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Tilting combinatorics for Brauer graph algebras

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1 Introduction

The study of derived categories have been one of the central subjects in representation theory. From Morita theoretic perspective, tilting complexes play an important role because the endomorphism algebras are derived equivalent to the original algebra. It is well-known that derived equivalences preserve various homological properties. Thus it is important to construct tilting complexes for a given algebra. As a method to construct tilting complexes, the authors in [AI] introduced the notion of mutation. Roughly speaking, mutation is an operation to construct a new tilting complex from a given one by replacing a direct summand. In this report, we study the structure of mutations of tilting complexes of Brauer graph algebras.

Notation. Throughout this report, K is an algebraically closed field. All algebras are assumed to be basic, indecomposable, and finite dimensional over K. We will often use Λ to denote such an algebra. We always work with finitely generated right modules, and use $\mathsf{mod}\,\Lambda$ to denote the category of finitely generated right Λ-modules. We denote by $\operatorname{proj}\Lambda$ the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of all finitely generated projective Λ-modules. We sometimes write $\Lambda = kQ/I$ where Q is a finite quiver with relations I. We denote by P_i (respectively, S_i) an indecomposable projective (respectively, simple) Λ-module corresponding to a vertex i of Q.

2 Tilting theory

In this section, we recall the notion of tilting complexes and mutation. Throughout this section, we assume that Λ is a symmetric algebra (*i.e.*, there exists an isomorphism of Λ-Λ-bimodules between Λ and $\operatorname{Hom}_K(\Lambda, K)$). We denote by $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$ the bounded homotopy category of $\operatorname{proj}\Lambda$ with the shift functor [1] and by $\operatorname{add} T$ the full subcategory of $\mathcal{T}$ whose objects are direct summands of finite direct sums of $T \in \mathcal{T}$. For a positive integer n, we call a complex $T = (T^i, d^i)_{i\in\mathbb{Z}} \in \mathcal{T}$ an *n-term* complex if $T^i = 0$ for all integers $i \neq 0, -1, -2, \ldots, -n+1$. Each indecomposable complex is isomorphic to an indecomposable complex $T = (T^i, d^i)_{i\in\mathbb{Z}}$ such that all differential maps d^i are radical maps in $\mathcal{T}$. Without loss of generality, we may assume that the differential maps of T are radical maps.

Definition 2.1. Let T be a complex in $\mathcal{T}$.

(1) We say that T is *pretilting* if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for all nonzero integers i.

(2) We say that T is *tilting* if it is pretilting and $\mathcal{T} = \mathrm{thick}\, T$, where $\mathrm{thick}\, T$ is the smallest full subcategory of $\mathcal{T}$ which contains T and is closed under cones, $[\pm 1]$ direct summands and isomorphisms.

Example 2.2. An algebra A is always a tilting complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$.

We denote by $\mathsf{tilt}\,\Lambda$ the set of isomorphism classes of basic tilting complexes in $\mathcal{T}$, by $\mathsf{n}\text{-}\mathsf{tilt}\,\Lambda$ the subset of $\mathsf{tilt}\,\Lambda$ consisting of n-term complexes, and by $\mathsf{n}\text{-}\mathsf{ptilt}\,\Lambda$ the set of isomorphism classes of indecomposable n-term pretilting complexes in $\mathcal{T}$.

In general, a Bongartz-type lemma does not hold for pretilting complexes. Namely, for some pretilting complex T in an algebra A, there exists no complex U such that $T \oplus U$ is a tilting complex. However, for two-term tilting complexes, we have the following result. We denote by $|P|$ the number of nonisomorphic indecomposable direct summands of $P \in \mathcal{T}$.

Lemma 2.3. [Ai1, Proposition 2.16] *Let T be a two-term pretilting complex. Then there exists a complex U such that $T \oplus U$ is a two-term tilting complex. In particular, T is a two-term tilting complex if and only if $|T| = |\Lambda|$. Moreover,* $2\text{-}\mathsf{tilt}\,\Lambda$ *is finite if and only if* $2\text{-}\mathsf{ptilt}\,\Lambda$ *is finite.*

We collect some results which are necessary in this report. We denote by $K_0(\mathcal{T})$ the Grothendieck group of $\mathcal{T}$ with the basis $[P_1], [P_2], \dots, [P_{|\Lambda|}]$.

Proposition 2.4. [AIR, Proposition 2.5 and 5.5] *Let T be a two-term pretilting complex.*

(1) *We have* $\mathrm{add}\, T^0 \cap \mathrm{add}\, T^{-1} = 0$.

