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Heegaard Floer homology for embedded bipartite graphs

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1 Introduction

Heegaard Floer theory defines topological invariants for many low-dimensional topological objects. The word "Heegaard" indicates that it is defined on a Heegaard diagram associated with the given topological object, and the word "Floer" comes from the fact that it is a special case of the Lagrangian intersection Floer homology, where the symplectic manifold and its Lagrangian pair are constructed from the given Heegaard diagram. In this note, we introduce a definition of the Heegaard Floer homology for an embedded balanced bipartite graph in a closed oriented 3-manifold. Details of the definition are included in the preprint [2]. For simplicity, we assume that the ambient manifold M is a rational homology 3-sphere.

In Section 2, we review the definition of a Heegaard diagram for M and that for a link in M. Then we show how to generalize the definition to a balanced bipartite graph in M. In Section 3, we brief the definition of the Heegaard Floer complex for M and for a link in it. Among various versions of the complex, we focus on the minus-version since it is the most useful case and can be used to reconstruct the other versions. Then we show a generalization of the definition to a balanced bipartite graph in M. In the last section, we discuss two combinatorial aspects of the theory, the definitions based on grid diagram and Kauffman state.

The topological invariance of the Heegaard Floer homology will not be discussed here. Please refer to the original papers for the proofs.

2 Heegaard diagram for a 3-manifold, a link, or a balanced bipartite graph

2.1 For a 3-manifold

Given a closed oriented 3-manifold M, let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a Heegaard diagram for M, where Σ is a genus g closed oriented surface, and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \cdots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots, \beta_g\}$

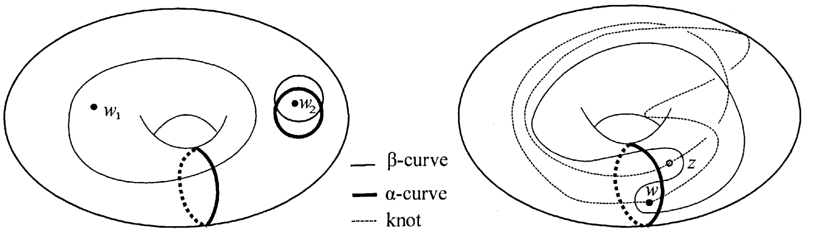

図 1: A 2-pointed Heegaard diagram for the 3-sphere (left figure), and a single-pointed Heegaard diagram for the right-handed trefoil knot (right figure).

are two sets of pairwisely disjoint simple closed curves on Σ. Ozsváth and Szabó [7] defined a topologicial invariant, called Heegaard Floer homology, for a 3-manifold by using its Heegaard diagram. In their definition, an extra datum $w \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, which is called a base point, is needed to make the invariant non-trivial. The data $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ is called a single-pointed Heegaard diagram for M.

It is usually convenient to define the chain complex on a multi-pointed Heegaard diagram, which can be obtained from a single-pointed Heegaard diagram by applying $(0,3)$ stabilizations. The definition is as follows.

Definition 2.1. Let $n \geq 1$ be an integer. An n-pointed Heegaard diagram for M is a quartet $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$, which satisfies the following conditions.

- Σ is a closed oriented genus g surface, which is called the Heegaard surface, and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \cdots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots, \beta_d\}$ are two sets of d pairwisely disjoint simple closed curves on Σ, where $n = d - g + 1$.

- Attaching 2-handles to Σ along curves in $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$), we get an n-punctured genus g handlebody $U_{\boldsymbol{\alpha}}$ (resp. $U_{\boldsymbol{\beta}}$). The union $U_{\boldsymbol{\alpha}} \bigcup_{\Sigma} U_{\boldsymbol{\beta}}$ is the 3-manifold M with $2n$-punctures. The orientation of Σ is induced from that of $U_{\boldsymbol{\alpha}}$, which in turn coincides with that of M.

- Let $\{A_i\}_{i=1}^n$ (resp. $\{B_i\}_{i=1}^n$) be the connected components of $\Sigma \setminus \boldsymbol{\alpha}$ (resp. $\Sigma \setminus \boldsymbol{\beta}$). Then $\boldsymbol{w} = \{w_1, w_2, \cdots, w_n\}$ is a set of n points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ so that $w_i \in A_i \cap B_i$ (by relabelling $\{B_i\}_{i=1}^n$ if necessary), which are called the base points.

