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Morse-Novikov numbers of 2-knots and surface-links

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1. Introduction

1.1. **A brief overview of the article.** In this paper we give a short presentation of our results on the Morse-Novikov theory for 2-knots and surface-links (see the articles arXiv:1502.06352 and arXiv:1605.04532 for more details and full proofs.)

Let $N^k \subset S^{k+2}$ be a closed oriented submanifold, let $C(N) = S^{k+2} \backslash N$ be its complement. The orientation of N determines a cohomology class $\xi \in H^1(C(N)) \approx [C(N), S^1]$. We say that N is *fibred* if there is a Morse map $f : C(N) \to S^1$ homotopic to ξ which is regular nearby N (see Definition 1.1) and has no critical points. In general a Morse map $C(N) \to S^1$ has some critical points, the minimal number of these critical points will be called *the Morse-Novikov number of* N and denoted by $\mathcal{MN}(C(N))$.

In the first part of this paper we study this invariant in relation with constructions of spinning. The classical Artin's spinning construction [2] associates to each classical knot $K \subset S^3$ a 2-knot $S(K) \subset S^4$. A twisted version of this construction is due to E.C. Zeeman [12]. In [10] D. Roseman introduced a *frame spinning* construction, and G. Friedman [3] gave a generalization of D. Roseman's construction to include *twisting*. Let M be a framed closed submanifold of the $(m+k)$-dimensional sphere, K be an m-knot and $\lambda : M \to S^1$ a C^∞ map. The twist spinning construction associates to these data an n-knot $\sigma(M, K, \lambda)$ (where $n = k + m$). In Section 2 we give an upper bound for the Morse-Novikov number of the twist spun knot in terms of Morse-Novikov invariants of M and K.

Section 3 is about Morse-Novikov theory for surface-links. In Subsection 3.1 we introduce a related invariant of surface-links, namely the *saddle number* $sd(F)$ (Definition 3.1) and prove the formula

$$\mathcal{MN}(C(F)) \leqslant 2sd(F) + \chi(F) - 2. \tag{1}$$

In Subsection 3.2 we discuss the case of spun knots. In subsection 3.3 we determine the Morse-Novikov numbers of certain surface-links.

1.2. **Basic definitions.** We start with the definition of a regular Morse map.

Definition 1.1. Let $N^k \subset S^{k+2}$ be a closed oriented submanifold. Denote by $\xi \in H^1(C(N)) \approx [C(N), S^1]$ the cohomology class dual to the orientation class of N. A Morse map $f : C(N) \to S^1$ is said to be *regular* if there is an orientation preserving C^∞ trivialisation

$$\Phi : T(N) \to N \times B^2(0, \epsilon) \tag{2}$$

of a tubular neighbourhood $T(N)$ of N such that the restriction $f| \big(T(N)\backslash N\big)$ satisfies $f \circ \Phi^{-1}(x, z) = z/|z|$.

An f-gradient v of a regular Morse map $f : C(N) \to S^1$ will be called *regular* if there is a C^∞ trivialisation (2) such that $\Phi^*(v)$ equals $(0, v_0)$ where v_0 is the Riemannian gradient of the function $z \mapsto z/|z|$.

If f is a Morse map of a manifold to $\mathbf{R}$ or to S^1, then we denote by $m_p(f)$ the number of critical points of f of index p. The number of all critical points of f is denoted by $m(f)$.

Definition 1.2. The minimal number $m(f)$ where $f : C(N) \to S^1$ is a regular Morse map is called *the Morse-Novikov number of* N and denoted by $\mathcal{MN}(C(N))$.

