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Some topics in L^p-theory for second-order elliptic operators with unbounded coefficients

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1. Introduction

In this paper we deal with the second-order elliptic operators of the form

$$Au(x) := -\mathrm{div}(a(x)\nabla u(x)) + F(x)\cdot\nabla u(x) + V(x)u(x), \quad x\in\mathbb{R}^N, \tag{1.1}$$

where $N\in\mathbb{N}$ and the coefficients (a,F,V) satisfy the following condition:

(A1) ${}^t a = a \in C^1(\mathbb{R}^N;\mathbb{R}^{N\times N})$, $F\in C^1(\mathbb{R}^N;\mathbb{R}^N)$, $V\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ and $a(x)$ is positive-definite for every $x\in\mathbb{R}^N$, that is, $\langle a(x)\xi,\xi\rangle>0$ for every $x\in\mathbb{R}^N$, $\xi\in\mathbb{C}^N\setminus\{0\}$.

Here $\langle\cdot,\cdot\rangle$ is the usual Hermitian product. Under condition **(A1)** we define the minimal and maximal realization of A in $L^p=L^p(\mathbb{R}^N)$ $(1<p<\infty)$ respectively as

$$\begin{cases} A_{p,\min}u := Au, \\ D(A_{p,\min}) := C_0^\infty(\mathbb{R}^N), \end{cases}$$

$$\begin{cases} A_{p,\max}u := Au, \\ D(A_{p,\max}) := \{u\in L^p\cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)\,;\, Au\in L^p\}. \end{cases}$$

Our interest is the following properties of $A_{p,\min}$ and $A_{p,\max}$:

- essential m-accretivity of $A_{p,\min}$
- m-accretivity of $A_{p,\max}$
- m-sectoriality of $A_{p,\max}$

(see e.g., Goldstein [7]). There properties are strongly related to the evolution equation

$$\frac{du}{dt} + A_{p,\max}u = 0, \quad t\in(0,\infty), \quad u(0)=u_0. \tag{1.2}$$

The m-accretivity of $A_{p,\max}$ gives solvability of (1.2) and the m-sectoriality of $A_{p,\max}$ implies the smoothing effect of solutions to (1.2), which may be expected for equations of parabolic type. Here we only discuss the essential m-accretivity of $A_{p,\min}$.

As is well-known, second-order elliptic operators appear in the theories of non-relativistic quantum mechanics and stochastic analysis. In particular, some important models of them are written by using the operator A in (1.1) with unbounded coefficients. For instance, in non-relativistic quantum mechanics the Schrödinger operators

$$Su(x) = -\Delta u(x) + V(x)u(x), \quad x \in \mathbb{R}^N$$

($a = (\delta_{jk})_{jk}$ and $F \equiv 0$, where δ_{jk} is the Kronecker delta) describe the motion of a quantum mechanical particle under the potential V. On the other hand, in stochastic analysis the Ornstein-Uhlenbeck operators

$$A_{OU}u(x) = -\Delta u(x) + Bx \cdot \nabla u(x), \quad x \in \mathbb{R}^N$$

($a = (\delta_{jk})_{jk}$, $F = (\sum_{j=1}^N B_{jk}x_j)_k$ and $V \equiv 0$) describe the process of random variables, where $B = (B_{jk})_{jk}$ is an $N \times N$-matrix. Our interest is the m-accretivity of operators which have the differential expression A.

There exist many investigations dealing with these problems for uniformly elliptic operators with bounded coefficients (see e.g., Kato [**9**, Example V.3.34], Fattorini [**5**, Chapter 3], Lunardi [**12**, Chapter 3] and their references).

For unbounded coefficients, the Schrödinger operators $-\Delta + V$ have been considered in many previous works (see e.g., Kato [**8**, **11**], Simon [**18**], Semenov [**17**], Okazawa [**15**, **16**] and others). The operators A with unbounded diffusion and drift are also dealt with (see e.g., Cupini–Fornaro [**3**], Metafune–Pallara–Prüss–Schnaubelt [**13**] and Fornaro–Lorenzi [**6**]).

Here we describe recent results for the (essential) m-accretivity of $A_{p,\min}$ and $A_{p,\max}$ with unbounded diffusions. In Eberle [**4**], it is shown that under

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \le \alpha\big(|x|\log(e+|x|)\big)^2, \quad x \in \mathbb{R}^N \setminus \{0\},$$

the operator $A_{p,\min}$ with $F \equiv 0$ and $V \equiv 0$ is essentially m-accretive in L^p. In Metafune–Pallara–Rabier–Schnaubelt [**14**], they proved the essential m-accretivity of $A_{p,\min}$ under the following conditions: there exist $\rho \in C^N(\mathbb{R}^N)$ satisfying $\lim_{|x|\to\infty} \rho(x) = \infty$ and $|\nabla\rho| \ne 0$ a.e. on $\mathbb{R}^N$ and constants $s, s' > 0$ satisfying $0 < s' < s$ and

$$V - \frac{\operatorname{div}F}{p} \ge 0; \tag{1.3}$$

$$V - \frac{\operatorname{div}F}{p} \ge -s\left(\langle F, \nabla\rho\rangle - \left(1 - \frac{2}{p}\right)\operatorname{div}(a\nabla\rho)\right) + s^2\langle a\nabla\rho, \nabla\rho\rangle; \tag{1.4}$$

$$e^{-p's'\rho}\langle F, \nabla\rho\rangle \in L^\infty(\mathbb{R}^N); \tag{1.5}$$

$$e^{-p's'\rho}\langle a\nabla\rho, \nabla\rho\rangle \in L^\infty(\mathbb{R}^N). \tag{1.6}$$

They also deal with the m-accretivity of $\widetilde{A}_{p,\min}$ under a similar restriction as (1.3)–(1.6) mentioned above. Although their result enables us to deal with general coefficients if $p = 2$, a certain restriction on the derivative of the diffusion a is required when $p \ne 2$ (see condition (1.4)). Because of this gap between $p = 2$ and $p \ne 2$, it seems to be unnatural from the view point of L^p-generalization.

