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Introduction to Heegaard Floer homology

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1 Atiyah-Floer conjecture and Ozsváth-Szabó's motivation

In [25, 26] Ozsváth and Szabó defined topological 3-manifold invariants by using Floer homology theory:

$$\widehat{HF}(Y,\mathfrak{s}),\ HF^{\infty}(Y,\mathfrak{s}),\ HF^{+}(Y,\mathfrak{s}),\ HF^{-}(Y,\mathfrak{s}).$$

Those invariants are new invariants in terms of the point that the invariants are categorifications of some topological invariants: Casson invariant (with correction term) Alexander polynomial and Turaev torsion invariant. The motivation of defining these invariants is that it is a symplectic counterpart of Seiberg-Witten Floer homology via Atiyah-Floer conjecture. Original Atiyah-Floer conjecture for Yang-Mills equation is the following:

Conjecture 1.1 (Atiyah-Floer conjecture) *Let Y be a closed oriented 3-manifolds. The instanton Floer homology on Y and the Lagrangian intersection Floer homology for flat connections are isomorphic each other:*

$$HF^{Inst}(\mathcal{M}_Y)\cong HF^{Symp}(\mathcal{M}_{\Sigma};\mathcal{L}_0,\mathcal{L}_1).$$

The instanton Floer homology $HF^{\mathrm{Inst}}(\mathcal{M}_Y)$ is a Morse homology on a moduli space $\mathcal{M}_Y$ of Yang-Mills equation on a 3-manifold Y up to gauge action. The generators are the flat connections on Y. The differentials are the counting of the moduli space of Yang-Mills solutions on the cylinder $Y\times I$.

On the other hand for a Heegaard decomposition $Y=H_0\cup_{\Sigma}H_1$ we consider the symplectic moduli space $\mathcal{M}_{\Sigma}$ on Σ of flat connections. Let $\mathcal{L}_0,\mathcal{L}_1$ in $\mathcal{M}_{\Sigma}$ be Lagrangian submanifolds extending to H_0 and H_1. The generators of symplectic side are intersection points of the two manifolds with a suitable general condition. The differentials are the counting of the holomorphic disks connecting two points. The boundary of a holomorphic disk lies in union of $\mathcal{L}_0$ and $\mathcal{L}_1$.

Heegaard Floer theory corresponds to the opposite side of the Seiberg-Witten Floer homology through analogy of Atiyah-Floer conjecture. Before Heegaard Floer homology appearing, no one would have succeeded a symplectic counterpart of Seiberg-Witten theory. Ozsváth and Szabó attacked that natural and essential problem to attain such Lagrangian intersection Floer homology. Instanton Floer homology had rediscovered topological invariants, Casson invariant [45]. Hence they should have expected applications of interesting topological invariants from the Heegaard Floer homology. More than expected, it turns out to be a powerful tool to study low-dimensional topology and be an attractable object for low-dimensional topologists. As a result, a whole lot of results have been built by Heegaard Floer homology, some new phenomena are discovered.

This article is a simple overview of Heegaard Floer homology up to now. But some interesting topics had to skip because of space limitation. If this article arouses your interests in your mind, as present Heegaard Floer topologists were so in the past, then you should begin with one in a pile of papers.

One can also read some excellent lecture notes [30] [31] [32] as the first study of Heegaard Floer homology.

2 Heegaard Floer homology

2.1 Definition of Heegaard Floer homologies

We define Heegaard Floer homology in this section. Let $Y = H_0 \cup_{\Sigma_g} H_1$ be a Heegaard decomposition of a 3-manifold Y. Since H_0, H_1 are two genus g handlebodies, there exist two tuples of g simple closed disjoint curves $\alpha_1, \cdots, \alpha_g \subset \partial H_0$ and $\beta_1, \cdots, \beta_g \subset \partial H_1$ each of which is compressing in each handlebody. Hence, on $\Sigma = \partial H_0 = \partial H_1$ we can see the diagram (Heegaard diagram) consisting of $2g$ curves $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$ (these are called α-curves and β-curves). Here we assume that all the intersection points between the α-curves and β-curves are transversal in the general position.