See Figure 1 (left) for an example of 2-pointed Heegaard diagram for the 3-sphere. A convenient way to understand the construction of a Heegaard diagram is to consider its corresponding Morse function. Choose a generic Riemannian metric $\mathfrak{g}$ on M and suppose $f : M \to \mathbb{R}$ is a self-indexed Morse function. Then $(f, \mathfrak{g})$ induces a multi-pointed Heegaard diagram for M, where the Heegaard surface $\Sigma := f^{-1}(3/2)$ and $\boldsymbol{\alpha}$ (resp. $\boldsymbol{\beta}$)

is the set of intersection curves of Σ with the ascending disks (resp. descending disks) of the index one (resp. two) critical points. The base points in $\boldsymbol{w}$ are chosen in a way that each component of $\Sigma \setminus \boldsymbol{\alpha}$ or $\Sigma \setminus \boldsymbol{\beta}$ contains exactly one base point. For any point $p \in \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, there exists a path $\gamma_p \subset M$ from an index zero critical point to an index three one so that $\mathfrak{g}(\dot{\gamma_p}, \cdot) = df$. Namely it is the flow line passing through p (see Figure 2 for an illustration). Since the base points come from different components of $\Sigma \setminus \boldsymbol{\alpha}$ or $\Sigma \setminus \boldsymbol{\beta}$, it is easy to see that $\{\gamma_p\}_{p \in \boldsymbol{w}}$ are pairwisely disjoint simple paths.

2.2 For a link

Consider an oriented link $L \subset M$. An n-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ for (M, L) is defined by a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric $\mathfrak{g}$ so that i) $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ is an n-pointed Heegaard diagram for M, ii) $\boldsymbol{z}$ is a finite set of points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{w})$ for which $L = \cup_{p \in \boldsymbol{w} \cup \boldsymbol{z}} \gamma_p$, and iii) the orientation of L makes the paths $\{\gamma_p\}_{p \in \boldsymbol{w}}$ direct downwards and $\{\gamma_p\}_{p \in \boldsymbol{z}}$ upwards. Note that in this case we always have $|\boldsymbol{w}| = |\boldsymbol{z}|$, so there are totally $2n$ base points on the Heegaard surface. See Figure 2 for a schematic illustration of the construction, and also see Figure 1 (right) for an example of a single-pointed Heegaard diagram of the right-handed trefoil knot in S^3.

Remark 2.2. *Note that a single-pointed Heegaard diagram of a knot K is nothing but a 1-bridge decomposition of the knot (see [6] for the definition). The minimal Heegaard genus is a knot invariant which is closely related to the tunnel number of the knot. Nevertheless, as far as the author knows, no research is known about the relation between Heegaard Floer homology and these geometric invariants.*

2.3 For a bipartite graph

A graph G with the vertex set V and the edge set E is called a *bipartite graph* if V is a disjoint union of two non-empty sets V_1 and V_2 so that every edge in E is incident to both V_1 and V_2. We use G_{V_1,V_2} to denote the graph with the choice of (V_1, V_2). If $|V_1| = |V_2|$, the graph G_{V_1,V_2} is called *balanced.* Furthermore, we call an orientation for G_{V_1,V_2} *balanced* if there are $n := |V_1|$ edges $\{e_i\}_{i=1}^n$ directing from V_1 to V_2 and the endpoints of which occupy V_1 and V_2, and the other edges direct from V_2 to V_1. See Figure 4 (left) for an example, where the solid line edges are $\{e_1, e_2\}$. We assume that all the graphs in this paper have no isolated vertices and single-valency vertices.

Two smooth embeddings $f_i : G_{V_1,V_2} \hookrightarrow M$ are said to be *ambient isotopic* if there is a homeomorphism of M isotopic to the identity map, sending $f_1(G_{V_1,V_2})$ to $f_2(G_{V_1,V_2})$ and

