On handlebody-links and Milnor’s link-homotopy invariants

Yuka Kotorii

Graduate School of Mathematical Science, The University of Tokyo

1 Introduction

This is a survey of the joint work [13] with Atsuhiko Mizusawa. A handlebody-link [11, 27] is a disjoint union of embeddings of handlebodies in the 3-sphere $S^3$ (Figure 1). Two handlebody-links are equivalent if there is an ambient isomorphism which transforms one to the other. An HL-homotopy is an equivalence relation on handlebody-links, which is analogous to link-homotopy of links. Here, link-homotopy is generated by ambient isotopies and self-crossing changes. In [22], Mizusawa and Nikkuni showed that the HL-homotopy classes of 2-component handlebody-links were classified completely by the linking numbers for handlebody-links, which was defined by Mizusawa in [21]. In [13], we construct HL-homotopy invariants for handlebody-links by using Milnor’s $\overline{\mu}$-invariants for links. We then give a necessary and sufficient condition of that a handlebody-link is HL-homotopic to a separable one by the extended Milnor’s $\overline{\mu}$-invariants. Here, a handlebody-link is separable if there exists a disjoint union of 3-balls such that each component of the handlebody-link is contained in a distinct 3-ball. Moreover, we give a bijection between the set of HL-homotopy classes of $n$-component handlebody-links with some assumption and a quotient of a tensor product of $\mathbb{Z}$-modules by the action of the general linear group.
2 Preliminaries

J. Milnor defined a family of invariants for an ordered oriented link in \( S^3 \) as a generalization of the linking numbers, in [19, 20]. These invariants are called Milnor's \( \overline{\mu} \)-invariants. For an ordered oriented \( n \)-component link \( L \), Milnor’s \( \overline{\mu} \)-invariant is specified by a sequence \( I \) of indices in \( \{ 1, 2, \ldots, n \} \) and denoted by \( \overline{\mu}_I(I) \). If the sequence is with distinct indices, then this invariant is also link-homotopy invariant and called Milnor’s link-homotopy invariant.

We introduce the definition of Milnor’s link-homotopy invariants, and to give invariants for handlebody-links, we show that these are additive under a bound sum for components.

Let \( L = L_1 \cup \cdots \cup L_n \) be an ordered oriented \( n \)-component link in \( S^3 \). Consider the link group \( \pi = \pi_1(S^3 \setminus L_1 \cup \cdots \cup L_{n-1}) \) of \( L_1 \cup \cdots \cup L_{n-1} \) and denote the \( i \)-th meridian by \( m_i \) for \( i \) \((1 \leq i \leq n-1)\).

Given a finitely generated group \( G \), the reduced group \( \overline{G} \) is defined as the quotient of \( G \) by its normal subgroup generated by \( [g, hgh^{-1}] \) for any \( g, h \in G \), where \([a, b] \) means the commutator of \( a \) and \( b \). Then \( \overline{\pi} \) is generated by the meridians \( m_1, m_2, \ldots, m_{n-1} \).

Let \( \mathbb{Z}[[X_1, \ldots, X_{n-1}]] \) be the non-commutative formal power series ring generated by \( X_1, \ldots, X_{n-1} \). Denote by \( \hat{\mathbb{Z}} \) its quotient ring by the two-side ideal generated by all monomials in which at least one of the generators appear at least twice. The Magnus expansion \( \varphi \) is a homomorphism from the free group \( F(m_1, \ldots, m_{n-1}) \) generated by \( m_1, \ldots, m_{n-1} \) into \( \mathbb{Z}[[X_1, \ldots, X_{n-1}]] \), defined by sending \( m_i \) to \( 1 + X_i \) and \( m_i^{-1} \) to \( 1 - X_i + X_i^2 - \cdots \). It induces a homomorphism from \( \overline{F(m_1, \ldots, m_{n-1})} \) into \( \hat{\mathbb{Z}} \). Let \( w_n \in \overline{F(m_1, \ldots, m_{n-1})} \) be a word representing \( L_n \) in \( \overline{\pi} \). We then define \( \mu_L(i_1i_2\ldots i_r) \) for distinct indices \( i_1, i_2, \ldots, i_r, n \) as the coefficient of the Magnus expansion of \( w_n \) in \( \hat{\mathbb{Z}} \):

\[
\varphi(w_n) = 1 + \sum \mu_L(i_1i_2\ldots i_r)X_{i_1}X_{i_2}\ldots X_{i_r},
\]