In this situation 'our moduli space' is the g-th symmetric product

$$\mathrm{Sym}^g(\Sigma_g) := \Sigma^g / S_g.$$

It is a smooth complex manifold by resolving the singular set by the S_g-action. Let $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ denote the image of

$$\alpha_1 \times \alpha_2 \times \cdots \times \alpha_g, \quad \beta_1 \times \beta_2 \times \cdots \times \beta_g$$

by the quotient map $\Sigma^g \to \Sigma^g / S_g$. The images in $\mathrm{Sym}^g(\Sigma_g)$ are g-dimensional tori. We denote the g curves by $\boldsymbol{\alpha} = \{\alpha_1, \cdots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \cdots, \beta_g\}$. Let z be a base point in Σ_g in the complement of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Naturally these are *totally real* submanifold ($\overset{\text{def}}{\Leftrightarrow}$ $Y \subset X$ and $i \cdot T_pY \cap T_pY = \{0\}$). Floer's theory works for the case of a pair of totally real submanifolds in a complex (or almost complex) manifold. This tuple of these data $(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is called *pointed Heegaard diagram.* Since $2\dim Y = \dim X$ holds, the intersection points $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are finite points.

Here we verify the homology of the symmetric product is isomorphic to the homology of Y.

$$\frac{H_1(\mathrm{Sym}^g(\Sigma_g))}{H_1(\mathbb{T}_\alpha) \oplus H_1(\mathbb{T}_\beta)} \cong H_1(Y, \mathbb{Z})$$

This means that the $\mathrm{Sym}^g(\Sigma_g)$ is regarded as an 'expounded body' of a 3-manifold. One direction is the ordinary homology and another sharpener direction is the Heegaard Floer homology.

The intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and base point z determine a spin^c structure. The point $\mathbf{x}$ implies g Morse trajectories from index 2 to 1 with respect to the Heegaard splitting. The point z is regarded as the trajectory from index 3 to 0 naturally. Thus removing the neighborhoods of the $g+1$ trajectories, we obtain a non-zero vector field in a holed 3-manifold. The homotopy classes of such fields up to homologous coincide with the spin^c structures on Y. The 'homologous' means that the two fields are homotopic after deleting several balls. Hence, we get the map:

$$s_z : \mathbb{T}_\alpha \cap \mathbb{T}_\beta \to \mathrm{Spin}^c(Y).$$

Let $\mathbf{x}, \mathbf{y}$ be intersection points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Let $\pi_2(\mathbf{x}, \mathbf{y})$ be the homotopy classes connecting $\mathbf{x}$ and $\mathbf{y}$. The class $[u] \in \pi_2(\mathbf{x}, \mathbf{y})$ is represented by a continuous map $u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$ satisfying

$$\begin{cases} u(\{\mathrm{Re} \geq 0\} \cap \partial\mathbb{D}) \subset \mathbb{T}_\alpha \\ u(i) = \mathbf{y} \\ u(\{\mathrm{Re} \leq 0\} \cap \partial\mathbb{D}) \subset \mathbb{T}_\beta \\ u(-i) = \mathbf{x}, \end{cases}$$

where $\mathbb{D}$ is the unit disk in the complex plane. For any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ let $n_z(\phi)$ be the algebraic count $\#((z \times \mathrm{Sym}^{g-1}(\Sigma_g) \cap u(\mathbb{D}))$. $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$ is the moduli space of holomorphic disk which is divided by natural $\mathbb{R}$-action. Then the differential $\hat{\partial}$ are defined to be

$$\hat{\partial}\mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), n_z(\phi) = 0, \mu(\phi) = 1} \widehat{\mathcal{M}}(\phi) \cdot \mathbf{y}.$$

Here μ is the Maslov index. We put $\widehat{CF}(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) = \mathbb{Z}\langle \mathbf{x} | \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle$. Then Ozsváth and Szabó proved the following:

Theorem 2.1 *$(\widehat{CF}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \hat{\partial})$ is a chain complex and its homology is a topological 3-manifold invariant.*

We denote the chain complex and homology by $\widehat{CF}(Y)$ and $\widehat{HF}(Y)$ respectively. The proof of this theorem is supported by an analytical argument on the moduli space of holomorphic disks. The argument is important, however, skip here. Actually, to hold this topological invariance, we need some general conditions: admissibility in Section 4.2.2 in [25].