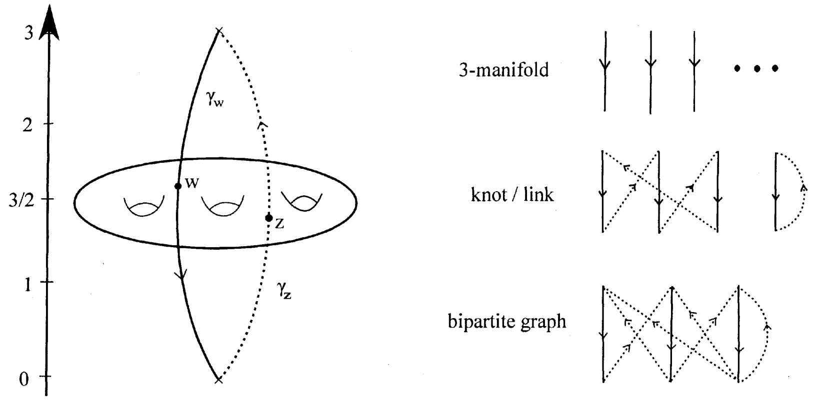

図 2: The schematic diagram for a Morse function of a 3-manifold. The flow lines constitute a link or a graph embedded in the manifold.

$f_1(V_i)$ to $f_2(V_i)$ for $i = 1, 2$. The Heegaard Floer homology to be defined is an invariant under the ambient isotopy.

To define the homology, we consider a multi-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w} := \{w_i\}_{i=1}^n, \boldsymbol{z})$ for (M, G_{V_1,V_2}) with a balanced orientation. It is defined by a Morse function $f : M \to \mathbb{R}$ and a Riemannian metric $\mathfrak{g}$ so that i) $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ is an n-pointed Heegaard diagram for M, ii) $\boldsymbol{z}$ is a finite set of points in $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{w})$ for which $G_{V_1,V_2} = \cup_{p \in \boldsymbol{w} \cup \boldsymbol{z}} \gamma_p$ and iii) the orientation of G_{V_1,V_2} makes $\{e_i\}_{i=1}^n = \{\gamma_p\}_{p \in \boldsymbol{w}}$ direct downwards and $\{\gamma_p\}_{p \in \boldsymbol{z}}$ upwards. Since G_{V_1,V_2} has no isolated vertices and single-valency vertices, we have $|\boldsymbol{z}| \geq |\boldsymbol{w}|$, but the position of $\boldsymbol{w}$ and $\boldsymbol{z}$ can be complicated. See Figure 2 for an illustration.

3 The Heegaard Floer complex $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$

3.1 Nearly symmetric almost complex structures

Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ be an n-pointed Heegaard diagram of M as in Def. 2.1. Consider the d-fold symmetric product of Σ,

$$\mathrm{Sym}^d(\Sigma) = \Sigma^{\times d}/S_d,$$

where S_d is the symmetric group of degree d, and let

$$\mathbb{T}_{\boldsymbol{\alpha}} = \alpha_1 \times \alpha_1 \times \cdots \times \alpha_d \text{ and } \mathbb{T}_{\boldsymbol{\beta}} = \beta_1 \times \beta_1 \times \cdots \times \beta_d.$$

Then $\mathrm{Sym}^d(\Sigma)$ is a $2d$-dimensional smooth manifold, whose local coordinate chart can be defined by the correspondence between coefficients and roots of a d-variable complex polynomial. Then $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ are two d-dimensional submanifolds of $\mathrm{Sym}^d(\Sigma)$. Suppose that $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ are in general position.

Let $(\eta, \mathfrak{j})$ be a Kähler form on Σ. Then it induces a Kähler form $(\eta^{\times d}, \mathfrak{j}^{\times d})$ on $\Sigma^{\times d}$. Consider the quotient map $\pi : \Sigma^{\times d} \to \mathrm{Sym}^d(\Sigma)$. The complex structure $\mathfrak{j}$ gives rise to a complex structure $\mathrm{Sym}^d(\mathfrak{j})$ on $\mathrm{Sym}^d(\Sigma)$ for which π is a holomorphic map. Let D be the diagonal of $\mathrm{Sym}^d(\Sigma)$. Namely

$$D := \{\{x_1, x_2, \cdots, x_n\} \in \mathrm{Sym}^d(\Sigma) | x_i = x_j \text{ for some } i \neq j\}.$$

Then π induces a covering map away from the diagonal. Since $\eta^{\times d}$ is invariant under the action of S_d, we get a Kähler form $(\mathrm{Sym}^d(\eta), \mathrm{Sym}^d(\mathfrak{j}))$ on $\mathrm{Sym}^d(\Sigma) \setminus D$. Note that the submanifolds $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ keep away from the diagonal, and are two Lagrangian and totally real tori with respects to $(\mathrm{Sym}^d(\eta), \mathrm{Sym}^d(\mathfrak{j}))$.

