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Some contractions and the Poncelet property of their numerical ranges

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1. A special class of contractions

In 1814, a French mathematician Jean-Victor Poncelet [26] described his famous closure theorem: Let C and D be two conics on the complex projective plane. If there exists a closed n-gon inscribed in D and circumscribed to C then, starting at an arbitrary point of D, there is a closed n-gon inscribed in D and circumscribed to C (cf. [13]). A rigid proof of Poncelet's closure theorem was given by Jacobi [20] based on the elliptic function theory (cf. [27]). For a pair of two conics C and D on the plane, a point $P \in C$ and a point $Q \in D$ have a relation $P \sim Q$ if there is a tangent line of C at P passing through Q. By this relation the space curve

$$L = \{(P, Q) \in C \times D : P \sim Q\}$$

has a parametrization by elliptic functions with common modular invariants (cf. [4]). In this sense, L is an elliptic curve. From a matrix theoretic view point, the Poncelet property arises in the boundary of the numerical range of some contraction matrices. Let A be an $n \times n$ matrix. The numerical range of A is defined as

$$W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, ||x|| = 1\}, \tag{1.1}$$

and the rank-k numerical range of A is introduced and defined in [7] as the set

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P\},$$

$1 \leq k \leq n$. In the case $k = 1$, $\Lambda_k(A)$ reduces to $W(A)$. The rank-k numerical range $\Lambda_k(A)$ is a compact convex set and $\Lambda_k(A) \neq \emptyset$ if $3k \leq n+2$

(cf. [3, 7, 8, 22]). If A is a contraction, i.e., $||Ax|| \leq ||x||$ for any $x \in \mathbb{C}^n$, then its numerical range $W(A)$ is contained in the closed unit disc. Mirman [23] found an important class S_n of $n \times n$ matrices for which the boundary $C = \partial W(A)$ of the numerical range of a matrix $A \in S_n$ and the unit circle $D = \{z \in \mathbb{C} : |z| = 1\}$ form a Poncelet pair. Gau-Wu [15] independently found the Poncelet property for the S_n class. For a survey on numerical range and the Poncelet property, see for instance [17], and for recent works on the Poncelet property, see [10, 16, 24]. A formulation of the Poncelet property of a matrix $A \in S_n$ using the complex algebraic geometry was given in [2, 25]. An $n \times n$ matrix A is in S_n if A is a contraction, A has no eigenvalue with modulus 1, and $\operatorname{rank}(I_n - A^*A) = 1$.

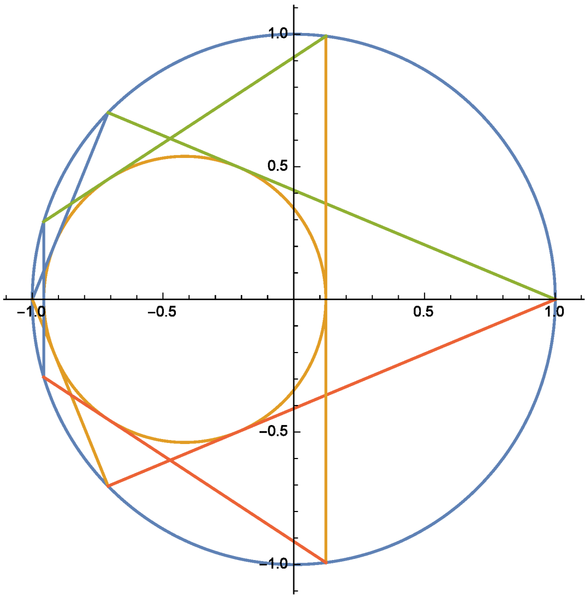

Figure1

In Figure 1, we provide an example of the boundary of the numerical range of a matrix A in S_3. We present two quadrilaterals inscribed in the unit circle and circumscribed to $\partial W(A)$.

The following result characterizes the class of S_n matrices.