In [19], the m-accretivity of $A_{p,\max}$ and coincidence of $A_{p,\max}$ and the closure of $A_{p,\min}$ are proved if the coefficients a, F and V satisfy that there exists a nonnegative auxiliary function $\Psi_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N; [0,\infty))$ such that

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \le (1+\Psi_p(x))^{1-\frac{2}{r}} f(|x|)^2, \qquad x \in \mathbb{R}^N \setminus B_R, \tag{1.7}$$

$$\frac{|\langle F(x), x\rangle|}{|x|} \le (1+\Psi_p(x))^{1-\frac{1}{r}} f(|x|), \qquad x \in \mathbb{R}^N \setminus B_R, \tag{1.8}$$

$$V(x) - \frac{\operatorname{div} F(x)}{p} \ge \Psi_p(x), \qquad x \in \mathbb{R}^N \tag{1.9}$$

with $f(r) = r \log r$ for constants $R > 1$ and $r \in [2,\infty)$, where B_R is the N-dimensional ball with center at the origin and radius R. More generally, the case where

$$f \in \mathcal{F}_R := \left\{ f \in C([R,\infty); (0,\infty)) \; ; \; \int_R^\infty f(s)^{-1}\, ds = \infty \right\}$$

is dealt with in [20]; note that the case where $V \equiv 0$ and $\Psi_p \equiv 0$ is proved in [4, Theorem 2.3] and $\mathcal{F}_R$ essentially appears in its proof. The optimality of $\mathcal{F}_R$ is shown in [2, Example 3.5]. This result may be regarded as a natural generalization from $p = 2$ to $p \neq 2$. In [19], the following view point is crucial.

Proposition 1.1 (see [19, Section 1]). *Assume that* **(A1)** *is satisfied. Then for every* $1 \le q \le \infty$, $w \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ *and* $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}\int_{\mathbb{R}^N} (A\psi)\overline{w}\, dx = &\int_{\mathbb{R}^N} \left[\langle a\nabla\psi, \nabla w\rangle + \left(V - \frac{\operatorname{div} F}{q}\right)\psi\overline{w} \right] dx \\ &+ \int_{\mathbb{R}^N} \left[\frac{1}{q'}\langle \overline{w}\nabla\psi, F\rangle - \frac{1}{q}\langle \psi\nabla\overline{w}, F\rangle \right] dx,\end{aligned} \tag{1.10}$$

where $q' = \frac{q}{q-1}$ *(the Hölder conjugate of* q*).*

The equality (1.10) may be regarded as a generalization in L^p of decomposition formula for sesquilinear form in L^2 into symmetric and anti-symmetric parts.

Recently, in [21], the endpoint case $r = \infty$ of [19] is discussed under an additional condition similar to the oscillation condition $|\nabla\Psi_p|^2 \le \gamma\Psi_p^3$ (see (1.15) below).

On the other hand, in Kato [10], the essential selfadjointness of the Schrödinger operators $(-\operatorname{div}(a\nabla\cdot) + V)_{2,\min}$ with the following coefficients is posed:

$$\textbf{(K)} \qquad \begin{cases} \dfrac{\langle a(x)x, x\rangle}{|x|^2} \le k(1+|x|)^{\ell+2}, & x \in \mathbb{R}^N \setminus B_R, \\ V(x) \ge c|x|^\ell, & x \in \mathbb{R}^N, \end{cases}$$

with $k, c, \ell > 0$. This problem is partially solved in [14] under the additional condition $c > \ell^2/4$. In the case $c < \ell^2/4$, the negative answer (counterexample) is given in [22] which is written in L^p-framework.

The first purpose of this paper is to prove the assertions of the essential m-accretivity of $A_{p,\min}$ in [**19**] and [**21**] via a unified approach. The second is to give a summary of answer to Kato's selfadjointness problem and its proof in L^2-framework (which is simpler than [**22**]).

Here we introduce the main assumption of this paper (almost the same setting as (1.7)–(1.9)).

(A2) There exist constants $\alpha, \beta > 0, r \in [2,\infty], R > 0$ and a nonnegative auxiliary function $\Psi_p \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ such that

$$\frac{\langle a(x)x, x\rangle}{|x|^2} \le \alpha(1+\Psi_p(x))^{1-\frac{2}{r}}\left(|x|\log|x|\right)^2 \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R; \tag{1.11}$$

$$\frac{\langle F(x), x\rangle}{|x|} \le \beta(1+\Psi_p(x))^{1-\frac{1}{r}}\left(|x|\log|x|\right) \quad \text{a.a. } x \in \mathbb{R}^N \setminus B_R; \tag{1.12}$$

$$V - \frac{\mathrm{div} F}{p} \ge \Psi_p \quad \text{a.e. on } \mathbb{R}^N. \tag{1.13}$$

Now we are in a position to state our main result. The first theorem is the assertion for essential m-accretivity in the case where $r \in [2,\infty)$.