For a finite set of points $\{p_\lambda\}_{\lambda \in \Lambda} \subset \Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$, find an open set $V \subset \mathrm{Sym}^d(\Sigma)$ keeping away from $\mathbb{T}_{\boldsymbol{\alpha}}$ and $\mathbb{T}_{\boldsymbol{\beta}}$ so that

$$\{p_\lambda\}_{\lambda \in \Lambda} \times \mathrm{Sym}^{d-1}(\Sigma) \cup D \subset V.$$

An almost complex structure J on $\mathrm{Sym}^d(\Sigma)$ is called $(\mathfrak{j}, \eta, V)$-nearly symmetric if i) J is compatible with $\mathrm{Sym}^d(\eta)$ on $\mathrm{Sym}^d(\Sigma) \setminus V$ and ii) $J = \mathrm{Sym}^d(\mathfrak{j})$ over V. The space of $(\mathfrak{j}, \eta, V)$-nearly symmetric almost complex structures is denoted $\mathcal{J}(\mathfrak{j}, \eta, V)$. Obviously $\mathrm{Sym}^d(\mathfrak{j}) \in \mathcal{J}(\mathfrak{j}, \eta, V)$.

3.2 The chain complex for a manifold

For an n-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$ of M, the definition of the chain complex $(\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}), \partial^-)$ is an analogue of that of Lagrangian intersection Floer complex. It is a free $\mathbb{F}[U_1, U_2, \cdots, U_n]$-module generated by the intersection points $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, where $\mathbb{F}[U_1, U_2, \cdots, U_n]$ is the n-variable polynomial ring with coefficient $\mathbb{F} := \mathbb{Z}/2\mathbb{Z}$. The differential is defined by counting pseudo-holomorphic disks connecting two generators. Precisely, for $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$,

$$\partial^-(x) = \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi) = 1\}} \sharp \widehat{\mathcal{M}}_{J_s}(\phi) \cdot U_1^{n_{w_1}(\phi)} U_2^{n_{w_2}(\phi)} \cdots U_n^{n_{w_n}(\phi)} y. \tag{1}$$

The notations $\pi_2(x, y)$, $\mu(\phi)$, $\widehat{\mathcal{M}}_{J_s}(\phi)$ and $n_{w_i}(\phi)$ are defined as follows. Let $\mathbb{D}$ be the unit disk on the complex plane. A smooth map

$$u : (\mathbb{D}, -i, i) \to (\mathrm{Sym}^d(\Sigma), x, y)$$

sending $\{s + it \in \partial\mathbb{D} | s \geq 0\}$ (resp. $\{s + it \in \partial\mathbb{D} | s \leq 0\}$) to $\mathbb{T}_{\boldsymbol{\alpha}}$ (resp. $\mathbb{T}_{\boldsymbol{\beta}}$) is called a Whitney disk from x to y. Let $\pi_2(x, y)$ be the set of homotopy classes of all Whitney disks from x to y. Then $\mu(\phi)$ is the Maslov index (also called the formal dimension or expected dimension in some literatures) of ϕ. Fix a path $J_s \subset \mathcal{J}(\mathfrak{j}, \eta, V)$, where $\{p_\lambda\}_{\lambda\in\Lambda}$ in the definition of V is chosen to be $\boldsymbol{w}$. Let

$$\mathcal{M}_{J_s}(\phi) = \big\{u: \text{ a whitney disk from } x \text{ to } y \;\big|\; [u] = \phi, \partial_s u + J_s \partial_t u = 0\big\}.$$

The translation action of $\mathbb{R}$ on $\mathbb{D}$ induces an action of $\mathbb{R}$ on $\mathcal{M}_{J_s}(\phi)$. Let $\widehat{\mathcal{M}}_{J_s}(\phi) := \mathcal{M}_{J_s}(\phi)/\mathbb{R}$, which is called the unparameterized moduli space of ϕ. Finally, $n_{w_i}(\phi) := \sharp\phi^{-1}(\{w_i\} \times \mathrm{Sym}^{d-1}(\Sigma))$ is called the local multiplicity of w_i in ϕ. When ϕ is a J_s-holomorphic representative, we have $n_{w_i}(\phi) \geq 0$ (Lemma 3.2 [7]).