Proposition 1.1[Mirman ; Gau,P. Y. Wu] Let $A_0 \in S_n$. Then there exists an $n \times n$ unitary matrix U so that $UA_0U^* = (a_{ij})$ is an upper triangular matrix given by

$$a_{ij} = \begin{cases} a_j, & \text{if } i = j \ ; \\ (1-|a_i|^2)^{1/2}(1-|a_j|^2)^{1/2}, & \text{if } i = j-1; \\ \prod_{k=i+1}^{j-1}(-\overline{a_k})(1-|a_i|^2)^{1/2}(1-|a_j|^2)^{1/2}, & \text{if } i < j-1; \\ 0, & \text{if } i > j; \end{cases} \tag{1.2}$$

for some $|a_j| < 1$, $j = 1, 2, \ldots, n$.

The numerical range of a matrix $A \in S_n$ can be expressed as

$$W(A) = \bigcap \{W(U) : U \,\text{is an}\, (n+1)\text{−dimensional unitary dilation of}\, A\}$$

(cf. [15, 23]) which also gives a partial answer to Halmos' conjecture, namely,

$$\text{closure}(W(T)) = \bigcap \{\text{closure}(W(U)) : U \,\text{is a unitary dilation of}\, T\},$$

for a contraction operator T on a complex Hilbert space (cf. [1]). A general answer is given by Choi and Li [9]. Moreover, it is shown in [14, Theorem 1.2] that an $n \times n$ contraction A with $\text{rank}(I_n - A^*A) = k$ has a general consequence:

$$\Lambda_k(A) = \bigcap \{W(U) : U \,\text{is an}\, (n+k)\text{−dimensional unitary dilation of}\, A\}.$$

Proposition 1.2 [Gau, Wu]. Let $A = (a_{ij})$ be a S_n matrix (1.2). Then any $(n+1) \times (n+1)$ unitary dilation of A is unitarily equivalent to a member of a one-parameter family of unitary matrices $B(\lambda) = (b_{ij}(\lambda))$ given by

$$b_{ij}(\lambda) = \begin{cases} a_{ij}, & \text{if } 1 \le i, j \le n; \\ \lambda(1-|a_j|^2)^{1/2}, & \text{if } i = n+1, j = 1; \\ \lambda\Big(\prod_{k=1}^{j-1}(-\overline{a_k})\Big)(1-|a_j|^2)^{1/2}, & \text{if } i = n+1,\ 2 \le j \le n; \\ (1-|a_i|^2)^{1/2}, & \text{if } j = n+1, i = n; \\ \Big(\prod_{k=i+1}^{n}(-\overline{a_k})\Big)(1-|a_i|^2)^{1/2}, & \text{if } j = n+1,\ 1 \le i \le n-1; \\ \lambda\prod_{k=1}^{n}(-\overline{a_k}), & \text{if } i = j = n+1; \end{cases} \tag{1.3}$$

where λ is a parameter on the unit circle $|z| = 1$.

2.The algorithm generating new Poncelet pairs

In [2], a complex algebraic formulation was given for $A \in S_n$. In [6], new Poncelet pairs are found. Let A be a S_n matrix (1.2) and $B(\lambda)$ its unitary dilation matrix (1.3). We present an algorithm that computes the defining polynomial $L(X,Y)$ which produces a new part $C_P : L(X,Y) = 0$ of the new Poncelet curve with respect to the boundary generating curve of $W(A)$.