The homology $\widehat{HF}(Y)$ is decomposed as follows:

$$\widehat{HF}(Y) = \oplus_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} \widehat{HF}(Y, \mathfrak{s}),$$

because the differential keep the map s_z, i.e., $s_z(\mathbf{x}) = s_z(\mathbf{y})$ when $\pi_2(\mathbf{x}, \mathbf{y}) \neq \emptyset$. The Euler number of $\widehat{HF}(Y, \mathfrak{s})$ is

$$\chi(\widehat{HF}(Y)) = \begin{cases} |H_1(Y, \mathbb{Z})| & b_1(Y) = 0 \\ 0 & b_1(Y) \neq 0. \end{cases}$$

Namely, $\widehat{HF}(Y)$ is a categorification of the ordinary homology $H_1(Y) = H^2(Y)$.

Let $(Y, \mathfrak{s})$ be a spin^c 3-manifold with $b_1 > 0$. Then the Euler characteristic of HF^+ agrees with the Turaev torsion.

$$\chi(HF^+(Y, \mathfrak{s})) = -\tau(Y, \mathfrak{s})$$

This equality makes sense in non-torsion spin^c case, but this equality holds even for any torsion spin^c structure by taking truncated Euler characteristic. In particular if Y is 0-surgery of a knot K, then the i-th Turaev torsion $t_i(K)$ is computed by the coefficients of the Alexander polynomial of K as in [33].

Furthermore by counting intersection numbers with $z \times \mathrm{Sym}^{g-1}(\Sigma_g)$, we define the following chain complex

$$CF^\infty(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) = \mathbb{Z}\langle U^n \cdot \mathbf{x} | \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, n \in \mathbb{Z}\rangle.$$

The differential ∂^∞ is a U-equivariant map and

$$\partial^\infty \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\mathbf{x}, \mathbf{y}), \mu(\phi) = 1} \#\widehat{\mathcal{M}}(\phi) U^{n_z(\phi)} \mathbf{y}.$$

There exists a subchain complex $CF^-(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) = \mathbb{Z}\langle U^n \cdot \mathbf{x} | \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, n \in \mathbb{Z}_{<0}\rangle$. The quotient complex CF^∞ / CF^- denotes by $CF^+(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$. Then we have the following:

Theorem 2.2 ([25]) *These chain complexes give topological invariants*

$$HF^\infty(Y), HF^-(Y) \text{ and } HF^+(Y).$$

Furthermore, these homology are $\mathbb{Z}[U] \otimes \Lambda^* H_1(Y, \mathbb{Z})/Tors$-module.

By the definition we have the following short exact sequence:

$$0 \to CF^-(Y) \xrightarrow{i} CF^\infty(Y) \xrightarrow{\pi} CF^+(Y) \to 0.$$

Thus we induce the long exact sequence:

$$\cdots \to HF^-(Y) \to HF^\infty(Y) \to HF^+(Y) \to HF^-(Y) \to \cdots$$

The cokernel of π is denoted by $HF_{\mathrm{red}}(Y)$. Since the chain complex $\widehat{CF}(Y)$ is embedded in the $n = 0$ level in $CF^\infty(Y)$, the short exact sequence:

$$0 \to \widehat{CF}(Y) \to CF^+(Y) \xrightarrow{\times U} CF^+(Y) \to 0$$

Here the second is the U-multiple map. Thus we induce the long exact sequence:

$$\cdots \to \widehat{HF}(Y) \to HF^+(Y) \to HF^+(Y) \to \widehat{HF}(Y) \to \cdots$$

2.2 Holomorphic disks

Next step of Heegaard Floer homology is a dimensional reduction of holomorphic disks in $\mathrm{Sym}^g(\Sigma_g)$. One of most remarkable points of Heegaard Floer theory is computable by using words of low-dimensional topology.

Proposition 2.3 ([25]) *For a continuous map $u : \mathbb{D} \to \mathrm{Sym}^g(\Sigma_g)$, there exists a continuous map from a branched cover $\hat{u} : \widehat{\mathbb{D}} \to \Sigma_g$ satisfying the following commutative diagram:*

$$\begin{array}{ccccc} \widehat{\mathbb{D}} & \longrightarrow & \{(x,v) \in \Sigma \times \mathrm{Sym}^g(\Sigma_g) | x \in v\} & \longrightarrow & \Sigma \\ \big\downarrow{\scriptstyle g\text{-fold branched cover}} & & \big\downarrow & & \\ \mathbb{D} & \longrightarrow & \mathrm{Sym}^g(\Sigma_g) & & \end{array}$$

This proposition asserts the holomorphic disk in $\mathrm{Sym}^g(\Sigma_g)$ is understood as a surface mapped in Σ_g. The condition $\partial\mathbb{D} \subset \mathbb{T}_\alpha \cup \mathbb{T}_\beta$ corresponds to $\hat{u} \subset \boldsymbol{\alpha} \cup \boldsymbol{\beta}$. The Maslov index in the definition of the Heegaard Floer homology is analytical information data of holomorphic disk.