Theorem 1.1 ([**19**, Theorem 1.1]). *Let $1 < p < \infty$. Assume that* **(A1)** *and* **(A2)** *are satisfied with $r \in [2,\infty)$. Then $A_{p,\min}$ is essentially m-accretive in L^p, that is,*

$$\mathrm{Re}\int_{\mathbb{R}^N} (A_{p,\min}u)\overline{u}|u|^{p-2}\,dx \ge 0 \;\; \forall u \in D(A_{p,\min}), \qquad \overline{R(1+A_{p,\min})} = L^p, \tag{1.14}$$

where $R(1+A_{p,\min})$ is the range of $1+A_{p,\min}$.

The second is the assertion for essential m-accretivity in the endpoint case $r=\infty$ of Theorem 1.1.

Theorem 1.2 ([**21**, Theorem 1.1]). *Let $1 < p < \infty$. Assume that* **(A1)** *and* **(A2)** *are satisfied with $r=\infty$. Assume further that $\Psi_p \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and there exists $\gamma_p > \frac{p-1}{4}$ such that*

$$\Psi_p \ge \frac{1}{p'}\frac{\langle F, \nabla\Psi_p\rangle}{1+\Psi_p} + \gamma_p\frac{\langle a\nabla\Psi_p, \nabla\Psi_p\rangle}{(1+\Psi_p)^2} \quad \textit{a.e. on } \mathbb{R}^N. \tag{1.15}$$

Then $A_{p,\min}$ is essentially m-accretive in L^p.

The conditions (1.13) and (1.15) in Theorem 1.2 can be replaced with a weaker condition.

Theorem 1.3. *Let $1 < p < \infty$. Assume that* **(A1)** *and* (1.11) *and* (1.12) *are satisfied with $r=\infty$. Assume further that $\Phi_p \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and there exists $\gamma_p > \frac{p-1}{4}$ such that*

$$V - \frac{\mathrm{div}F}{p} \ge \max\left\{0, \frac{1}{p'}\frac{\langle F, \nabla\Psi_p\rangle}{1+\Psi_p} + \gamma_p\frac{\langle a\nabla\Psi_p, \nabla\Psi_p\rangle}{(1+\Psi_p)^2}\right\} \quad \textit{a.e. on } \mathbb{R}^N. \tag{1.16}$$

Then $A_{p,\min}$ is essentially m-accretive in L^p.

The last theorem is a summary of the answer to Kato's selfadjointness problem.

Theorem 1.4 ([**21**, Theorem 3.1] and [**22**, Theorem 1.1], $p = 2$). *The following assertions hold*:

(i) *If* (a, V) *satisfies* **(A1)** *and* **(K)** *with* $c > \frac{k\ell^2}{4}$, *then* $A_{2,\min}$ *is essentially selfadjoint.*

(ii) *If* $N \geq 5$ *and* $c_0 < \frac{k_0\ell^2}{4}$, *then there exists a pair* (a, V) *such that* (a, V) *satisfies* **(A1)** *and* **(K)** *with* $k = k_0$ *and* $c = c_0$ *and* $A_{2,\min}$ *is not essentially selfadjoint.*

The plan of this paper is as follows. Theorems 1.1, 1.2 are proved in Section 2 via a unified approach. In Section 3, We prove Theorem 1.4 **(i)** and **(ii)**. The proof of **(i)** is based on Theorem 1.3 and the other is a simplified version of that in [**22**].

2. Proofs of Theorems 1.1 and 1.3

First we show that $A_{p,\min}$ is accretive in L^p (the first part of (1.14)).

Proof of Theorems 1.1 and 1.3 (accretivity). Let $u \in C_0^\infty(\mathbb{R}^N)$. If $2 \leq p < \infty$, then taking the real part of (1.10) with $q = p$, $w = |u|^{p-2}u$ and $\psi = u$, we see from **(A1)** and $V - \frac{\operatorname{div} F}{p} \geq 0$ that

$$\begin{aligned}\operatorname{Re}\int_{\mathbb{R}^N}(Au)\overline{u}|u|^{p-2}\,dx = (p-1)\int_{\mathbb{R}^N}&|u|^{p-4}\langle a\operatorname{Re}(\overline{u}\nabla u), \operatorname{Re}(\overline{u}\nabla u)\rangle\,dx\\ &+\int_{\mathbb{R}^N}|u|^{p-4}\langle a\operatorname{Im}(\overline{u}\nabla u), \operatorname{Im}(\overline{u}\nabla u)\rangle\,dx\\ &+\int_{\mathbb{R}^N}\left(V - \frac{\operatorname{div}F}{p}\right)|u|^p\,dx \geq 0.\end{aligned}$$

If $1 < p < 2$, then we use (1.10) with $q = p$, $w = (|u|^2 + \varepsilon)^{\frac{p-2}{2}}u$ ($\varepsilon > 0$) and $\psi = u$. Letting $\varepsilon \downarrow 0$, we obtain the accretivity of $A_{p,\min}$ for $1 < p < 2$. □

Next we prove the (essential) maximality of $A_{p,\min}$, that is, $R(1 + A_{p,\min})$ (the range of $1 + A_{p,\min}$) is dense in L^p. We only prove the case $2 \leq p' < \infty$ ($1 < p \leq 2$) in order to avoid the complicated computation. The case $1 < p' < 2$ can be verified via a procedure similar to the other case with ε-regularization as in the proof of accretivity (see [**19**, Theorem 1.1], [**20**, Theorem 1.1] and [**21**, Theorem 1.1]).