Algorithm

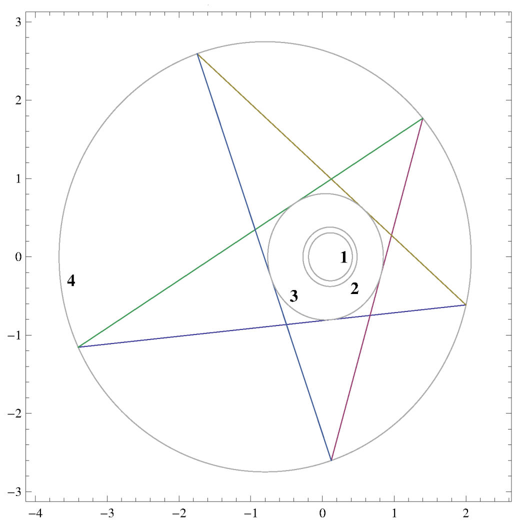

Figure 2; new Poncelet pair

- **Step 1** Compute $F_{B(\lambda)}(t,x,y)$ associated with the matrix $B(\lambda)$ of the form (1.3).
- **Step 2** Substitute $y = -1/Y - xX/Y$ into $F_{B(\lambda)}(t,x,y)$ and define a polynomial

$$\begin{aligned} H(x,X,Y:\lambda) &= Y^{n+1}F_{B(\lambda)}(1,x,-1/Y-xX/Y) = F_{B(\lambda)}(Y,xY,-1-xX) \\ &= c_{n+1}(X,Y)x^{n(n+1)} + \cdots + c_0(X,Y). \end{aligned}$$

- **Step 3** Take the resultant $R(X,Y:\lambda)$ of $H(x,X,Y:\lambda)$ and $H_x(x,X,Y:\lambda)$ with respect to x.
- **Step 4** Find a factor polynomial $K(X,Y:\lambda)$ of the resultant $R(X,Y:\lambda)$ of total degree $(n+1)n/2$ in X,Y with multiplicity 2.

- **Step 5** Substitute $\lambda = ((1-s^2)+2is)/(1+s^2)$ into $K(X,Y;\lambda)$ and $K_X(X,Y;\lambda)$.
- **Step 6** Take the respective numerators $\tilde{K}(X,Y;s)$ and $\tilde{K}_X(X,Y;s)$ of $K(X,Y;s)$ and $K_X(X,Y;s)$.
- **Step 7** Compute the Sylvester's resultant $S(X,Y)$ of $\tilde{K}(X,Y;s)$ and $\tilde{K}_X(X,Y;s)$ with respect to s.
- **Step 8** Find a factor $L(X,Y)$ of $S(X,Y)$ with multiplicity 2.

In Figure , we present the graphic of a new Poncelet curve for a matrix A in S_4. The union of the curve labeled 4 and the curve labeled 2 is $L(X,Y) = 0$. The curve labeled 1 is $\partial\Lambda_2(A)$. The curve labeled 3 is $\partial W(A)$.

Example. Let $n = 3$ and

$$B(\lambda) = \begin{pmatrix} a & 1-a^2 & -a\sqrt{1-a^2} & a^2\sqrt{1-a^2} \\ 0 & a & 1-a^2 & -a\sqrt{1-a^2} \\ 0 & 0 & a & \sqrt{1-a^2} \\ \lambda\sqrt{1-a^2} & -\lambda a\sqrt{1-a^2} & \lambda a^2\sqrt{1-a^2} & -\lambda a^3 \end{pmatrix}$$

for a is a positive real number less than 1. Then the polynomial $L(X,Y)$ which gives the equation $L(X,Y) = 0$ of the new Poncelet curve is given by

$$L(X,Y) = 6a(-a^2+1)XY^2 + (a^6+3a^2-4)Y^2 + 2a(a^2+3)X^3$$
$$+(a^6-21a^2-4)X^2 + 6a(-a^4+3a^2+2)X + (a^6-9a^2).$$

3. Matrices unitarily similar to complex symmetric matrices

In this section we present a result related with the inverse problem for the shape of a numerical range (cf. [18]).

Theorem 3.1(cf.[5]). Every matrix in S_n is unitarily similar to a complex symmetric matrix.