Theorem 3.1 (Theorems 3.4, 3.18 [7]). *For a generic choice of $J_s \subset \mathcal{J}(\mathfrak{j}, \eta, V)$, we have*

1. *$\mathcal{M}_{J_s}(\phi)$ is an oriented manifold of dimension $\mu(\phi)$, and $\widehat{\mathcal{M}}_{J_s}(\phi)$ is an oriented manifold of dimension $\mu(\phi) - 1$, and*
2. *when $\mu(\phi) = 1$, $\widehat{\mathcal{M}}_{J_s}(\phi)$ is compact,*

for any $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $\phi \in \pi_2(x, y)$.

From this theorem, we see that for a generic J_s, the set $\widehat{\mathcal{M}}_{J_s}(\phi)$ appeared in $\partial^-(x)$ is a compact 0-dim manifold. Therefore we can count its number $\sharp\widehat{\mathcal{M}}_{J_s}(\phi)$ modulo two. Here we only consider the number modulo two to avoid the discussion of coherent orientation of the moduli spaces.

In order to make the right hand side of (1) a finite sum for each y, the Heegaard diagram is required to satisfy a technic condition, called weak admissibility. For details of the definition, please refer to [8, Section 3.4]. For this reason, we hereafter assume that the Heegaard diagrams are weakly admissible.

Finally we state that the differential defined in (1) respects $\mathrm{Spin}^c(M)$, the set of spin^c-structures of M. In the context of Heegaard Floer theory, a spin^c-structure indicates the homology class of a nowhere vanishing vector field over M. Two nowhere vanishing vector fields are said to be *homologous* if they are homotopic through nowhere vanishing vector fields on the complement of a 3-ball in M. The base points $\boldsymbol{w}$ provide a map

$$s_{\boldsymbol{w}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \mathrm{Spin}^c(M),$$

for the definition of which one can refer to [7, Section 3.3]. We have $s_{\boldsymbol{w}}(x) = s_{\boldsymbol{w}}(y)$, when y appears in the differential of x. In this way, we have the splitting

$$\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}) = \bigoplus_{s\in\mathrm{Spin}^c(M)} \mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}; s).$$

3.3 The chain complex for a link

Consider a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ for a link $L \subset M$. The additional base points $\boldsymbol{z}$ are used here to construct a filtration to the chain complex $(\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}), \partial^-)$. It is defined by the map

$$s_{\boldsymbol{w},\boldsymbol{z}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \underline{\mathrm{Spin}}^c(M, L),$$

where $\underline{\mathrm{Spin}}^c(M, L)$ is the set of relative spin^c-structures defined as follows. Let v be a nowhere vanishing unit vector field over $M \setminus \mathrm{int}(N(L))$ for which the restriction of v on $\partial N(L)$ coincides with the canonical vector field, which is defined by the condition that its orbit on each component of $\partial N(L)$ is a frame zero longitude of the link component. Let $\chi(M, L)$ be the set of such vector fields. Then v can be extended to M so that L is a closed orbit of the extension. Two elements in $\chi(M, L)$ are said to be *homologous relative to* L $(\sim_{(M,L)})$ if they are homotopic through nowhere vanishing vector fields on the complement of a 3-ball in $M \setminus \mathrm{int}(N(L))$. Let

$$\underline{\mathrm{Spin}}^c(M, L) := \chi(M, L)/ \sim_{(M,L)} .$$

For the construction of $s_{\boldsymbol{w},\boldsymbol{z}}$, please refer to [8, Section 3.6]. The extension of $v \in \chi(M, L)$ to M defines a map $\xi : \underline{\mathrm{Spin}}^c(M, L) \to \mathrm{Spin}^c(M)$. We have $\xi \circ s_{\boldsymbol{w},\boldsymbol{z}} = s_{\boldsymbol{w}}$. When L is null-homologous in M, we have an affine identification $\underline{\mathrm{Spin}}^c(M, L) \cong \mathrm{Spin}^c(M) \oplus \mathbb{Z}^l$, where l is the component number of L. The restriction on the second factor gives rise to a $\mathbb{Z}^l$-filtration to $(\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}), \partial^-)$.

The associated graded chain complex is denoted by $(\mathrm{CFL}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}), \partial_L^-)$. We see that $\mathrm{CFL}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is the same as $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$, a free $\mathbb{F}[U_1, U_2, \cdots, U_n]$-module generated by the intersection points $\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$. The differential is

$$\partial_L^-(x) = \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1, n_{\boldsymbol{z}}(\phi)=\{0\}\}} \sharp \widehat{\mathcal{M}}_{J_s}(\phi) \cdot U_1^{n_{w_1}(\phi)} U_2^{n_{w_2}(\phi)} \cdots U_n^{n_{w_n}(\phi)} y, \tag{2}$$

where $n_{\boldsymbol{z}}(\phi)$ is the set of local multiplicities at ϕ of points in $\boldsymbol{z}$.