Here we need the following lemma.

Lemma 1. *Let* $v \in L^{p'}$ *be real-valued. Assume that* **(A1)** *and*

$$\int_{\mathbb{R}^N} v(\varphi + A\varphi)\,dx = 0 \quad \forall\varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.1}$$

Then $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. *Moreover, if* $\Phi \in W^{1,\infty}(\mathbb{R}^N)$ *has a compact support in* $\mathbb{R}^N$, *then*

$$\begin{aligned}\int_{\mathbb{R}^N}&\left[(p'-1)\langle a\nabla v, \nabla v\rangle\Phi|v|^{p-2} + \langle a\nabla v, \nabla\Phi\rangle v|v|^{p-2}\right]dx\\ &+\int_{\mathbb{R}^N}\left[\frac{1}{p'}\langle F, \nabla\Phi\rangle + \left(1 + V - \frac{\operatorname{div}F}{p}\right)\Phi\right]|v|^{p'}\,dx = 0.\end{aligned}$$

Proof. Using the elliptic regularity (see e.g., Agmon [**1**, Lemma 5.1]) iteratively, we see $v \in H^1_{\mathrm{loc}}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. Then using (1.10) with $q = p$ and $w = v$ and $\psi = \varphi$, we deduce

$$\int_{\mathbb{R}^N} \left[\langle a\nabla v, \nabla\varphi\rangle + \frac{1}{p'}\langle F, v\nabla\varphi\rangle - \frac{1}{p}\langle F, \varphi\nabla v\rangle + \left(1 + V - \frac{\mathrm{div}F}{p}\right) v\varphi \right] dx = 0.$$

The above equality is verified even for $\varphi \in H^1(\mathbb{R}^N)$ with a compact support. Here we choose $\varphi = \Phi v|v|^{p'-2}$. Then noting that

$$\frac{1}{p'} v\nabla\varphi - \frac{1}{p}\varphi\nabla v = \frac{1}{p'}\nabla\Phi|v|^{p'},$$

we obtain the desired assertion. □

Proof of Theorem 1.1 (maximality). Assume (2.1) for $v \in L^{p'}(\mathbb{R}^N)$. It suffices to prove that $v = 0$ a.e. on $\mathbb{R}^N$. We may assume without loss of generality that v is real-valued. We take the cut-off functions $\{\zeta_n\}_n \subset W^{1,\infty}(\mathbb{R}^N)$ as

$$\zeta_n(x) := \begin{cases} 1 & \text{if } |x| < \exp\exp n, \\ 0 & \text{if } > \exp\exp(n+1), \\ n + 1 - \log\log|x| & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$. Applying Lemma 1 with $\Phi = \zeta_n^r$, we deduce that

$$\begin{aligned} (p'-1) \int_{K_n} \zeta_n^r \langle a\nabla v, \nabla v\rangle |v|^{p'-2}\, dx + r \int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle a\nabla v, \nabla\zeta_n\rangle |v|^{p'-2} v\, dx \\ + \frac{r}{p'} \int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle |v|^{p'}\, dx + \int_{K_n} \zeta_n^r \left(1 + V - \frac{\mathrm{div}F}{p}\right) |v|^{p'}\, dx = 0, \end{aligned}$$

where $K_n := \operatorname{supp}\zeta_n$. By the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned} \int_{K_n} \zeta_n^r \left(1 + V - \frac{\mathrm{div}F}{p}\right) |v|^{p'}\, dx \le \frac{r^2}{4(p'-1)} \int_{K_n\setminus K_{n-1}} \zeta_n^{r-2}\langle a\nabla\zeta_n, \nabla\zeta_n\rangle |v|^{p'}\, dx \\ - \frac{r}{p'} \int_{K_n\setminus K_{n-1}} \zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle |v|^{p'}\, dx. \end{aligned} \tag{2.2}$$

On the other hand, note that

$$\nabla\zeta_n(x) = \begin{cases} \dfrac{-x}{|x|\log|x|} & \text{if } x \in K_n \setminus K_{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus it follows from (1.11), (1.12) of the condition **(A2)** and Young's inequality that there exist constants $C_1, C_2 > 0$ such that for every $n \ge \log\log R$,

$$\begin{aligned} \zeta_n^{r-2}\langle a\nabla\zeta_n, \nabla\zeta_n\rangle &= \frac{\zeta_n^{r-2}\langle a(x)x, x\rangle}{|x|^4(\log|x|)^2} \le \alpha\zeta_n^{r-2}(1+\Psi_p)^{1-\frac{2}{r}} \le \frac{2(p'-1)}{r^2}(C_1 + \zeta_n^r(1+\Psi_p)), \\ -\zeta_n^{r-1}\langle F, \nabla\zeta_n\rangle &= \frac{\zeta_n^{r-1}\langle F(x), x\rangle}{|x|^2\log|x|} \le \beta(1+\Psi_p)^{1-\frac{1}{r}}\zeta_n^{r-1} \le \frac{p'}{2r}(C_2 + \zeta_n^r(1+\Psi_p)). \end{aligned}$$