where the summation is over all sequences \( i_1i_2\ldots i_r \) with distinct indices between 1 and \( n-1 \). Similarly, we define \( \mu_L(i_1i_2\ldots i_s) \) for any distinct indices between 1 and \( n \). We define \( \overline{\mu}_L(i_1i_2\ldots i_r) \) as the residue class of \( \mu_L(i_1i_2\ldots i_r) \) modulo the indeterminacy \( \Delta_L(i_1i_2\ldots i_r) \) which is the greatest common divisor of \( \mu_L(j_1j_2\ldots j_s) \)'s, where \( j_1j_2\ldots j_s \) ranges over all sequences obtained by deleting at least one of the indices \( i_1, i_2, \ldots, i_r, n \) and permuting the remaining ones cyclically. Moreover we define \( \Delta_L(i_1n) = 0 \). Similar to this, for any \( n \)-component link \( L \), we can define \( \overline{\mu}_L(I) \) for any sequence \( I \) of distinct indices in \( \{ 1, 2, \ldots, n \} \).

**Theorem 2.1** ([19, 20]). If \( L \) and \( L' \) are link-homotopic, then \( \overline{\mu}_L(I) = \overline{\mu}_{L'}(I) \) for any sequence \( I \) with distinct indices.
Lemma 2.2 ([20]). Let $L$ be an ordered oriented link. Then the following relations hold.

1. $\overline{\mu}_L(i_1 i_2 \ldots i_m) = \overline{\mu}_L(i_2 \ldots i_m i_1)$
2. If the orientation of the $k$-th component of $L$ is reversed, then $\overline{\mu}_L(i_1 i_2 \ldots i_m)$ is multiplied by $+1$ or $-1$ according as the sequence $i_1 i_2 \ldots i_m$ contains $k$ an even or odd number of times.

The following lemma is used for Proposition 3.4. This lemma is showed by using the definition of Milnor’s link-homotopy invariants.

Lemma 2.3. Let $L = L_1 \cup L_2 \cup \cdots \cup L_{n-1}$ be an $(n-1)$-component link in $S^3$. Let $K$ and $K'$ be disjoint knots in $S^3 \setminus L$. Let $I$ be a sequence with distinct indices in $\{1, 2, \ldots, n\}$. If $I$ contains the index $n$,

$$\mu_{L \cup (K \#_b K')}(I) \equiv \mu_{L \cup K}(I) + \mu_{L \cup K'}(I) \mod \gcd(\Delta_{L \cup K}(I), \Delta_{L \cup K'}(I)),$$

where $K \#_b K'$ is a band sum of $K$ and $K'$ with respect to any band, and $L \cup (K \#_b K')$, $L \cup K$ and $L \cup K'$ are $n$-component links whose $n$-th components are $K \#_b K'$, $K$ and $K'$, respectively.

Remark 2.4. By a property of the $\overline{\mu}$-invariant, we can obtain the same result for a band sum of the $i$-th component instead of the $n$-th component.

Remark 2.5. In [14], V. S. Krushkal showed Milnor’s $\overline{\mu}$-invariants are additive under connected sum for links which are separated by a 2-sphere.

3 Milnor’s $\overline{\mu}$-invarinats for handlebody-links

In this section, we define the HL-homotopy, which is an equivalence relation on handlebody-links and construct HL-homotopy invariants for handlebody-links by using Milnor’s $\overline{\mu}$-invariants.

Definition 3.1 (HL-homotopy). Let $H_0$ be $n$ handlebodies and $H_i$ $(i = 1, 2)$ two $n$-component handlebody-links obtained by embedding $H_0$ to $S^3$ by $f_i$. Two handlebody-links $H_1$ and $H_2$ are called HL-homotopic if there is homotopy $h_t$ from $f_1$ to $f_2$ where the components of $h_t(H_0)$ are mutually disjoint at any $0 \leq t \leq 1$.

Remark 3.2. In [22], the notation of neighborhood homotopy of spatial graphs was introduced. A spatial graph is an embedding of graph in $S^3$. We can represent the HL-homotopy of handlebody-links by the neighborhood homotopy of spatial graphs.