To obtain lower bounds for numbers $m_p(f)$ one uses the *Novikov homology.* Let $L = \mathbb{Z}[t, t^{-1}]$; denote by $\widehat{L} = \mathbb{Z}((t))$ and $\widehat{L}_{\mathbb{Q}} = \mathbb{Q}((t))$ the rings of all series in one variable t with integer (respectively rational) coefficients and finite negative part. Recall that $\widehat{L}$ is a PID, and $\widehat{L}_{\mathbb{Q}}$ is a field. Consider the infinite cyclic covering $\overline{C(N)} \to C(N)$; the Novikov homology of $C(N)$ is defined as follows:

$$\widehat{H}_*(C(N)) = H_*(\overline{C(N)}) \underset{L}{\otimes} \widehat{L}.$$

The rank and torsion number of the $\widehat{L}$-module $\widehat{H}_k(C(N))$ will be denoted by $\widehat{b}_k(C(N))$, respectively $\widehat{q}_k(C(N))$. For any regular Morse function f there is a Novikov complex $\mathcal{N}_*$ over $\widehat{L}$ generated in degree k by critical points of f of index k and such that $H_*(\mathcal{N}_*) \approx \widehat{H}_*(C(N))$ (see [8]). Therefore we have the Novikov inequalities

$$\sum_k \Big(\widehat{b}_k(C(N)) + \widehat{q}_k(C(N)) + \widehat{q}_{k-1}(C(N))\Big) \leqslant \mathcal{MN}(C(N)).$$

These inequalities, which are far from being exact in general, are however very useful in the case of surface-links (see Section 3).

2. Spinning and related constructions

2.1. **Frame twist spun knots: the construction.** In this subsection we recall the Artin-Zeeman-Roseman-Friedman frame twist spinning construction. The input data for this construction is:

(TFS1) A closed manifold $M^k \subset S^{m+k}$ with trivial (and framed) normal bundle.
(TFS2) An m-knot $K^m \subset S^{m+2}$.
(TFS3) A C^∞ map $\lambda : M \to S^1$.

To these data one associates an n-knot $\sigma(M, K, \lambda)$, where $n = k + m$. When λ is a constant map we denote this knot by $\sigma(M, K)$; this is the Roseman's *frame spun knot.*

Let $a \in K^m$. Removing a small open disk $D(a)$ from S^{m+2} we obtain an embedded (knotted) disk K_0 in the disk $D^{m+2} \approx S^{m+2} \backslash D(a)$. We identify D^{m+2} with the standard Euclidean disk of radius 1 and center 0 in $\mathbb{R}^{m+2}$, then $\partial D^{m+2} = S^{m+1}$. We have the usual diffeomorphism

$$\chi : S^{m+1} \times]0, 1] \xrightarrow{\approx} D^{m+2} \backslash \{0\}, \quad (x, t) \mapsto tx.$$

We can assume that $K_0 \cap \partial D^{m+2}$ is an equatorial sphere † S^{m-1} in $\partial D^{m+2} = S^{m+1}$. Moreover, we can assume that the intersection of K_0 with a neighbourhood of ∂D^{m+2} is also standard, that is,

$$K_0 \cap \chi\big(S^{m+1} \times [1-\epsilon, 1]\big) = \chi\big(S^{m-1} \times [1-\epsilon, 1]\big).$$

We have a framing of M in S^n (recall that $n = m + k$); combining this with the standard framing of S^n in S^{n+2} we obtain a diffeomorphism

$$\Phi : N(M, S^{n+2}) \xrightarrow{\approx} M \times D^m \times D^2$$

where $N(M, S^{n+2})$ is a regular neighbourhood of M in S^{n+2}. We can assume that the restriction of Φ to $N(M, S^n)$ is a diffeomorphism

$$\Phi : N(M, S^n) \xrightarrow{\approx} M \times D^m \times \{0\}$$

induced by the given framing of M. The Euclidean disc D^{m+2} is a subset of $D^m \times D^2$, so that $K_0 \subset D^m \times D^2$.

For $\theta \in S^1$ denote by R_θ the rotation of D^2 around its center. The disc $D^{m+2} \subset D^m \times D^2$ is invariant with respect to this rotation as well as the intersection of K_0 with a small neighbourhood of ∂D^{m+2}. We have $\Phi\big(S^n \cap N(M, S^{n+2})\big) = M \times D^m \times \{0\}$. Let

$$Z = \{(x, y, z) \mid (y, z) \in R_{\lambda(x)}(K_0)\}.$$

This is an n-dimensional submanifold of $M \times D^m \times D^2$. We define $\sigma(M, K, \lambda)$ as follows

$$\sigma(M, K, \lambda) = \Big(S^{n+2} \backslash N(M, S^{n+2})\Big) \cup \Phi^{-1}(Z).$$

This is the image of an embedded n-sphere, knotted in general.