Therefore, combining (2.2), (1.13) and the above estimates, we have

$$\int_{K_n} \zeta_n^r(1+\Psi_p)|v|^{p'}\,dx \le (C_1+C_2)\int_{K_n\setminus K_{n-1}} |v|^{p'}\,dx + \int_{K_n\setminus K_{n-1}} \zeta_n^r(1+\Psi_p)|v|^{p'}\,dx.$$

Consequently, we see that

$$\int_{K_{n-1}} |v|^{p'}\,dx \le (C_1+C_2)\int_{K_n\setminus K_{n-1}} |v|^{p'}\,dx \to 0$$

as $n\to\infty$. This implies that $v=0$ a.e. on $\mathbb{R}^N$, that is, $R(1+A_{p,\min})$ is dense in L^p. □

Proof of Theorem 1.3 (maximality). Assume (2.1) for real-valued function $v\in L^{p'}$. As in the proof of Theorem 1.1, we prove that $v=0$ a.e. on $\mathbb{R}^N$. Applying Lemma 1 with $\Phi=\Theta_p^{-1}\zeta_n^2$ ($\Theta_p := 1+\Psi_p$), we deduce that

$$\begin{aligned}
(2.3)\qquad &(p'-1)\int_{K_n}\frac{\zeta_n^2\langle a\nabla v,\nabla v\rangle|v|^{p'-2}}{\Theta_p}\,dx + 2\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle a\nabla v,\nabla\zeta_n\rangle|v|^{p'-2}v}{\Theta_p}\,dx\\
&-\int_{K_n}\frac{\zeta_n^2\langle a\nabla v,\nabla\Psi_p\rangle|v|^{p'-2}v}{\Theta_p^2}\,dx - \frac{1}{p'}\int_{K_n\setminus K_{n-1}}\frac{\zeta_n^2\langle F,\nabla\Psi_p\rangle|v|^{p'}}{\Theta_p^2}\,dx\\
&+\frac{2}{p'}\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle F,\nabla\zeta_n\rangle|v|^{p'}}{\Theta_p}\,dx + \int_{K_n}\frac{\zeta_n^2}{\Theta_p}\left(1+V-\frac{\operatorname{div}F}{p}\right)|v|^{p'}\,dx = 0.
\end{aligned}$$

By the Cauchy-Schwarz and Young inequalities, we have

$$\begin{aligned}
&\int_{K_n}\frac{\zeta_n^2}{\Theta_p}\left(1+V-\frac{\operatorname{div}F}{p}-\frac{1}{p'}\frac{\langle F,\nabla\Psi_p\rangle}{\Theta_p}-\gamma_p\frac{\langle a\nabla\Psi_p,\nabla\Psi_p\rangle}{\Theta_p^2}\right)|v|^{p'}\,dx\\
&= -(p'-1)\int_{K_n}\frac{\zeta_n^2\langle a\nabla v,\nabla v\rangle|v|^{p'-2}}{\Theta_p}\,dx - \gamma_p\int_{K_n}\frac{\zeta_n^2\langle a\nabla\Psi_p,\nabla\Psi_p\rangle|v|^{p'}}{\Theta_p^3}\,dx\\
&\quad-\int_{K_n}\frac{\zeta_n^2\langle a\nabla v,\nabla\Psi_p\rangle|v|^{p'-2}v}{\Theta_p^2}\,dx - 2\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle a\nabla v,\nabla\zeta_n\rangle|v|^{p'-2}v}{\Theta_p}\,dx\\
&\quad-\frac{2}{p'}\int_{K_n\setminus K_{n-1}}\frac{\zeta_n^2\langle F,\nabla\zeta_n\rangle|v|^{p'}}{\Theta_p}\,dx\\
&\le -\left(p'-1-\frac{1}{4\gamma_p}\right)\int_{K_n}\frac{\zeta_n^2\langle a\nabla v,\nabla v\rangle|v|^{p'-2}}{\Theta_p}\,dx\\
&\quad-2\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle a\nabla v,\nabla\zeta_n\rangle|v|^{p'-2}v}{\Theta_p}\,dx - \frac{2}{p'}\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle F,\nabla\zeta_n\rangle|v|^{p'}}{\Theta_p}\,dx\\
&\le\left(p'-1-\frac{1}{4\gamma_p}\right)^{-1}\int_{K_n\setminus K_{n-1}}\frac{\langle a\nabla\zeta_n,\nabla\zeta_n\rangle|v|^{p'}}{\Theta_p}\,dx - \frac{2}{p'}\int_{K_n\setminus K_{n-1}}\frac{\zeta_n\langle F,\nabla\zeta_n\rangle|v|^{p'}}{\Theta_p}\,dx.
\end{aligned}$$

Therefore we see from (1.11), (1.12) and (1.16) that

$$\int_{K_n}\frac{\zeta_n^2}{\Theta_p}|v|^{p'}\,dx \le \left[\alpha\left(p'-1-\frac{1}{4\gamma_p}\right)^{-1}+\frac{2\beta}{p'}\right]\int_{K_n\setminus K_{n-1}}|v|^{p'}\,dx \to 0$$

as $n\to\infty$. This implies that $v=0$ a.e. on $\mathbb{R}^N$. This completes the proof. □

Remark 2.1. The difference between the proof of Theorem 1.1 and one of Theorem 1.1 is only the choice of sequence of Φ. If $r \in [2, \infty)$, then we do not need to assume the differentiability of Ψ_p. On the other hand, if $r = \infty$, then the differentiability of Ψ_p is required.