3.4 The chain complex for a graph

In this section we give a paralleled description as Section 3.3 while keeping in mind the differences. For details about the contents in this section, please see [2]. Consider a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ for a balanced bipartite graph $G_{V_1,V_2} \subset M$. The additional base points $\boldsymbol{z}$ can be used here to construct a relative grading to the chain complex $(\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}), \partial^-)$. Similar with the case of link, it is defined by the map

$$s_{\boldsymbol{w},\boldsymbol{z}} : \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}} \to \underline{\mathrm{Spin}}^c(M, G_{V_1,V_2}),$$

where $\underline{\mathrm{Spin}}^c(M, G_{V_1,V_2})$ is the set of relative spin^c-structures. We give its definition below.

Let $X := M \setminus \mathrm{int}(N(G_{V_1,V_2}))$ and consider the decomposition $\partial X = \partial_+ X \cup N(\mathfrak{m}) \cup \partial_- X$, where $N(\mathfrak{m})$ is a tubular neighborhood of the meridian set $\mathfrak{m}$ of the edges on ∂X and $\partial_+ X$ (resp. $\partial_- X$) is the intersection of ∂X with a neighborhood of V_1 (resp. V_2) in M. Define v_G to be the unit vector field on $TX|_{\partial X}$ as follows: 1) $v_G|_{\partial_+ X} \perp T\partial_+ X$ and points outwards; 2) $v_G|_{\partial_- X} \perp T\partial_- X$ and points inwards; 3) $v_G|_{N(\mathfrak{m})} = \dfrac{\partial}{\partial t}(\mathfrak{m} \times \{t\})$ under the identification $N(\mathfrak{m}) = \mathfrak{m} \times [-1, 1]$. We perturb v_γ around $\partial N(\mathfrak{m})$ to make it continuous. Note that v_G only depends on the topology of G. Let v be a nowhere vanishing unit vector field over X for which $v|_{\partial X} = v_G$, and we denote the set of such vector fields $\chi(M, G_{V_1,V_2})$. Two elements in $\chi(M, G_{V_1,V_2})$ are said to be *homologous relative to* G_{V_1,V_2} ($\sim_{(M,G_{V_1,V_2})}$) if they are homotopic through nowhere vanishing vector fields on the complement of a 3-ball in X. Let

$$\underline{\mathrm{Spin}}^c(M, G_{V_1,V_2}) := \chi(M, G_{V_1,V_2}) / \sim_{(M,G_{V_1,V_2})} .$$

Note that there is a free and transitive action of $H_1(X;\mathbb{Z}) \cong H^2(X, \partial X; \mathbb{Z})$ on the set $\underline{\mathrm{Spin}}^c(M, G_{V_1,V_2})$. For $[v], [w] \in \underline{\mathrm{Spin}}^c(M, G_{V_1,V_2})$, let $[v] - [w] \in H_1(X;\mathbb{Z})$ be their difference.

Define the Alexander grading on $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$. It is a relative $H_1(X;\mathbb{Z})$-grading defined by the map $A : (\mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}) \cup \{U_i\}_{i=1}^n \to H_1(X;\mathbb{Z})$ which satisfies the relations

$$\begin{aligned} A(x) - A(y) &= s_{\boldsymbol{w},\boldsymbol{z}}(x) - s_{\boldsymbol{w},\boldsymbol{z}}(y), \text{ and} \\ A(U_i) &= [m_{w_i}]. \end{aligned}$$

for any $x, y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$, where m_{w_i} is the meridian of the edge that contains w_i.