(2) *The map $T \mapsto [T^0] - [T^1]$ induces an injection from the set of isomorphism classes of two-term pretilting complexes in $\mathcal{T}$ to $K_0(\mathcal{T})$.*

We recall the definition of mutation. Let U be an object in $\mathcal{T}$ and $f : X \to U'$ a morphism in $\mathcal{T}$. We say that f is a *left* $\mathrm{add}\, U$*-approximation* of X if U' belongs to $\mathrm{add}\, U$ and $\mathrm{Hom}_{\mathcal{T}}(f, U'')$ for any $U'' \in \mathrm{add}\, U$. Moreover, it is called *minimal left* $\mathrm{add}\, U$*-approximation* of X if any morphism $g : U' \to U'$ satisfying $gf = f$ is an isomorphism. Dually, we define a *(minimal) right* $\mathrm{add}\, U$*-approximation.*

Let T be a basic tilting complex in $\mathcal{T}$ and decompose $T = X \oplus U$. We take a triangle

$$X \xrightarrow{f} U' \longrightarrow Y \longrightarrow X[1],$$

where f is a minimal left $\mathrm{add}\, U$-approximation of X: It is possible because $\mathrm{Hom}_{\mathcal{T}}(X, U)$ is finite dimensional. We call $\mu_X^+(T) := Y \oplus U$ a *left mutation* of T with respect to X. Dually, we define a *right mutation* $\mu_X^-(T)$. A mutation means a left or right mutation. In addition, we say that it is *irreducible* if X is indecomposable.

Proposition 2.5. [AI, Theorem 2.31] *If T is a basic tilting complex, then so is $\mu_X^{\pm}(T)$.*

We visualize the structure of mutations.

Definition 2.6. (1) The tilting quiver $\mathcal{Q}_\Lambda$ of Λ is defined as follows:

- The vertex set is $\mathsf{tilt}\,\Lambda$.
- We draw an arrow $T \to U$ if U is a irreducible left mutation of T.

(2) A symmetric algebra is said to be *tilting-connected* if the tilting quiver is connected.

A class of tilting-connected symmetric algebras is very important because all tilting complexes are obtained from Λ by iterated mutations. We remark that examples of symmetric algebras which are not tilting-connected was given by Aihara-Grant-Iyama. Thus we have a natural question.

Question 2.7. *When is a symmetric algebra tilting-connected?*

It is difficult to check that a given symmetric algebra is tilting-connected or not. However, it is known that the following class of algebras is a reasonable class of tilting-connected algebras.

Definition 2.8. We say that Λ is *tilting-discrete* if for each positive integer n, the set $\mathsf{n\text{-}tilt}\,\Lambda$ is finite.

Proposition 2.9. [Ai1, Corollary 3.9] *If Λ is tilting-discrete, then it is tilting-connected.*

We give a criterion of tilting-discreteness for a given symmetric algebra. We say that Λ is *locally tilting-discrete* if $\mathsf{2\text{-}tilt}\,\Lambda$ is a finite set. A connected component of $\mathcal{Q}_\Lambda$ is said to be *canonical* if it contains Λ.

Proposition 2.10. [AAC, AM] *Let Λ be a symmetric algebra. If the endomorphism algebra $\mathrm{End}_{\mathcal{T}}(T)$ of each tilting complex T in the canonical component of $\mathcal{Q}_\Lambda$ is locally tilting-discrete, then Λ is tilting-discrete.*

At the end of this section, we give a remark on tilting quivers.

Remark 2.11. [AI, Ai1] The set $\mathsf{tilt}\,\Lambda$ has the structure of a partially ordered set: For two objects $T, U \in \mathsf{tilt}\,\Lambda$, we write $M \geq N$ if $\mathrm{Hom}_{\mathcal{T}}(M, N[>0]) = 0$. Then we have

$$\mathsf{n\text{-}tilt}\,\Lambda = \{T \in \mathsf{tilt}\,\Lambda \mid \Lambda \geq T \geq \Lambda[n-1]\}.$$

Moreover, the tilting quiver coincides with the Hasse quiver of $\mathsf{tilt}\,\Lambda$.

3 Brauer graph algebras

Our aim of this section is to give a graph theoretic interpretation of mutations of tilting complexes for a Brauer graph algebra.

Throughout of this report, a graph is always assumed to be connected. A finite graph is said to be *locally embedded* if for each vertex v there exists a cyclic ordering of the edges incident with v, described by the counter-clockwise direction of the plane. A *Brauer graph* is a locally embedded graph G with a multiplicity function $m : V \to \mathbb{Z}_{>0}$, where V is the vertex set of G. A vertex v is called *exceptional* if $m(v) = 1$.