Proposition 2.4 ([13]) *The Maslov index of a disk u in $[u] = \phi \in \pi_2(\mathbf{x}, \mathbf{y})$ is determined by a topological information of the region mapped in Σ_g.*

2.3 The TQFT viewpoint and absolute grading

Floer theory is defined in the framework of topological quantum field theory (TQFT). Any element in Floer homology can be delivered by the cobordism to an element in Floer homology of another side. Let $(W, \mathfrak{s})$ be a spin^c 4-dimensional cobordism from Y_1 to Y_2. Then there exists a U-equivariant map:

$$F^\infty_{W,\mathfrak{s}} : HF^\infty(Y_1, \mathfrak{t}_1) \to HF^\infty(Y_2, \mathfrak{t}_2),$$

where $\mathfrak{s}|_{Y_i} = \mathfrak{t}_i$. This map is induced to other homologies HF^+, HF^- and $\widehat{HF}$ in the same way. For the composition of two cobordisms $X = X_1 \cup X_2$ we have $F^\circ_{X_1,\mathfrak{s}_1} \circ F^\circ_{X_2,\mathfrak{s}_2} = \sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(X) | \mathfrak{s}|_{X_i} = \mathfrak{s}_i\}} F^\circ_{X,\mathfrak{s}}$, where $\circ = \infty, \pm, \wedge$.

Let Y be a rational homology sphere. In general, $(Y, \mathfrak{s})$ is a torsion spin^c 3-manifold, where $c_1(\mathfrak{s})$ is a torsion element. Then the Heegaard Floer homology admits an absolute $\mathbb{Q}$-grading gr. This grading is defined as follows. First for a generator $\mathbf{x}_0 \in \widehat{HF}(S^3)$ $\mathrm{gr}(\mathbf{x}_0) = 0$ holds. The difference of the gradings by $F^\circ_{W,\mathfrak{s}}$ ($\circ = \infty, \pm, \wedge$) is computed by

$$\mathrm{gr}(F^\circ_{W,\mathfrak{s}}(\mathbf{x})) - \mathrm{gr}(\mathbf{x}) = \frac{c_1^2(\mathfrak{s}) - 2\chi(W) - 3\sigma(W)}{4}.$$

The U-action lowers the degree by -2. These properties determine the absolute $\mathbb{Q}$-grading on the Heegaard Floer homology of $(Y, \mathfrak{s})$ uniquely.

Heegaard Floer homology relates to smooth 4-manifold invariant. Ozsváth and Szabó in [37] defined the mixed invariant as a map $F^{\mathrm{mix}}_{W,\mathfrak{s}} : HF^-(Y_1, \mathfrak{t}_1) \to HF^+(Y_2, \mathfrak{t}_2)$ by using 3+1 dimensional TQFT. First, we cut a 4-dimensional cobordism into 2 pieces V_1, V_2 along a suitable 3-manifold N. Suppose that each of V_1, V_2 has at least one positive eigenvalue in the intersection form and $\delta H_1(N)$ is 0 in $H^2(W, \partial W)$. The spin^c 4-manifolds $(V_1, \mathfrak{s}|_{V_1})$ and $(V_2, \mathfrak{s}|_{V_2})$ are spin^c cobordisms from $(Y_1, \mathfrak{t}_1)$ to $(N, \mathfrak{s}|_N)$ and from $(N, \mathfrak{s}|_N)$ to $(Y_2, \mathfrak{t}_2)$ respectively. Then $F^{\mathrm{mix}}_{W,\mathfrak{s}}$ is a composition as follows:

$$F^{\mathrm{mix}}_{W,\mathfrak{t}_1} : HF^-(Y_1, \mathfrak{t}_1) \xrightarrow{F^-_{V_1,\mathfrak{s}|_{V_1}}} HF^+(N, \mathfrak{s}|_N) \to HF_{\mathrm{red}}(N, \mathfrak{s}|_N) \subset HF^+(N, \mathfrak{s}|_N) \xrightarrow{F^+_{V_2,\mathfrak{s}|_{V_2}}} HF^+(Y, \mathfrak{t}_2).$$

The reduced part of $HF^+(N,\mathfrak{s}|_N)$ is also embedded in $HF^-(N,\mathfrak{s}|_N)$ naturally.