The map ∂^- does not preserve the grading in general. We define a chain complex $(\mathrm{CFG}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}), \partial_G^-)$ that preserves the grading, where $\mathrm{CFG}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z})$ is the same as $\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w})$. The differential is

$$\partial_G^-(x) = \sum_{y \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}} \sum_{\{\phi \in \pi_2(x,y) | \mu(\phi)=1, n_{\boldsymbol{z}}(\phi)=\{0\}\}} \sharp \widehat{\mathcal{M}}_{J_s}(\phi) \cdot U_1^{n_{w_1}(\phi)} U_2^{n_{w_2}(\phi)} \cdots U_n^{n_{w_n}(\phi)} y. \tag{3}$$

There are mainly two algebraic differences with the case of link. For a graph the relative grading takes value on $H_1(X;\mathbb{Z})$, on which there is no canonical order in general. Therefore it is hard to discuss a non-trivial filtration on it. Another difference is that the chain complex $(\mathrm{CFG}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}, \boldsymbol{z}), \partial_L^-)$ in general is not the associated graded complex of $(\mathrm{CF}^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{w}), \partial^-)$. Precisely we mean that there might be some terms in $\partial^-(x) - \partial_G^-(x)$ having the same $H_1(X;\mathbb{Z})$-grading with x for given $x \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$.

Remark 3.2. *The chain complex considered here can be regarded as a special case of the definition in [1]. Here we choose the sutured manifold to be the graph complement and*

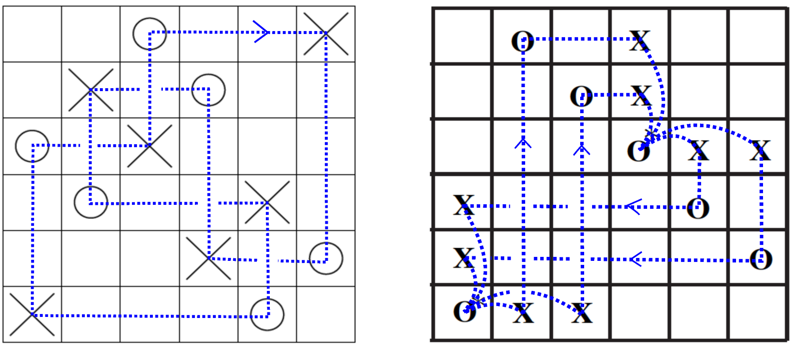

図 3: Two grid diagrams separately from [5] and [3] for a link and for a transverse graph. The dotted curves are the corresponding link and graph.

define the algebra for the boundary so that each edge e_i for $1 \leq i \leq n$ is associated with a variable U_i and the other edges zero. Then the algebra becomes $\mathbb{F}[U_1, U_2, \cdots, U_n]$.

4 Combinatorial aspects of the homology

4.1 grid diagram

The Heegaard Floer homology for a link in the 3-sphere S^3 has a completely combinatorial definition, which was introduced in [5]. The chain complex is defined on a grid diagram of the given link. The generators are the bijections between the set of horizontal circles and that of the vertical circles, and the differential counts empty rectangles between two generators. In a similar vein, Harvey and O'Donnol [3] defined the grid diagram for a transverse graph (the definition of which was first introduced there) and constructed the Heegaard Floer homology for it. See Figure 3 for the examples of both cases.

Note that an $n \times n$ grid diagram of a link is in particular an n-pointed Heegaard diagram for the link. For this Heegaard diagram, the $\boldsymbol{\alpha}$- and $\boldsymbol{\beta}$-curves are the horizental and vertical circles respectively, and the pesudo-holomorphic disks are the empty rectangles. Therefore the combinatorial definition coincides with the definition in Section 3.3.

We remark that a balanced bipartite graph with a balanced orientation naturally gives rise to a transverse graph, and vise versa. See Figure 4 for an example. As in the case of link, if we regard the grid diagram of a transverse graph as a Heegaard diagram of its corresponding bipartite graph, the definition in [3] and ours coincide with each other.

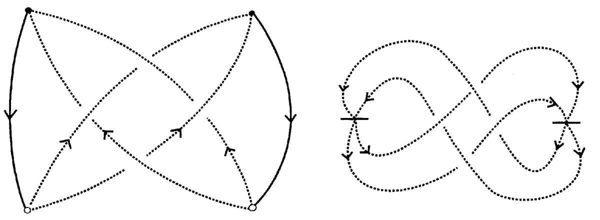

図 4: shrinking the solid line edges, whose orientation reverses that of the others, of the bipartite graph on the left, one gets a transverse graph on the right. Conversely, inserting an edge at each bar of the transverse graph on the right, we get the bipartite one.