We recall the definition of Brauer graph algebras.

Definition 3.1. Let $G = (G, m)$ be a Brauer graph.

(1) We define a finite quiver Q_G given by a Brauer graph as follows.

- There exists a one-to-one correspondence between the vertices of Q_G and the edges of G.
- For two distinct vertices i and j in Q_G corresponding to edges e_i and e_j in G, there is an arrow $i \to j$ in Q_G if the edges e_j is a direct successor of the edges e_i in the cyclic ordering around a common vertex in G. If the valency of an endpoint of e_i is equal to one, there is an arrow $i \to i$ in Q_G.

(2) We call the algebra $\Lambda_G = KQ_G/I_G$ the *Brauer graph algebra* of G, where I_G is the ideal generated by the following relations:

- We call the cycle in Q_G associated with the cyclic ordering of edges around a vertex a *Brauer cycle*. We denote by $C_{v,e}$ the Brauer cycle starting at the vertex in Q_G corresponding to an edge e incident with v. Let $C_{v,e}^{m(v)}$ be the $m(v)$-th power of $C_{v,e}$. Then for an edge e with the endpoints u, v in G, we have the relation $C_{u,e}^{m(u)} - C_{v,e}^{m(v)}$.
- All paths of length two where respective two arrows belong to different Brauer cycles are relations.

(3) A Brauer graph algebra is called a Brauer graph algebra *of type odd* if G includes at most one odd-cycle and no even cycle. A Brauer graph algebra is called a *generalized Brauer tree algebra* if G is a tree. A generalized Brauer tree algebra is called a *Brauer tree algebra* if there exists at most one exceptional vertex.

Example 3.2. Let G be the Brauer graph

$$a \overset{1}{\text{———}} b \overset{2}{\text{———}} c$$

with the multiplicity function $m(a) = 1, m(b) = 2, m(c) = 3$. Then the Brauer algebra $\Lambda_G = KQ_G/I_G$ is isomorphic to KQ'_G/I'_G, where Q_G, I_G, Q'_G, I'_G are given by

$$Q_G = \begin{tikzcd} 1 \arrow[loop above, "\alpha_1"] \arrow[r, shift right, "\gamma"'] & 2 \arrow[loop above, "\alpha_2"] \arrow[l, shift right, "\beta"'] \end{tikzcd}, \quad I_G = \langle \alpha_1 - (\gamma\beta)^2,\ (\beta\gamma)^2 - \alpha_2^3,\ \beta\alpha_1,\ \alpha_1\gamma,\ \alpha_2\beta,\ \gamma\alpha_2 \rangle,$$

$$Q'_G = \begin{tikzcd} 1 \arrow[r, shift right, "\gamma"'] & 2 \arrow[loop above, "\alpha_2"] \arrow[l, shift right, "\beta"'] \end{tikzcd}, \quad I'_G = \langle (\beta\gamma)^2 - \alpha_2^3,\ \beta(\gamma\beta)^2,\ (\gamma\beta)^2\gamma,\ \alpha_2\beta,\ \gamma\alpha_2 \rangle.$$

We collect some results for Brauer graph algebras.

Proposition 3.3. (1) [Ro, An] *Every Brauer graph algebra is a symmetric special biserial algebra, and vice versa.*

(2) *A Brauer graph algebra is representation-finite if and only if it is a Brauer tree algebra.*

It is well-known that each indecomposable nonprojective module of a special biserial algebra is either a string module or a band module. If an indecomposable module M is a *band* module, then the corresponding minimal projective presentation P_M is not pretilting. Hence, for an indecomposable two-term complex $P \in \mathcal{T}$, if the 0-th cohomology $H^0(P)$ is band, then P is not pretilting. In this report, since we are interested in (pre)tilting complexes, we will not give any more details about band modules and concentrate only in the string modules.