3. Proof of Theorem 1.4

Proof of Theorem 1.4 **(i)**. To apply Theorem 1.2 with $p = 2$, we put

$$\Psi_2(x) := \max\{1, c|x|^\ell\} - 1.$$

Then we see from **(K)** that for every $x \in \mathbb{R}^N$ satisfying $|x| > c^{-1/\ell}$,

$$\begin{aligned}\frac{\langle a(x)\nabla\Psi_2(x), \nabla\Psi_2(x)\rangle}{(1+\Psi_2(x))^2} &= \ell^2\frac{\langle a(x)x, x\rangle |x|^{2\ell-4}}{c^2|x|^{2\ell}} \\ &\le k\ell^2|x|^\ell \\ &\le \frac{k\ell^2}{c}\Psi_p(x).\end{aligned}$$

Therefore if $c > k\ell^2/4$, then Theorem 1.2 is applicable to (a, V), that is, $A_{2,\min}$ is essentially m-accretive in L^2. Since $A_{2,\min}$ is symmetric, we obtain the essential selfadjointness of $A_{2,\min}$. □

To give a clear proof, we show Theorem 1.4 **(ii)** for $\ell \in (0, 2]$ (For the other case, see [**22**]). Before starting, we give a strategy of the proof. The proof is divided into the following three parts.

Step 1. We consider the Schrödinger operators

$$B = -\Delta_y + \frac{\lambda}{y_N^2} + W(y) \quad \text{in } \mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, \infty) \tag{3.1}$$

with $0 \le W \in C(\overline{\mathbb{R}^N_+})$ and prove that $B_{2,\min}$ (B defined on $C_0^\infty(\mathbb{R}^N_+)$) is nonnegative but not essentially selfadjoint in $L^2(\mathbb{R}^N_+)$ when $\lambda \in (-\frac{1}{4}, \frac{3}{4})$.

Step 2. Using diffeomorphism $\Phi : \mathbb{R}^N \to \mathbb{R}^N$, we translate the operator B in $\mathbb{R}^N_+$ into an operator A in $\mathbb{R}^N$; remark that A is not essentially selfadjoint in $L^2(\mathbb{R}^N)$.

Step 3. We construct (a, V) such that **(K)** is satisfied with $k = k_0$ and $c = c_0$ and corresponding operator $A_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}^N)$.

3.1. Step 1

Lemma 2. *Assume $-\frac{1}{4} < \lambda < \frac{3}{4}$. Then $B_{2,\min}$ is nonnetgative but not essentially selfadjoint in $L^2(\mathbb{R}^N_+)$.*

Proof. **(nonegativity)** Let $v \in C_0^\infty(\mathbb{R}^N_+)$. Then by integration by parts and the one-dimensional Hardy inequality (with respect to y_N), we have

$$\begin{aligned}
\int_{\mathbb{R}^N_+} (B_{2,\min} v)\overline{v}\, dy &= \int_{\mathbb{R}^N_+} |\nabla v|^2\, dy + \lambda \int_{\mathbb{R}^N_+} \frac{|v|^2}{y_N^2}\, dy \\
&\geq \int_{\mathbb{R}^N_+} \left|\frac{\partial v}{\partial y_N}\right|^2 dy + \lambda \int_{\mathbb{R}^N_+} \frac{|v|^2}{y_N^2}\, dy + \int_{\mathbb{R}^N_+} W|v|^2\, dy \\
&\geq \left(\lambda + \frac{1}{4}\right) \int_{\mathbb{R}^N_+} \frac{|v|^2}{y_N^2}\, dy + \int_{\mathbb{R}^N_+} W|v|^2\, dy \\
&\geq 0.
\end{aligned}$$

Therefore we can find the Friedrichs extension B_F of $B_{2,\min}$. We remark that $D(B_F) \subset D(1/y_N)$. More precisely,

$$\left(\lambda + \frac{1}{4}\right) \int_{\mathbb{R}^N_+} \frac{|v|^2}{y_N^2}\, dy \leq \|v\|_{L^2(\mathbb{R}^N_+)} \|B_F v\|_{L^2(\mathbb{R}^N_+)} \quad \forall v \in D(B_F).$$

(Non-selfadjointness) First we prove Fix $\eta \in C_0^\infty(\mathbb{R}; [0,1])$ such that $\eta(s) = 1$ for $|s| \leq 1$ and $\eta(s) = 0$ for $|s| \geq 2$ and set

$$\psi(y) = (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} \prod_{j=1}^{N} \eta(y_j).$$

Then noting that $\lambda \in (-\frac{1}{4}, \frac{3}{4})$, we have $\psi \in L^2(\mathbb{R}^N_+)$ and $y_N^{-1}\psi \notin L^2(\mathbb{R}^N_+)$ and therefore $\psi \notin D(B_F)$. Define

$$f := \psi + B\psi.$$

Then noting that $(-\frac{d^2}{ds^2} + \frac{\lambda}{s^2}) s^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}} = 0$, we have

$$|B\psi(y)| \leq C \left(1 + (y_N)^{\frac{1}{2} - \sqrt{\lambda + \frac{1}{4}}}\right) \chi_{(-2,2)^{N-1} \times (0,1)}$$

and hence $f \in L^2(\mathbb{R}^N_+)$. Setting $\psi_W := \psi - (1 + B_F)^{-1} f \notin D(B_F)$, we have $\psi_W + B\psi_W = 0$. This yields

$$\int_{\mathbb{R}^N_+} (v + B_{2,\min} v)\psi_W\, dx = \int_{\mathbb{R}^N_+} v(\psi_W + B\psi_W)\, dx = 0.$$

Hence $B_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}^N_+)$. □

3.2. Step 2

The following elementary lemma gives a transform of operators in $\mathbb{R}^N$ into one in $\mathbb{R}^N_+$.