Let X be a closed smooth 4-manifold with $b_2^+(X) > 1$. Then deleting two 4-balls in X we built a cobordism W from S^3 to S^3. Applying $U^n \cdot \Theta^-$ in $HF^-(S^3)$ to the mixed invariant through this cobordism, we obtain an invariant

$$F^{\text{mix}}_{W,\mathfrak{s}}(U^n \cdot \Theta^-) =: \Phi_{X,\mathfrak{s}} \cdot \Theta^+ \in HF^+(S^3)$$

Here Θ^- (Θ^+) is the top (bottom) generator in $HF^-(S^3)$ ($HF^+(S^3)$) and $n = (c_1(\mathfrak{s})^2 - 2\chi(X) - 3\sigma(X))/4$ (it agrees with the dimension of the moduli space of Seiberg-Witten solutions). The number $\Phi_{X,\mathfrak{s}} \in \mathbb{Z}$ is a smooth 4-manifold invariant (smooth OS-invariant) and conjecturely, the number coincides with the Seiberg-Witten invariant $SW_{X,\mathfrak{s}}$.

Definition 2.5 (correction term [33]) *Let $(Y,\mathfrak{s})$ be a spin^c 3-manifold. Then $d(Y,\mathfrak{s})$ is defined to be the minimal grading in the image $\pi : HF^\infty(Y,\mathfrak{s}) \to HF^+(Y,\mathfrak{s})$. This invariant is called correction term (or d-invariant).*

In the case of integral homology sphere the invariant is denoted by $d(Y) \in 2\mathbb{Z}$. From the property of gr the correction term d is a spin^c rational homology cobordism invariant. Suppose that W is a rational homology cobordism between Y_1, Y_2. Let $\mathfrak{s}$ be a spin^c structure on W with $\mathfrak{s}|_{Y_i} = \mathfrak{t}_i$. Then we have

$$d(Y_1,\mathfrak{t}_1) = d(Y_2,\mathfrak{t}_2).$$

Let $(Y,\mathfrak{s})$ be a spin^c rational homology sphere. Then

$$HF^+(Y,\mathfrak{s}) \cong T^+_{(d(Y,\mathfrak{s}))} \oplus HF_{\text{red}}(Y,\mathfrak{s}).$$

Here $T^+_{(d)}$ is isomorphic to T^+ and the minimal grading is d. The d-invariant and Euler number of HF_{red} compute the Casson invariant.

Theorem 2.6 *Let Y be an integral homology sphere. Then the Casson invariant $\lambda(Y)$ is computed by*

$$\lambda(Y) = \chi(HF_{red}(Y)) - \frac{1}{2}d(Y).$$

The Heegaard Floer homology of (branched) covering space $\Sigma_n(K)$ is less known so far. Casson invariant formula of double branched cover is known by Mullins [18] as follows:

$$\lambda(\Sigma_2(K)) = \frac{1}{8}\text{sign}(K) - \frac{1}{12}V_K'(-1)/V_K(-1),$$

where $\Sigma_2(K)$ is a homology sphere and V_K is the Jones polynomial. On the other hand Ozsváth and Szabó show the following in [35]:

Theorem 2.7 *Let L a link in S^3. There exists a spectral sequence with E_2-term the Khovanov homology of L such that the sequence converges to the $\mathbb{F}_2$-coefficient homology $\widehat{HF}(\Sigma_2(L),\mathbb{F})$. Here $\mathbb{F}$ is a field isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

The Khovanov homology is a categorification of Jones polynomial, and $\widehat{HF}$ and Kh have surgery exact sequence. Do this spectral sequence interpret Mullins' formula? Definitely, covering space is a weak point for Casson invariant and Heegaard Floer homology. Dehn surgery formula for these invariants have been studied ever, however there are not so much research for covering space.