Definition 3.4. An indecomposable nonprojective Λ_G-module M is a *string* module if its minimal projective presentation can be written in one of the following forms:

(I)
$$\begin{tikzcd}
P_{e_{i_1}} \arrow[dr, "d_{1,1}"] & \\
& P_{e_{j_1}} \\
P_{e_{i_2}} \arrow[ur, "d_{1,2}"] \arrow[dr, "d_{2,2}"] & \\
\vdots & P_{e_{j_2}} \\
P_{e_{i_{n-1}}} \arrow[dr, "d_{m,n-1}"] & \vdots \\
& P_{e_{j_m}} \\
P_{e_{i_n}} \arrow[ur, "d_{m,n}"] &
\end{tikzcd}$$

(II)
$$\begin{tikzcd}
P_{e_{i_1}} \arrow[dr, "d_{1,1}"] & \\
& P_{e_{j_1}} \\
P_{e_{i_2}} \arrow[ur, "d_{1,2}"] \arrow[dr, "d_{2,2}"] & \\
\vdots & P_{e_{j_2}} \\
& P_{e_{j_{m-1}}} \\
P_{e_{i_n}} \arrow[ur, "d_{m-1,n}"] \arrow[dr, "d_{m,n}"] & \\
& P_{e_{j_m}}
\end{tikzcd}$$

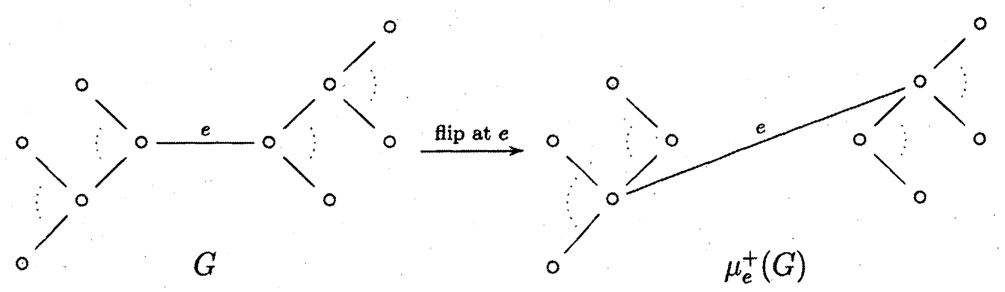
(III)
$$\begin{tikzcd}
& P_{e_{j_1}} \\
P_{e_{i_1}} \arrow[ur, "d_{1,1}"] \arrow[dr, "d_{2,1}"] & \\
& P_{e_{j_2}} \\
P_{e_{i_2}} \arrow[ur, "d_{1,2}"] & \vdots \\
\vdots & P_{e_{j_{m-1}}} \\
P_{e_{i_n}} \arrow[ur, "d_{m-1,n}"] \arrow[dr, "d_{m,n}"] & \\
& P_{e_{j_m}}
\end{tikzcd}$$

where $d_{j,i}$ is a nonzero morphism in $\mathsf{mod}\Lambda$.

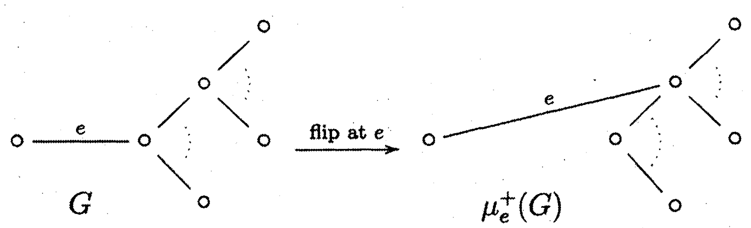
Next, we recall the notion of flip of locally embedded graphs. For the definition of flip, we refer to [Ai2, Ai3]. Roughly speaking, flip is an operation to construct a new locally embedded graph from a given one by replacing an edge. For an edge e of G, we denote by $\mu_e^+(G)$ the locally embedded graph obtained by the flip at e. We give typical examples of flip.

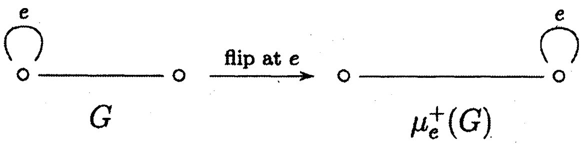
Example 3.5. Let e be an edge of G.

(1)



(2)



(3) 

Dually, we define $\mu_e^-(G)$ by $\mu_e^-(G) := (\mu_e^+(G^{\mathrm{op}}))^{\mathrm{op}}$, where G^{op} is the opposite Brauer graph (*i.e.*, its cyclic ordering is described by clockwise).

In general, flip has the following properties.

Lemma 3.6. *Let G be a locally embedded graph and e an edge of G.*

(1) *We have $\mu_e^+\mu_e^-(G) = G = \mu_e^-\mu_e^+(G)$.*

(2) *If G includes at most one odd-cycle and no even-cycle, then so is $\mu_e^{\pm}(G)$. In particular, G is a tree if and only if so is $\mu_e^{\pm}(G)$.*

In the context of Brauer graph algebras, mutation of tilting complexes can be interpreted as flip of edges of a Brauer graph.