Examples and particular cases.

1) Let $\dim M = 0$, so that M is a finite set; denote by p its cardinality. Then the n-knot $\sigma(M, K, \lambda)$ is equivalent to the connected sum of p copies of K.
2) If M is the equatorial circle of the sphere S^2, which is in turn considered as an equatorial sphere of S^4, and $\lambda(x) = 1$ for all x, we obtain the classical Artin's construction. If $\lambda : S^1 \to S^1$ is a map of degree d, we obtain the Zeeman's twist-spinning construction [12].
3) If $\lambda(x) = 1$ for all $x \in M$ we obtain the Roseman's construction of spinning around the manifold M [10]. In this case we will denote $\sigma(M, K, \lambda)$ by $\sigma(M, K)$.

2.2. Morse-Novikov numbers of twist spun knots.

Theorem 2.1.

$$\mathcal{MN}\Big(C(\sigma(M, K, \lambda))\Big) \leqslant \mathcal{MN}(C(K)) \cdot \mathcal{MN}(M, [\lambda]).$$

(where $[\lambda] \in H^1(M, \mathbb{Z}) \approx [M, S^1]$ is the homotopy class of λ).

†By *equatorial sphere* in $S^N \subset \mathbb{R}^{N+1}$ we mean the intersection of a linear subspace $L \subset \mathbb{R}^{N+1}$ with S^N; this intersection is a Euclidean sphere of dimension $\dim L - 1$.

Corollary 2.2. *Let $K \subset S^3$ be a classical knot, denote by $S(K)$ the spun knot of K. Then*

$$\mathcal{MN}(C(S(K))) \leqslant 2\mathcal{MN}(C(K)). \tag{3}$$

Proof. In this case $M = S^1$ and $[\lambda] = 0$. We have $\mathcal{MN}(S^1, 0) = 2$ and the result follows. □

The classical theorems concerning fibrations of spun knots follow from Theorem 2.1:

Corollary 2.3. *(D. Roseman [10]) If K is fibred, then $\sigma(M, K)$ is fibred.*

Proof. Since $\mathcal{MN}(C(K)) = 0$, Theorem 2.1 implies $\mathcal{MN}(C(\sigma(M, K))) = 0$. □

Corollary 2.4. *(E.C. Zeeman [12]) The d-twist spun knot of any classical knot K is fibred for $d \geqslant 1$.*

Proof. Let Σ be an equatorial circle in S^2. The d-twist spun knot of K is by definition the 2-knot $\sigma(\Sigma, K, \lambda)$ in S^4 where $\lambda : \Sigma \to \Sigma$ is a map of degree d. The assertion follows, since $\mathcal{MN}(S^1, [\lambda]) = 0$. □

Remark 2.5. The Zeeman's theorem above generalizes immediately to the following statement: If $\mathcal{MN}(M, [\lambda]) = 0$, then the knot $\sigma(M, K, \lambda)$ is fibred for any knot K.

2.3. **Rotation.** In this subsection we present one more geometric construction related to spinning techinques. Let Σ be an equatorial n-sphere of S^{n+1}. We can view the sphere S^{n+1} as the union of two $(n+1)$-dimensional discs $D_+ \cup D_-$ intersecting by Σ. Consider S^{n+1} as the equatorial sphere of S^{n+2}. The sphere S^{n+2} can be considered as the result of rotation of the disc D_+ around its boundary Σ. We have the (linear orthogonal) action of S^1 on S^{n+2}, such that Σ is the fixed point set of the action, and the action is free on the rest of the sphere S^{n+2}. Let K^{n-1} be an $(n-1)$-knot in S^{n+1}. We can assume that $K^{n-1} \subset \text{Int } D_+$. Rotation of K^{n-1} around Σ gives a submanifold $R(K)$ of codimension 2 in S^{n+2}. The manifold $R(K)$ is diffeomorphic to $S^1 \times K$. We call this constrution *rotation*. When $\dim K = 1$, the manifold $R(K)$ is sometimes called the *spun torus* of K. In this section we relate the Morse-Novikov numbers of $R(K)$ with those of K.