Lemma 3. *Let* $\Phi \in C^\infty(\mathbb{R}^N_+;\mathbb{R}^N)$ *be a diffeomorphism with* $\det D\Phi(y) = y_N^{1/2}$. *Set* $J : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N_+)$ *as*

$$Ju(y) := y_N^{1/4} u(\Phi(y)). \tag{3.2}$$

Then J *can be extended to an isometry from* $L^2(\mathbb{R}^N)$ *to* $L^2(\mathbb{R}^N_+)$ *and for every* $u \in C_0^\infty(\Omega_2)$,

$$-\mathrm{div}(a^\Phi \nabla u) = J^{-1}\left(-\Delta - \frac{1}{4y_N^2}\right) Ju, \tag{3.3}$$

where $a^\Phi \in C^\infty(\Omega_2, \mathbb{R}^{N^2})$ *is defined as* $a^\Phi(x) = (D\Phi D\Phi^*)(\Phi^{-1}(x))$.

3.3. Step 3

Now we consider the operator

$$A^\Phi u = -\mathrm{div}(a^\Phi \nabla u) + c|x|^\ell u,$$

where Φ is determined later. Then we have

Lemma 4. *Condition* **(K)** *for* a^Φ *is equivalent to the following condition:*

$$\frac{1}{4}\left|\nabla|\Phi(y)|^2\right|^2 \le k(1+|\Phi(y)|)^{\ell+2}|\Phi(y)|^2 \quad \textit{if } |\Phi(y)| \ge R. \tag{3.4}$$

Now we introduce a suitable Φ. We define $\Phi \in C^\infty(\mathbb{R}^N_+;\mathbb{R}^N)$ as follows: for $y = (y', y_N)$ with $y' \in \mathbb{R}^{N-1}$ and $y_N > 0$,

$$\Phi_j(y) := \left(\frac{(y_N)^{1/2}}{F'(y_N)}\right)^{\frac{1}{N-1}} y', \quad \Phi_N(y) := F(y_N). \tag{3.5}$$

Here $F \in C^\infty(\mathbb{R}_+;\mathbb{R})$ satisfies $F'(t) > 0$ and

$$F(t) = \begin{cases} -\left(\frac{\ell}{2}t\right)^{-\frac{2}{\ell}} & \text{if } 0 < t < \frac{2}{\ell}, \\ t^{3/2} & \text{if } \frac{4}{\ell} < t < \infty. \end{cases} \tag{3.6}$$

Remark 3.1. In view of (3.4), the choice of $\Phi_N = F$ may be essential because of

$$\frac{1}{4}\left|\frac{d}{dt}|F(t)|^2\right|^2 = |F(t)|^{\ell+4}, \qquad t \in \left(0, \frac{2}{\rho-2}\right).$$

This property will be used in Lemma 5.

Next we verify **(K)** for a^Φ with precise constant $k > 0$.

Lemma 5. *If* $N \ge 5$, *then there exists* $R_0 > 0$ *such that* a^Φ *satisfies* **(K)** *with* $k = 1$.

Proof. Observe that $N \geq 3 + 2\beta$ yields $4\widetilde{\beta} \leq 1$ for every $\rho > 2$ and hence $k_0 = 1$. Therefore we only prove the general case.

By virtue of Lemma 4, it suffices to prove **(K)** that (3.4) holds with $k = 1$ and for some R_0. By the definition of Φ, we see that

$$\nabla|\Phi(y)|^2 = 2\begin{pmatrix} \left(\dfrac{(y_N)^{1/2}}{F'(y_N)}\right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \dfrac{1}{N-1}\left(\dfrac{1}{2y_N} - \dfrac{F''(y_N)}{F'(y_N)}\right)|\hat{\Phi}(y)|^2 + F'(y_N)\Phi_N(y) \end{pmatrix}, \tag{3.7}$$

where $\hat{\Phi}(y) = (\Phi_1(y), \ldots, \Phi_{N-1}(y))$. Here we prove (3.4) by dividing its proof into three cases:

(The case $y_N \geq \frac{4}{\ell}$). In this case $F(y_N) = (y_N)^{\beta+1}$ and hence we have

$$\nabla|\Phi(y)|^2 = 2\begin{pmatrix} \left(\dfrac{2}{3}\right)^{\frac{1}{N-1}} \hat{\Phi}(y) \\ \frac{3}{2}|\Phi_N(y)|^{\frac{4}{3}} \end{pmatrix}, \tag{3.8}$$

Therefore there exists $R_1 > 0$ such that (3.4) holds with $k = 1$ if $|\Phi(y)| \geq R_1$.