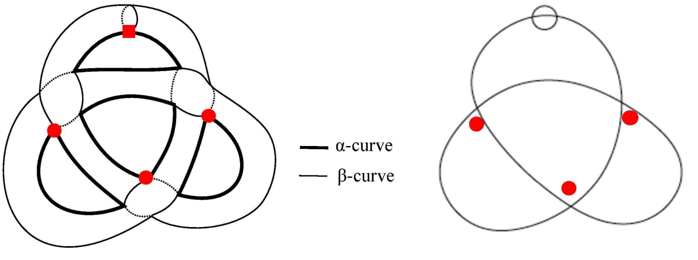

図 5: A knot diagram for the right-handed trefoil knot and its associted Heegaard diagram. The square intersection point appears in every generator and therefore is omitted on the right.

4.2 Kauffman state

For a knot in S^3, its knot diagram in S^2 defines a Heegaard diagram [10]. See Figure 5 for an example. The set of dots provides a Kauffman state [4] and also corresponds to a generator of the Heegaard Floer chain complex. For an alternating diagram, [10] showed that the differential of the complex is trivial, and the Heegaard Floer homology is determined by the Alexander polynomial and the signature of the given knot. But for general diagram, a combinatorial description of the differential is unknown.

For a bipartite graph in S^3, we also constructed in [2] a Heegaard diagram from its diagram in S^2. We proved that the generators of the Heegaard Floer chain complex for the graph are the states of the diagram (see Figure 6). Compare with the case of knot, there are still many questions to be solved.

Question 4.1 (Y. Bao). *Can any two states be connected by transpositions of type I and II? Is it possible to calculate the (relative) Alexander grading combinatorially? For "alternating" (there is no standard definition) bipartite graphs, is the Heegaard Floer complex completely determined by the Alexander polynomial (up to overall shifts of the gradings)?*

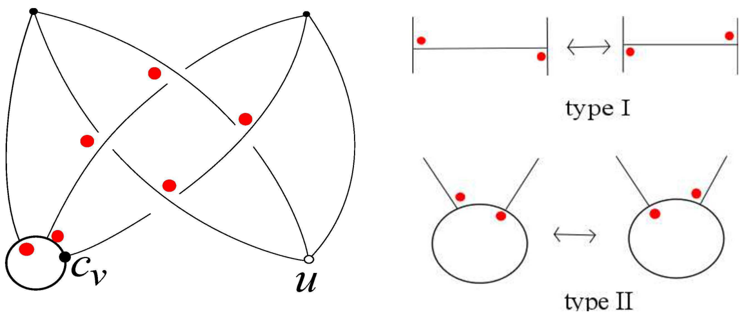

図 6: The set of dots represents a generator of the Heegaard Floer complex biult on the diagram (left figure). Transpositions of type I and II between two generators (right figure).

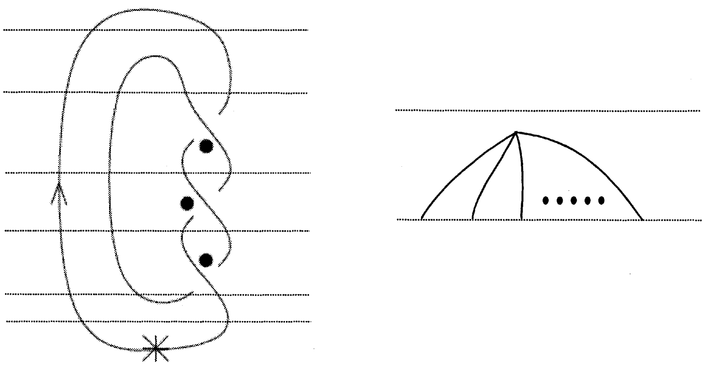

図 7: Slice a knot diagram.

In a recent paper [9], from a knot diagram, Ozsváth and Szabó constructed a bigraded chain complex over $\mathbb{F}[U]$, the homology of which is shown to be isomorphic to the Heegaard Floer homology (minus version) of the given knot. It is freely generated by the Kauffman states, and its differential is defined algebraically, built on bordered Floer homology.

The brief idea is as follows. Slice the knot diagram at different heights so that each piece contains exactly one cap, cup, or crossing. See Figure 7 (left). Ozsváth and Szabó announced that for each piece they constructed a differential graded algebra for the top boundary and an A^∞ algebra for the bottom boundary and a bimodule for the piece in between and proved that the tensor product of all pieces reproduces the Heegaard Floer homology of the given knot. For the case of graph, we may ask the following question.

Question 4.2. *For a bipartite graph, or a general graph, is it possible to construct such a complex? The first step might be how to define a module for the piece in Figure 7(right).*

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