(\mathcal{H})$, let $\Delta_2(T) = 0$. If T is self-adjoint or $\Delta_1(T)$ is p-hyponormal, then $\Delta_1(T) = 0$.*

Corollary 5.7 *Let C be a conjugation operator on $\mathcal{H}$ and let H and K be self-adjoint operators. Suppose that $T = H + iK \in \mathcal{L}(\mathcal{H})$ satisfies $HCK = KCH$ and $CRC \geq R$, where $R = i(HK - KH)$. If $\Delta_2(T) = 0$, then $\Delta_1(T) = 0$.*

Theorem 5.8 *If $\Delta_m(T)$ is hyponormal and $\Delta_{m+1}(T) = 0$, then $ker(\Delta_m(T) - \lambda) \cap ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.*

Corollary 5.9 *Let C be a conjugation on $\mathcal{H}$. Let self-adjoint operators H and K satisfy $HCK = KCH$ and $CRC \geq R$, where $R = i(HK - KH)$. For an operator $T = H + iK$, if $\Delta_2(T) = 0$, then $ker(\Delta_1(T) - \lambda) = \{0\}$ for any nonzero $\lambda \in \mathbb{C}$.*

References

[1] P. Aiena, *Fredholm and local spectral theory with applications to multipliers*, Kluwer Academic Pub. 2004.

[2] T. Bermudez, A. Martinon, V. Muller, and J. Noda, *Perturbation of m-isometries by nilpotent operators*, Abstr. Appl. Anal. **2**(2014), 313-328.

[3] S. K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc., **13**(1962), 111-114.

[4] M. Chō, S. Ota, K. Tanahashi, and M. Uchiyama, *Spectral properties of m-isometric operators*, Functional Analysis, Application and Computation 4:2 (2012), 33-39.

[5] I. Colojoara and C. Foias, *Theory of generalized spectral operators*, Gordon and Breach, New York, 1968.

[6] M. Chō, E. Ko and J. Lee, *On m-complex symmetric operators*, Mediterranean J. Math., in press.

[7] _________, *On m-complex symmetric operators* II, Mediterranean J. Math., accepted.

[8] _________, *On (m, C)-isometric operators*, preprint.

[9] _________, *Perturbation of m-complex symmetric operator by nilpotent operators*, preprint.

[10] J. Eschmeier, *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, J. Funct. Anal. **94**(1990), 196-222.

[11] S. R. Garcia, *Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **60**(2008), 357-367.

[12] S. R. Garcia and M. Putinar, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358**(2006), 1285-1315.

[13] _________, *Complex symmetric operators and applications* II, Trans. Amer. Math. Soc. **359**(2007), 3913-3931.

[14] S. R. Garcia, E. Prodan, and M. Putinar, *Mathematical and physical aspects of complex symmetric operators,* J. Phys. A: Math. Gen. **47** (2014), 353001.

[15] S. R. Garcia and W. R. Wogen, *Some new classes of complex symmetric operators*, Trans. Amer. Math. Soc. **362**(2010), 6065-6077.

[16] J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc. Janos Bolyai **5**, Hilbert Space Operators, Tihany, Hungary (1970), 279-287.

[17] S. Jung, E. Ko, M. Lee, and J. Lee, *On local spectral properties of complex symmetric operators*, J. Math. Anal. Appl. **379**(2011), 325-333.

[18] S. Jung, E. Ko, and J. Lee, *On scalar extensions and spectral decompositions of complex symmetric operators*, J. Math. Anal. Appl. **382**(2011), 252-260.

[19] ________, *On complex symmetric operator matrices*, J. Math. Anal. Appl. **406**(2013), 373-385.

[20] K. Laursen and M. Neumann, *An introduction to local spectral theory*, Clarendon Press, Oxford, 2000.

[21] X. Wang and Z. Gao, *A note on Aluthge transforms of complex symmetric operators and applications*, Int. Eq. Op. Th. **65**(2009), 573-580.

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