Proof. Let $A \in S_n$. Then by [16, Corollary 1.3] (see also [23] [Theorem 4]), the matrix A has a canonical upper triangular form. The matrix A also dilates to an $(n+1)\times(n+1)$ unitary matrix W with distinct eigenvalues (cf. [15, Lemma 2.2]). We assume the distinct eigenvalues of W are given by $c_1, c_2, \ldots, c_{n+1}$, and their respective corresponding eigenvectors are $f_1, f_2, \ldots, f_{n+1}$. Let P be the n-dimensional orthogonal projection satisfying $A = (PWP)|_{\mathbf{C}^n}$. By replacing f_j by $\exp(i\theta_j)f_j$ for some angles $\theta_1, \ldots, \theta_{n+1}$, the space $\mathbf{C}^n = P(\mathbf{C}^{n+1})$ is expressed as

$$\begin{aligned} \mathbf{C}^n &= \{z_1 f_1 + z_2 f_2 + \cdots + z_{n+1} f_{n+1} : (z_1, \ldots, z_{n+1}) \in \mathbf{C}^{n+1}, \\ &\quad b_1 z_1 + b_2 z_2 + \cdots + b_{n+1} z_{n+1} = 0\} \end{aligned}$$

for some non-negative real numbers $b_1, b_2, \ldots, b_{n+1}$. Since the modulus of any eigenvalue of A is strictly less than 1, the numbers b_j are positive. Then the space $\mathbf{C}^n = P(\mathbf{C}^{n+1})$ consists of the linear spans of

$$\{b_1 f_2 - b_2 f_1,\ b_1 f_3 - b_3 f_1,\ \ldots,\ b_1 f_{n+1} - b_{n+1} f_1\}. \tag{3.1}$$

Let $\{\xi_1, \xi_2, \ldots, \xi_n\}$ be an orthonormal basis of $\mathbf{C}^n = P(\mathbf{C}^{n+1})$ obtained by the Gram-Schmidt orthonormalization of n independent vectors in (3.1). The vectors ξ_j are expressed as

$$\xi_j = \xi_{j,1} f_1 + \xi_{j,2} f_2 + \cdots + \xi_{j,n+1} f_{n+1}$$

for some real numbers $\xi_{j,k}$ with $\xi_{j,j+1} > 0$ and $\xi_{j,j+2} = \xi_{j,j+3} = \cdots = 0$, $j = 1, 2, \ldots, n$. With respect to the orthonormal basis $\{\xi_1, \ldots, \xi_n\}$, the operator A on the n-dimensional Hilbert space $\mathbf{C}^n$ satisfies the property

$$\langle A\xi_\ell, \xi_k\rangle = \sum_{j=1}^{n+1} c_j \xi_{\ell,j} \xi_{k,j} = \sum_{j=1}^{n+1} c_j \xi_{k,j} \xi_{\ell,j} = \langle A\xi_k, \xi_\ell\rangle.$$

Thus the operator A has a symmetric matrix representation with respect to this orthonormal basis $\{\xi_1, \ldots, \xi_n\}$. □

We are interested in matrices unitarily similar to complex symmetric matrices. In [18] Helton and Spitkovsky proved that every $n \times n$ complex matrix A has an $n \times n$ complex symmetric matrix B satisfying $W(A) = W(B)$. This result follows from the followin theorem.

Theorem 3.2(Helton and Vinnikov [19]). Suppose that $F(x, y, z)$ is a degree n ternary homogeneous polynomial with real coefficients for which the equation $F(\cos\theta, \sin\theta, z) = 0$ in z has n real solutions for every angle $0 \le \theta \le 2\pi$ and $F(0, 0, 1) = 1$. Then there exist $n \times n$ real symmetric matrices G, H satisfyng

$$F(x, y, z) = \det(xH + yG + zI_n).$$

This result proved that the conjecture posed by P. Lax [21](page 184) is true. In [11], page 95, M. Fiedler posed a similar conjecture by relaxing H, G by Hermitian matrices. In [12], Fiedler proved the assertion of Theorem 3.2 in the case $F(x, y, z) = 0$ is a rational curve.

In [28] T. Takagi proved that every Toeplitz matrix is unitarily symmetric to a complex symmetric matrix.

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