(The case $\frac{2}{\ell} \leq y_N < \frac{4}{\ell}$). In this case, F', $1/F'$ and F'' are uniformly bounded on $[\frac{2}{\rho-2}, \frac{4}{\rho-2}]$. Hence

$$\frac{1}{2}|\nabla|\Phi(y)|^2| \leq C_1|\Phi(y)| + C_2|\hat{\Phi}(y)|^2 \leq (C_1 + C_2|\Phi(y)|)|\Phi(y)|,$$

where

$$C_1 = \max\left\{ \sup_{t\in[\frac{2}{\ell},\frac{4}{\ell}]} \left(\frac{t^{a/2}}{F'(t)}\right)^{\frac{1}{N-1}}, \sup_{t\in[\frac{2}{\ell},\frac{4}{\ell}]} F'(t), \right\},$$
$$C_2 = \frac{1}{N-1}\sup_{t\in[\frac{2}{\ell},\frac{4}{\ell}]}\left|\frac{1}{2t} - \frac{F''(t)}{F'(t)}\right|.$$

Thus there exists $R_2 > 0$ such that (3.4) holds with $k = 1$ if $|\Phi(y)| \geq R_2$.

(The case $0 < y_N < \frac{2}{\ell}$) Observe that

$$F'(t) = |F(t)|^{\frac{\ell+2}{2}}, \quad F''(t) = -\frac{\rho}{2}|F(t)|^{\ell+1}.$$

Hence

$$\nabla|\Phi(y)|^2 = 2\begin{pmatrix} \left(\dfrac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta}\hat{\Phi}(y) \\ \beta|\Phi_N(y)|^{\frac{\ell}{2}}|\hat{\Phi}(y)|^2 - 2|\Phi_N(y_N)|^{2+\frac{\ell}{2}} \end{pmatrix},$$

where $\beta = \frac{3\ell+4}{N-1}$. Noting that $|\Phi_N(y)| \geq 1$, we have

$$\left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi_N(y)|^{-\beta}|\hat{\Phi}(y)| \leq \left(\frac{\ell}{2}\right)^{-\frac{1}{2(N-1)}} |\Phi(y)|.$$

On the other hand, putting $|\Phi_N(y)| = |\Phi(y)|\sqrt{s}$ and $|\Phi_j(y)'| = |\Phi(y)|\sqrt{1-s}$ ($s \in [0,1]$), we have

$$\frac{\partial}{\partial x_N}|\Phi(y)|^2 = 2|\Phi(y)|^{\frac{\ell}{2}+1}\left(\beta s^{\frac{\rho-2}{4}}(1-s) - s^{\frac{2+\rho}{4}}\right).$$

By the standard computation, we have

$$\left|\beta s^{\frac{\rho-2}{4}}(1-s) - s^{\frac{2+\rho}{4}}\right| \leq 1.$$

Hence we can choose $R_3 > 0$ such that (3.4) holds with $k = 1$ if $|\phi(y)| \geq R_3$. Consequently, taking $R_0 = \max\{R_1, R_2, R_3\}$, we have (3.4) with $k = k_0$ and $R = R_0$. □

Next we define

$$W_\ell(y) := |\Phi(y)|^\ell, \quad y \in \mathbb{R}^N_+.$$

Then we have

Lemma 6. *Let $\ell > 0$. Then there exists $M > 0$ depending only on ℓ such that if $y \in \mathbb{R}^N_+$ satisfies $y_N \leq \frac{2}{\ell}$, then*

$$\frac{4}{\ell^2}\frac{1}{(y_N)^2} \leq W_\ell(y) \leq \frac{4}{\ell^2}\frac{1}{(y_N)^2} + M(y_N)^{\frac{3\ell+4}{2(N-1)}}|y'|^2. \tag{3.9}$$

Proof. By the definition of Φ and F, we have

$$W_\ell(y) = \left(\left(\frac{\ell}{2}\right)^{\frac{2(\ell+2)}{\ell(N-1)}} (y_N)^{\frac{1}{N-1}+\frac{2(\ell+2)}{\ell(N-1)}}|y'|^2 + \left(\frac{\ell}{2}y_N\right)^{-\frac{4}{\ell}}\right)^{\frac{\ell}{2}}.$$

Noting that $\ell \leq 2$, the triangle inequality yields

$$\begin{aligned}\left(\frac{\ell}{2}y_N\right)^{-2} \leq W_\ell(y) &\leq \left(\frac{\ell}{2}y_N\right)^{-2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}}(y_N)^{\frac{3\ell+4}{2(N-1)}}|y'|^2 \\ &\leq \frac{4}{\ell^2}\frac{1}{(y_N)^2} + \left(\frac{\ell}{2}\right)^{\frac{\ell+2}{N-1}}(y_N)^{\frac{\ell(\beta+1)+2}{N-1}}|y'|^2.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.4 **(ii)**. Setting $\widetilde{W_\ell} \geq 0$ as

$$\widetilde{W_\ell}(y) := W_\ell(y) - \frac{4}{\ell^2}\frac{1}{y_N^2} \in C(\overline{\mathbb{R}^N_+}),$$

we deduce

$$A^\Phi u = J^{-1}B^\Phi Ju$$

with

$$B^{\Phi} = -\Delta + \left(\frac{4c}{\ell^2} - \frac{1}{4}\right)\frac{1}{y_N^2} + c\widetilde{W}_\ell(y).$$

Apllying Lemma 2 to $\lambda = \frac{4c}{\ell^2} - \frac{1}{4} \in (-\frac{1}{4}, \frac{3}{4})$ and $W = c\widetilde{W}_\ell$, we see that $B^{\Phi}_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}^N_+)$. Since J is an isometry, $A^{\Phi}_{2,\min}$ is not essentially selfadjoint in $L^2(\mathbb{R}^N)$. This completes the proof. □

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