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Loewy Structure as a q-analog of Composition Factors

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Abstract

The Loewy structure of a module can be viewed as a q-analog of its composition factors. From this point of view we define q-composition multiplicity, q-composition length, and the q-Cartan matrix. By way of example, group algebras of finite p-groups and path algebras of finite acyclic quivers are investigated. Some known results for these algebras are stated in terms of the q-Cartan matrix.

Keywords Loewy structure, Composition factor, q-analog, Cartan matrix, Group algebra, the Jennings theorem, Path algebra.

Mathematical Subject Classification (MSC) 16G10; 20C20, 16G20, 05E10.

1 Introduction

The Loewy structure of a module enables us to visualize and gain some intuition about the module. In fact, such visualization is enough to obtain submodules or homological invariants, provided sufficient additional information is available [1, 3]. To establish Loewy structures several studies has been done such as [2, 8, 11]. We explore another way of dealing with Loewy structures in this report.

2 Notation and Terminology

In this section, we define q-composition multiplicity, q-composition length, and the q-Cartan matrix. We follow the notation and terminology of [10] unless otherwise stated. The term module refers to a finitely generated right module.

Definition 2.1. Let V be a module over a right artinian ring. The Jacobson radical of V is denoted by $\operatorname{rad} V$. For an integer $n \geq 0$, the *nth radical* of V is defined inductively by $\operatorname{rad}^0 V = V$ and

$$\operatorname{rad}^n V = \operatorname{rad}(\operatorname{rad}^{n-1} V)$$

if $n > 0$. We then write

$$\operatorname{rad}_n V = \operatorname{rad}^n V / \operatorname{rad}^{n+1} V$$

and call it the *nth radical layer* of V. (Note that these are different from the custom.) The decomposition of semisimple modules $\operatorname{rad}_n V$ into simple modules is referred to as the *Loewy structure* of V and may be visualized as (3.2) for instance.

Definition 2.2. Let R be a right artinian ring and $S_1, \ldots, S_k$ the representatives of simple R-modules. For an R-module V and its composition series

$$0 = V_0 < V_1 < \cdots < V_t = V \qquad (t \geq 0)$$

we call

$$c_i(V) := \#\{\, 1 \leq s \leq t \mid V_s/V_{s-1} \cong S_i \,\} \qquad (1 \leq i \leq k)$$

the *composition multiplicity* of S_i in V and $t = c_1(V) + \cdots + c_k(V)$ the *composition length* of V. The latter is denoted by $\ell(V)$. We then call its q-analog defined by

$$c_i(V; q) := \sum_{n \geq 0} c_i(\operatorname{rad}_n V) q^n \in \mathbb{Z}[q]$$

the *q-composition multiplicity* of S_i in V and $c_1(V; q) + \cdots + c_k(V; q)$ the *q-composition length* of V. The latter is denoted by $\ell(V; q)$.

Remark 2.3. The coefficient of q^n in (2.2) represents the number of times that the simple module S_i appears in the decomposition of $\operatorname{rad}_n V$. Hence the Loewy structure of V corresponds to the vector

$$^T\big(c_1(V;q), \ldots, c_k(V;q)\big).$$

For illustration, see (3.2) and (3.3). Since $c_i(V) = \lim_{q \to 1} c_i(V;q)$, the Loewy structure of a module can be viewed as a q-analog of its composition factors.

Now it is possible to define a q-analog of anything that is defined in terms of composition multiplicity or composition length. We subsequently define and investigate a q-analog of the Cartan matrix.

Definition 2.4. Under the notation of Definition 2.2, let $P_1, \ldots, P_k$ be the projective covers of $S_1, \ldots, S_k$. We call

$$C_R := \big[c_i(P_j)\big]_{1 \le i,j \le k} \quad \text{and} \quad C_R(q) := \big[c_i(P_j;q)\big]_{1 \le i,j \le k}$$

the *Cartan matrix* of R and the *q-Cartan matrix* of R respectively.

Remark 2.5. Wilson [12] and Fuller [6] give more general definitions for similar concepts and call them *Cartan homomorphisms* for graded algebras and *F-filtered Cartan matrices* respectively in the context of the Cartan determinant conjecture. Bessenrodt and Holm [4] also give essentially the same definition for the factor algebra of a path algebra by a homogeneous ideal and call it a *q-Cartan matrix.*

The q-analogs defined above enable us to deal with Loewy structures algebraically and to ask questions involving terms of matrix theory such as determinant.

3 Group Algebras

Let us show an example that motivates the definition of these q-analogs.