Theorem 3.7. [Ai3, Theorem 5.8] *Let G be a Brauer graph and e an edge of G. Let P_e be an indecomposable projective Λ_G-module corresponding to e. Then we have an isomorphism*

$$\mathrm{End}_{\mathcal{T}}(\mu_{P_e}^{\pm}(\Lambda_G)) \simeq \Lambda_{\mu_e^{\pm}(G)}.$$

4 Main result

In this section, we give a proof of the following theorem.

Theorem 4.1. [AAC] *A Brauer graph algebra of type odd is tilting-discrete.*

Let G be a Brauer graph, and $\Lambda = \Lambda_G$ the Brauer graph algebra. To prove Theorem 4.1, we have only to show that $\mathrm{End}_{\mathcal{T}}(T)$ is locally tilting-discrete for each two-term tilting complex T in the canonical component of $\mathcal{Q}_\Lambda$ by Proposition 2.10.

First, we give a combinatorial description of indecomposable two-term pretilting complexes in Λ. In this report, we regard a walk of G as a sequence of edges of G. A *signed walk* of G is a walk $(e_1, e_2, \ldots, e_n)$ with a map $\epsilon : \{e_1, e_2, \ldots, e_n\} \to \{+1, -1\}$ such that $\epsilon(e_{i+1}) = -\epsilon(e_i)$ for all $i \in 1, 2, \ldots, n-1$. We denote by $\mathsf{SW}(G)$ the set of signed walks of G (up to reflection).

Proposition 4.2. *There is an injection*

$$\text{2-ptilt}\,\Lambda \longrightarrow \mathsf{SW}(G).$$

Proof. For an indecomposable two-term pretilting complex $T = (T^{-1} \to T^0)$, we define a walk W_T as follows: First, assume that $T^{-1} = 0$. Since T is indecomposable, there is an edge e such that $T^0 = P_e$. Then $W_T := (e^+)$. Similarly, if $T^0 = 0$, then there exists an edge e such that $T^{-1} = P_e$, and hence $W_T := (e^-)$. Next, assume that both $T^0 \neq 0$ and

$T^{-1} \neq 0$ hold. Since the 0-th cohomology $H^0(T)$ is a string module, W_T is given by the following tree types:

$$\begin{aligned}
\text{(I) } W_T &:= (e_{i_1}^-, e_{j_1}^+, \dots, e_{i_{n-1}}^-, e_{j_m}^+, e_{i_n}^-) \text{ if } H^0(T) \text{ is of type (I)},\\
\text{(II) } W_T &:= (e_{i_1}^-, e_{j_1}^-, \dots, e_{j_{m-1}}^+, e_{i_n}^-, e_{j_m}^+) \text{ if } H^0(T) \text{ is of type (II)},\\
\text{(III) } W_T &:= (e_{j_1}^+, e_{i_1}^-, \dots, e_{j_{m-1}}^+, e_{i_n}^-, e_{j_m}^+) \text{ if } H^0(T) \text{ is of type (III)}.
\end{aligned}$$

Since T^0 and T^{-1} have no nonzero direct summands in common by Proposition 2.4(1), W_T is a signed walk, and hence the map $T \mapsto W_T$ is well-defined. Moreover, we can easily check that it is an injection by Proposition 2.4(2). □

If $\mathsf{SW}(G)$ is finite, then $\mathsf{2\text{-}ptilt}\,\Lambda$ is clearly finite by Proposition 4.2, and hence Λ is locally tilting-discrete by Lemma 2.3.

Lemma 4.3. *If G includes at most one odd-cycle and no even-cycle, then $\mathsf{SW}(G)$ is finite.*

Proof. Assume that G is a locally embedded graph including at most one odd-cycle and no even-cycle. By the definition of signature ϵ, the same edge never appear more than once in a signed walk. Thus length of a signed walk is at most the number of edges of G. Hence the assertion follows from that G is finite. □

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let T be a two-term tilting complex in the canonical component of $\mathcal{Q}_\Lambda$ and $\Gamma := \mathrm{End}_{\mathcal{T}}(T)$. Then there is a finite sequence of mutations between Λ and T. By Lemma 3.6(2) and Proposition 3.7, if Λ is a Brauer graph algebra of type odd, then so is Γ. Hence Γ is locally tilting-discrete by Proposition 4.2 and Lemma 4.3. Thus the assertion follows from that Proposition 2.10. □

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