Theorem 2.6.

$$\mathcal{MN}(C(R(K))) \leqslant 2\mathcal{MN}(C(K)) + 2.$$

3. Morse-Novikov numbers of surface-links

In this section we develop circle-valued Morse theory for surface-links.

3.1. **Motion pictures and saddle numbers.** Let F be a surface-link, that is, a closed oriented 2-dimensional C^∞ submanifold of S^4. We can assume $F \subset \mathbb{R}^4$.

Choose a projection p of $\mathbb{R}^4$ onto a line. Assume that the critical points of the function $p|F$ are non-degenerate. Denote by $sdl(F)$ the minimal number of saddle points of $p|F$ over all the projections p.

Definition 3.1. A saddle number $sd(F)$ is the minimum of numbers $sdl(F')$ where F' ranges over all surface-links F' ambiently isotopic to F.

The invariant $sd(F)$ is closely related to the *ch-index* of F, introduced and studied by K. Yoshikawa in [11]. In particular, we have $sd(F) \leqslant ch(F)$. In order to relate the number $sd(F)$ to $\mathcal{MN}(S^4 \backslash F)$ we will reformulate the definition of the saddle number.

Let $F \subset S^4$ be a surface-link. The equatorial 3-sphere Σ^3 of the standard Euclidean sphere S^4 divides S^4 into two parts:

$$S^4 = D^4_+ \cup D^4_-, \quad \text{with} \quad D^4_+ \cap D^4_- = \Sigma^3.$$

We assume that F is included in $\mathrm{Int}(D^4_-)$ and F does not contain the centre of D^4_-. Perturbing the embedding $F \subset D^4_-$ if necessary, we can assume that the restriction $\rho = r|_F$ of the radius function $r : D^4_- \to [0,1]$ is a Morse function. The family $\{(r^{-1}(t), \rho^{-1}(t))\}_{t \in [0,1]}$ of possibly singular links can be drawn as a *motion picture* (see [5], Chapter 8). Each singularity of a link in the family corresponds to a critical point of ρ. A critical point of ρ of index 0 (1, 2, respectively) is called *minimal point* (*saddle point*, *maximal point*, respectively) of ρ, which is represented by a *minimal band* (*saddle band*, *maximal band*, respectively) in (a modification of) the motion picture.

It is clear that the minimal number of the saddle points for all such Morse functions ρ and all surface-links ambiently isotopic to F is equal to $sd(F)$.

Theorem 3.2. $\mathcal{MN}(C(F)) \leqslant 2\,sd(F) + \chi(F) - 2.$

Corollary 3.3. *Let $K \subset S^4$ be a 2-knot. Then $\mathcal{MN}(C(K)) \leqslant 2sd(K)$.* □

Proposition 3.4. *Let $F \subset S^4$ be the trivial k-component surface-link. Then* $\mathcal{MN}(C(F)) = 4k - 2 - \chi(F)$.

Proof. It is not diffcult to show that $\widehat{b}_1(C(F)) \geqslant k-1$, $\widehat{b}_3(C(F)) \geqslant k-1$. Therefore for every regular Morse map $f : C(F) \to S^1$ we have $m_1(f) + m_3(f) \geqslant 2(k-1)$. Assuming $m_0(f) = m_4(f) = 0$ we have $m_1(f) - m_2(f) + m_3(f) = 2 - \chi(F)$, and $\mathcal{MN}(C(F)) \geqslant 4k - 2 - \chi(F)$; this lower bound coincides with the upper bound derived from Theorem 3.2. □

3.2. **Spun knots.** Let K be a classical knot in S^3; denote by $S(K)$ the corresponding spun knot.

Proposition 3.5. *If K is a non-fibered knot of tunnel number 1, then* $\mathcal{MN}(S^4 \backslash S(K)) = 4$.