Example 3.1. Let G be a group of order 4 and consider the group algebra FG over a field F of characteristic 2. The composition length is not sufficient to distinguish the difference between the isomorphism classes of groups of order 4; namely $C_2 \times C_2$ and C_4. The q-composition length, on the other hand, is sufficient to distinguish the difference satisfactorily.

$$
\begin{tikzcd}[column sep=small]
 & F \arrow[dl, no head] \arrow[dr, no head] & \\
F \arrow[dr, no head] & & F \arrow[dl, no head] \\
 & F &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[row sep=small]
F \arrow[d, no head] \\
F \arrow[d, no head] \\
F \arrow[d, no head] \\
F
\end{tikzcd}
$$

$F[C_2 \times C_2]$ $\qquad\qquad$ $F[C_4]$

$$
\begin{aligned}
\ell(F[C_2 \times C_2]) &= 4 & \qquad \ell(F[C_4]) &= 4 \\
\ell(F[C_2 \times C_2]; q) &= (1+q) \times (1+q) & \qquad \ell(F[C_4]; q) &= 1 + q + q^2 + q^3
\end{aligned}
$$

In fact, these polynomials are naturally obtained as the generating functions of the dimensions of radical layers. This, of course, does not happen by chance; We restate the Jennings theorem in terms of q-composition length or the q-Cartan matrix and give some remarks.

Theorem 3.2 (Jennings [7, Theorem 3.7]). *Let p be a prime number, G a finite p-group, and F a field of characteristic p. Set*

$$K_n := \{\, g \in G \mid g - 1 \in \operatorname{rad}^n FG \,\}$$

for an integer $n \geq 0$. Then K_n / K_{n+1} is an elementary abelian p-group of rank $r_n \geq 0$ and

$$\ell(FG; q) = \prod_{n \geq 1} \left(\frac{1 - q^{np}}{1 - q^n} \right)^{r_n} = \det C_{FG}(q). \tag{3.1}$$

Remark 3.3. Let F be a field of characteristic $p > 0$ and G a finite group such that p divides its order. In modular representation theory of finite groups, it is well-known that the dimension of a projective FG-module is divisible by the order of the Sylow p-subgroup of G [9, Theorem 3.1.26]. The first equality of (3.1) can be viewed as a q-analog of this theorem. It is also well-known that $\det C_{FG}$ is a power of p [9, Lemma 3.6.31]. The second equality of (3.1) can be viewed as a q-analog of this theorem.

Therefore it is expected that some properties of Cartan matrices also hold for q-Cartan matrices. Unfortunately, well-known facts about Cartan matrices of group algebras are no longer hold naïvely for q-Cartan matrices.

For example, the group algebra FA_5 of the alternating group A_5 of degree 5 over an algebraically closed field F of characteristic 2 has the Loewy structures [5, p.52] and q-Cartan matrix below.

$$
\begin{tikzcd}[column sep=small]
 & S_1 \arrow[dl, dash] \arrow[dr, dash] & & S_2 \arrow[d, dash] & S_3 \arrow[d, dash] & \\
S_2 \arrow[d, dash] & & S_3 \arrow[d, dash] & S_1 \arrow[d, dash] & S_1 \arrow[d, dash] & \\
S_1 \arrow[d, dash] & & S_1 \arrow[d, dash] & S_3 \arrow[d, dash] & S_2 \arrow[d, dash] & \\
S_3 \arrow[dr, dash] & & S_2 \arrow[dl, dash] & S_1 \arrow[d, dash] & S_1 \arrow[d, dash] & \\
 & S_1 & & S_2 & S_3 & S_4 \\
 & P_1 & & P_2 & P_3 & P_4
\end{tikzcd} \tag{3.2}
$$

$$
C_{FA_5}(q) = \begin{bmatrix} 1+2q^2+q^4 & q+q^3 & q+q^3 & 0 \\ q+q^3 & 1+q^4 & q^2 & 0 \\ q+q^3 & q^2 & 1+q^4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}
$$

Hence $\ell(P_2; q)$ and $\det C_{FA_5}(q)$ are not divisible by $1+q$ in $\mathbb{Z}[q]$, unlike the p-group case.

Furthermore, in general q-Cartan matrices of group algebras need not be symmetric as Cartan matrices must [9, Theorem 2.8.21.ii], although it is the case in (3.3). For example, the group algebra FG of the group G defined by

$$G = \begin{bmatrix} 1 & \mathbb{F}_p \\ 0 & {\mathbb{F}_p}^{\times} \end{bmatrix}$$

over a field F of characteristic p has the following q-Cartan matrix.

$$C_{FG}(q) = \begin{bmatrix} 1+q^{p-1} & q & \dots & q^{p-2} \\ q^{p-2} & 1+q^{p-1} & \dots & q^{p-3} \\ \vdots & \vdots & \ddots & \vdots \\ q & q^2 & \dots & 1+q^{p-1} \end{bmatrix} \tag{3.4}$$

An appropriate generalization is expected.