Let $H = L_1 \cup \cdots \cup L_n$ be an $n$-component handlebody-link with genus $g_i$ for each $i$. Let $\{e^i_1, \ldots, e^i_{g_i}\}$ be a basis of $H_i(L_i; \mathbb{Z})$ and $\mathcal{B} = \{e^1_1, \ldots, e^1_{g_1}, \ldots, e^n_1, \ldots, e^n_{g_n}\}$. We can
regard an element of $\mathcal{B}$ as an embedded closed oriented circle in $S^3$. So the disjoint union $e_{k_1}^1 \cup e_{k_2}^2 \cup \cdots \cup e_{k_n}^n$ can be regarded as an ordered oriented link for each $k_i$ ($1 \leq k_i \leq g_i$). Let $I$ be a sequence of length $m$ ($m \leq n$) with distinct indices in $\{1, 2, \ldots, n\}$. For each $I$, we define an element $M_{H,B}(I)$ of tensor product space $(\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_1} \otimes \cdots \otimes (\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_n}$ as $\mathbb{Z}/\triangle_{I}\mathbb{Z}$-module defined by

$$M_{H,B}(I) := \sum_{k_1, \ldots, k_n=1}^{g_1, \ldots, g_n} \overline{\mu}_{e_{k_1}^1 \cup \cdots \cup e_{k_n}^n}(I) e_{k_1}^1 \otimes \cdots \otimes e_{k_n}^n,$$

where $\overline{\mu}_{e_{k_1}^1 \cup \cdots \cup e_{k_n}^n}(I)$ is in $\mathbb{Z}/\triangle_{I}\mathbb{Z}$, $\triangle_{I}$ is the greatest common divisor of all $\triangle_{e_{k_1}^1 \cup \cdots \cup e_{k_n}^n}(I)$ for all $k_1, \ldots, k_n$, where $\triangle_{e_{k_1}^1 \cup \cdots \cup e_{k_n}^n}(I)$ is indeterminacy of the original Milnor’s invariant for the link $e_{k_1}^1 \cup e_{k_2}^2 \cup \cdots \cup e_{k_n}^n$ and $e_{k_i}^i$ is the canonical basis $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $(\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g}$ as $\mathbb{Z}/\triangle_{I}\mathbb{Z}$-module.

**Remark 3.3.** If the first homology group of each component of $H$ is $\mathbb{Z}$, the $M_{H,B}(I)$ is identified with the original Milnor’s link-homotopy invariant for a link, essentially.

We consider a natural action of $GL(g_1, \mathbb{Z}) \times \cdots \times GL(g_n, \mathbb{Z})$ on $(\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_1} \otimes \cdots \otimes (\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_n}$ and denote by $M_H(I)$ the residue class of $M_{H,B}(I)$ by the action for $(\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_1} \otimes \cdots \otimes (\mathbb{Z}/\triangle_{I}\mathbb{Z})^{\otimes g_n}$.

**Proposition 3.4.** Let $H$ be an $n$-component handlebody-link. Then $M_H(I)$ is independent of a basis $\mathcal{B}$ of $H_1(H, \mathbb{Z})$ and an HL-homotopy invariant.

**Proof.** The proof is by induction on the length $m$ of sequence $I$. We can show it by using properties of $\overline{\mu}$-invariants for links (Lemma 2.2 and 2.3). See [13] for details. \hfill $\square$

**Example 3.5.** Let $H$ be a handlebody-link which are the regular neighborhood of graph illustrated in Figure 2. Let $I = 123$. Then, $\Delta_{e_{1}^1 \cup e_{2}^2 \cup e_{3}^3}(I) = \Delta_{e_{1}^1 \cup e_{2}^2 \cup e_{3}^3}(I) = 2$ and $\Delta_{e_{1}^1 \cup e_{2}^2 \cup e_{3}^3}(I) = 0$ in other cases. So $\Delta_{I} = 2$ and

$$M_H(I) = 1 \ e_{1}^1 \otimes e_{2}^1 \otimes e_{3}^1 + 1 \ e_{2}^1 \otimes e_{3}^2 \otimes e_{3}^3 \in (\mathbb{Z}_2)^2 \otimes (\mathbb{Z}_2)^2 \otimes (\mathbb{Z}_2)^2.$$

We can show the following corollary by using clasper theory introduced by Habiro [8].

**Corollary 3.6.** An $n$-component handlebody-link $H$ is HL-homotopic to a separable handlebody-link if and only if $M_H(I) = 0$ for any $I$.

**Remark 3.7.** T. Fleming defined a numerical invariant $\lambda_{\Phi}(H)$ of a pair of a spatial graph $\Phi$ and its subgraph $H$ under component homotopy in [3]. Now, we define $\Phi$ as a handlebody-link instead of a spatial graph and $H$ as its component instead of a subgraph. We then can naturally extend this invariant to a pair of a handlebody-link and its component under HL-homotopy. Then, the value of $\lambda_{\Phi}(H)$ is the length of first non-vanishing for $M_{\Phi}(I)$ such that $I$ contains the component number of $H$. 
4 Main Theorem

Let $\mathbb{H}[g_1, g_2, \cdots, g_n]$ be the set of $n$-component handlebody-links with genus $g_i$ for each $1 \leq i \leq n$ such that its any $(n-1)$-component subhandlebody-link is HL-homotopic to a separable handlebody-link. By Corollary 3.6, this condition is equivalent to that its any $M(I)$'s of length less than $n$ vanishes.