Proof. Recall that $\mathcal{MN}(S^4 \backslash S(K)) \leqslant 2\mathcal{MN}(K)$ (Corollary 2.2). In the paper [7] of the second author it is shown that $\mathcal{MN}(C(K)) \leqslant 2t(K)$, hence $\mathcal{MN}(C(S(K))) \leqslant 4$ by Corollary 2.2. Put $G = \pi_1(S^3 \backslash K)$, then $\pi_1(S^4 \backslash S(K)) \approx G$; let $H = [G,G]$. Let $f : S^4 \backslash S(K) \to S^1$ be a regular Morse map without minima and maxima. If $m_1(f) = 0$, then a standard Morse-theoretic argument applied to the infinite cyclic cover of $S^4 \backslash S(K)$ implies that H is finitely generated, which is impossible, since K is not fibred. Therefore $m_1(f) \geqslant 1$, and similarly, $m_3(f) \geqslant 1$, hence $m_2(f) \geqslant 2$ and the proposition is proved. □

3.3. **Surface-links of Yoshikawa's table.** A. Kawauchi, T. Shibuya and S. Suzuki [6] developed a method of representing surface-links by diagrams. Based on this method K. Yoshikawa [11] introduced a numerical invariant $ch(F)$ of surface-links F and enumerated all the (weakly prime) surface-links F with $ch(F) \leqslant 10$.

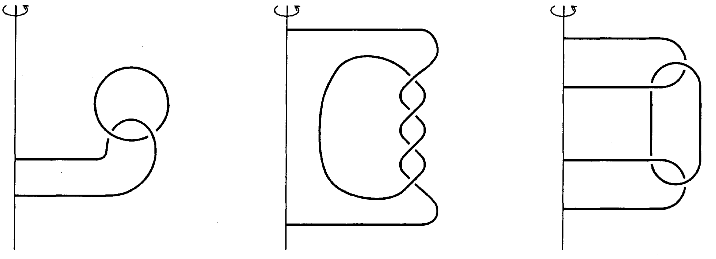

FIGURE 1

It is clear from the definition of the invariant $ch(F)$ that we have $sd(F) \leqslant ch(F)$. In the rest of this section we assume that the reader is familiar with Yoshikawa's work, and with his terminology. There are 6 two-knots in Yoshikawa's table, namely

$$0_1,\ 8_1, 9_1,\ 10_1,\ 10_2,\ 10_3.$$

The trivial 2-knot 0_1 is obviously fibred. The knots 8_1 and 10_1 are spun knots of the trefoil knot and respectively of the figure 8 knot, thus both 8_1 and 10_1 are fibred by [1].

The case of 9_1 is more complicated. The saddle number of this 2-knot is 2. Therefore $\mathcal{MN}(9_1) \leqslant 4$. Using the presentation of the fundamental group of the complement to 9_1 (see [11]) and Poincaré duality properties it is easy to compute the Novikov numbers of 9_1. Namely we have $\widehat{q}_1 = 1, \widehat{q}_2 = \widehat{q}_3 = 0$. Therefore

$$2 \leqslant \mathcal{MN}(9_1) \leqslant 4.$$

The 2-knot 10_2 is the 2-twist-spun knot of the trefoil knot, hence fibered by Zeeman's theorem [12]. Similarly, 10_3 is fibered, being the 3-twist spun of the trefoil knot.

The surface-link $6_1^{0,1}$ is the result of spinning of the Hopf link which is fibred (see the left of Figure 2) therefore $\mathcal{MN}(6_1^{0,1}) = 0$.

The surface-link $8_1^{1,1}$ is the spun torus of the Hopf link. Applying Theorem 2.6 we get the upper bound $\mathcal{MN}(8_1^{1,1}) \leqslant 2$. Computing the Euler charcateristic implis the inverse inequality, so $\mathcal{MN}(8_1^{1,1}) = 2$.

The same argument applies to the surface-link 10_1^1, which is the spun torus of the trefoil knot, see the figure 2 (middle), so that $\mathcal{MN}(10_1^1) = 2$.

The surface-link $10_1^{0,1}$ is the result of spinning of the link 4_1^2 which is fibred, therefore $\mathcal{MN}(10_1^{0,1}) = 0$.

The case of the surface-link $F = 10_1^{0,0,1}$ is more complicated. Applying a generalisation of spinning constructions we prove that $\mathcal{MN}(10_1^{0,0,1}) = 2$.

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