4 Path Algebras

We give a combinatorial interpretation of q-Cartan matrices for path algebras in Theorem 4.2. Let us begin with a simple example to observe how it should look.

Example 4.1. Let F be a field and Q the quiver defined by the following.

$$Q \quad = \quad 1 \quad \leftarrow \quad 2 \quad \leftarrow \quad \cdots \quad \leftarrow \quad k$$

The path algebra of Q over the field F is denoted by FQ. Write S_i for the simple FQ-module that corresponds to a vertex $1 \leq i \leq k$. The projective covers P_i of S_i have the Loewy structures described below.

$$\begin{array}{cccc}
 & & & S_k \\
 & & & | \\
 & & & \vdots \\
 & & & | \\
 & S_2 & & S_2 \\
 & | & \cdots & | \\
S_1 & S_1 & & S_1 \\
 & & & \\
P_1 & P_2 & \cdots & P_k
\end{array}$$

Hence FQ has the following Cartan matrix and q-Cartan matrix.

$$C_{FQ} = \begin{bmatrix} 1 & 1 & & 1 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{bmatrix} \qquad C_{FQ}(q) = \begin{bmatrix} 1 & q & & q^{k-1} \\ & 1 & \ddots & \\ & & \ddots & q \\ & & & 1 \end{bmatrix}$$

Readers might see in Example 4.1 that the coefficient of q^n of the (i,j)-entry of $C_{FQ}(q)$ agrees with the number of paths from j to i of length n. This phenomenon is generalized as follows. Recall that for a finite quiver Q the matrix with each of its (i,j)-entries equaling the number of arrows from i to j is called the *adjacency matrix* of Q.

Theorem 4.2. *Let F be a field and Q a finite acyclic quiver with vertex set $\{1,\dots,k\}$ and adjacency matrix A. Write S_i and P_i for the simple FQ-modules and their projective covers corresponding to vertices $1 \le i \le k$ respectively. Then the q-Cartan matrix $C_{FQ}(q) = \left[c_i(P_j;q)\right]$ can be expressed as the power series*

$$C_{FQ}(q) = \sum_{n\ge 0} ({}^{T}A)^n q^n. \tag{4.1}$$

Remark 4.3. The $q = 1$ case of (4.1) can be found in [10, Corollary I.11.6].

References

[1] Alperin, J. L. Diagrams for modules. *J. Pure Appl. Algebra 16*, 2 (1980), 111–119.

[2] Benson, D. The Loewy structure of the projective indecomposable modules for A_8 in characteristic 2. *Comm. Algebra 11*, 13 (1983), 1395–1432.

[3] Benson, D. J., and Carlson, J. F. Diagrammatic methods for modular representations and cohomology. *Comm. Algebra 15*, 1-2 (1987), 53–121.

[4] Bessenrodt, C., and Holm, T. q-Cartan matrices and combinatorial invariants of derived categories for skewed-gentle algebras. *Pacific J. Math. 229*, 1 (2007), 25–47.

[5] Carlson, J. F. *Modules and group algebras.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Notes by Ruedi Suter.

[6] Fuller, K. R. The Cartan determinant and global dimension of Artinian rings. In *Azumaya algebras, actions, and modules (Bloomington, IN, 1990)*, vol. 124 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 1992, pp. 51–72.

[7] Jennings, S. A. The structure of the group ring of a p-group over a modular field. *Trans. Amer. Math. Soc. 50* (1941), 175–185.

[8] Koshitani, S. On the Loewy series of the group algebra of a finite p-solvable group with p-length > 1. *Comm. Algebra 13*, 10 (1985), 2175–2198.

[9] Nagao, H., and Tsushima, Y. *Representations of finite groups.* Academic Press, Inc., Boston, MA, 1989. Translated from the Japanese.

[10] Skowroński, A., and Yamagata, K. *Frobenius algebras. I: Basic representation theory.* Zürich: European Mathematical Society (EMS), 2011.

[11] WAKI, K. The Loewy structure of the projective indecomposable modules for the Mathieu groups in characteristic 3. *Comm. Algebra 21*, 5 (1993), 1457–1485.

[12] WILSON, G. V. The Cartan map on categories of graded modules. *J. Algebra 85*, 2 (1983), 390–398.

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