Let $S$ be a permutation group on $\{2, 3, \ldots, n-1\}$. For any element $\sigma$ in $S$, we define $I_{ \sigma}$ as a sequence $1\sigma(23\cdots n-1)n$.

Theorem 4.1. For any element $\sigma$ in $S$, the map

$$
\varphi : \mathbb{H}[g_1, \cdots, g_n] \to \bigoplus_{\sigma \in S}(\mathbb{Z}^{g_1} \otimes \cdots \otimes \mathbb{Z}^{g_n})
$$

$$
H \mapsto (M_{H}(I_{ \sigma}))_{\sigma \in S}
$$

induces a bijection between the set of HL-homotopy classes of $\mathbb{H}[g_1, g_2, \cdots, g_n]$ and the residue class of $\bigoplus_{\sigma \in S}(\mathbb{Z}^{g_1} \otimes \cdots \otimes \mathbb{Z}^{g_n})$ by diagonal action of general linear group.

We give two examples.

Example 4.2. Let $I = 123$. Let $H_1$ and $H_2$ be two handlebody-links which are the regular neighborhood of graphs depicted in Figure 3. Then, $\Delta_I = 0$ and

$$
M_{H_1}(I) = 1 e_1^1 \otimes e_2^1 \otimes e_3^1 + 1 e_1^1 \otimes e_2^2 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1
$$

$$
+ 2 e_1^1 \otimes e_2^1 \otimes e_3^2 + 2 e_1^1 \otimes e_2^1 \otimes e_3^2 + 2 e_1^1 \otimes e_2^1 \otimes e_3^2 + 2 e_1^1 \otimes e_2^1 \otimes e_3^2
$$

$$
\in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2.
$$

$$
M_{H_2}(I) = 1 e_1^1 \otimes e_2^1 \otimes e_3^1 + 1 e_1^1 \otimes e_2^2 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1 + 1 e_1^1 \otimes e_2^3 \otimes e_3^1
$$

$$
+ 1 e_1^1 \otimes e_2^1 \otimes e_3^2 + 1 e_1^1 \otimes e_2^1 \otimes e_3^2 + 1 e_1^1 \otimes e_2^1 \otimes e_3^2 + 1 e_1^1 \otimes e_2^1 \otimes e_3^2
$$

$$
\in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2.
$$

We have that $M_{H_1}(I)$ is transformed to $M_{H_2}(I)$ by the diagonal action of general linear group. Therefore $H_1$ and $H_2$ are HL-homotopic.
Example 4.3. Let $I = 123$. Let $H_3$ and $H_4$ be two handlebody-links which are the regular neighborhood of graphs depicted in Figure 4. Then, $\Delta_I = 0$ and

\[
M_{H_3}(I) = e_1^1 \otimes e_1^2 \otimes e_1^3 + e_1^1 \otimes e_2^2 \otimes e_1^3 + e_1^1 \otimes e_3^2 \otimes e_1^3 + e_2^1 \otimes e_2^2 \otimes e_2^3 + e_2^1 \otimes e_3^2 \otimes e_1^3 + e_3^1 \otimes e_2^2 \otimes e_1^3 + e_3^1 \otimes e_3^2 \otimes e_2^3 + e_3^1 \otimes e_3^2 \otimes e_3^3 \in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2.
\]

\[
M_{H_4}(I) = 2 e_1^1 \otimes e_1^2 \otimes e_1^3 + 2 e_2^1 \otimes e_2^2 \otimes e_1^3 + 1 e_1^1 \otimes e_3^2 \otimes e_2^3 + 1 e_2^1 \otimes e_2^2 \otimes e_2^3 \in \mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^2.
\]

We can show that $H_1$ is not HL-homotopic to $H_2$ by using some invariants for the action of general linear group on the tensor product space. See [13] for details.
Acknowledgements

The author would like to thank Professor Tomotada Ohtsuki for inviting me the workshop “Intelligence of Low-dimensional Topology 2016”. She would also like to thank Professor Sadayoshi Kojima and Professor Mitsuhiro Takasawa for your advice.

References


Graduate School of Mathematical Science
The University of Tokyo
Tokyo 153-8914
JAPAN
E-mail address: kotorii@ms.u-tokyo.ac.jp

東京大学大学院数理科学研究科 小鳥居 祐香