For a rational homolog sphere Y, *the renormalized Euler characteristic* is defined as

$$\hat{\chi}(Y,\mathfrak{s})=\chi(HF_{\mathrm{red}}(Y,\mathfrak{s}))-\frac{1}{2}d(Y,\mathfrak{s}).$$

Then due to [43] we have

$$\sum_{\mathfrak{s}\in\mathrm{Spin}^c(Y)}\hat{\chi}(Y,\mathfrak{s})=|H_1(Y)|\cdot\lambda_{CW}(Y),$$

where λ_{CW} is the Casson-Walker invariant.

The correction term has the 4-dimensional information as follows:

Theorem 2.8 *Let $(Y,\mathfrak{t})$ be a spin^c rational homology sphere. If $(Y,\mathfrak{t})$ admits a negative definite bounding $(W,\mathfrak{s})$ with $\mathfrak{s}|_Y=\mathfrak{t}$, then we have the following inequality:*

$$c_1(\mathfrak{s})^2+b_2(W)\le 4d(Y,\mathfrak{t}).$$

The extended inequality of this theorem to a 3-manifold with $b_1>0$ can prove the Thom conjecture again [33].

2.4 L-spaces

Let Y be a rational homology sphere. If $\widehat{HF}(Y,\mathfrak{s})=\mathbb{Z}$ for any spin^c structures, then Y is called *an L-space.* Examples of L-space are lens spaces. The genus one Heegaard diagram of a lens space $L(p,q)$ consists of single α-curve and β-curve. Then the pointed Heegaard digram is (T^2,α,β,z). Thus the symmetric product is T^2 itself and we have $\mathbb{T}_\alpha=\alpha$ and $\mathbb{T}_\beta=\beta$. Hence, the generators $\alpha\cap\beta$ are p points. The differentials are all zero, because the p points belong to distinct spin^c structures. Thus we have the following:

Proposition 2.9 *Let Y be a lens space and $\mathfrak{s}$ any spin^c structure on Y.*

$$\widehat{HF}(Y,\mathfrak{s})\cong\mathbb{Z},\ HF^\infty(Y,\mathfrak{s})=\mathbb{Z}[U,U^{-1}],\ HF^+(Y,\mathfrak{s})\cong\mathbb{Z}[U,U^{-1}]/U\mathbb{Z}[U],\ HF^-(Y,\mathfrak{s})\cong\mathbb{Z}[U]$$

We put $T^\infty=\mathbb{Z}[U,U^{-1}]$, $T^+=\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U]$, $T^-=\mathbb{Z}[U]$. Another component as a $\mathbb{Z}[U]$-module is $\mathbb{Z}[U]/U^n$, which is a finite rank $\mathbb{Z}$-module and we denote it by $T^+(n)$.

Any elliptic manifolds (i.e., it is a finite fundamental group) are an L-space. In other words L-space is a generalization of such a manifold. In addition, L-spaces contain not elliptic 3-manifolds. Suppose that a hyperbolic knot K is an L-space knot (i.e., an integral Dehn surgery is an L-space), e.g., the $(-2,3,7)$-pretzel knot. Then a sufficiently large Dehn surgery of the knot is also an L-space. This is an easy result due to the surgery exact sequence of Heegaard Floer homology. In other examples, double cover of any alternating knot is also L-space [35]. [35] shows that the double cover of *quasi-alternating link* is L-space. The quasi-alternating knots $\mathcal{Q}$ is the smallest set defined inductively as follows: (1) unknot$\in\mathcal{Q}$; (2) if $L_0,L_1\in\mathcal{Q}$, $\det(L_0)\det(L_1)\neq 0$ and L_0,L_1 are two types of the resolution of a crossing of L and $\det(L)=\det(L_0)+\det(L_1)$, then $L\in\mathcal{Q}$.

L-space includes interesting geometric properties besides:

Theorem 2.10 ([29]) *Any L-space does not admit co-oriented taut foliation.*

This result is useful, because the nonexistence of taut foliation is hard to prove without L-space property. We do not find a method in Heegaard Floer theory to prove the nonexistence of tight contact structure, so far.

Recently developing noticeable property is left-orderbility of the fundamental group. '*Left-orderable*' (*LO*) means that the existence of a total order on a group that any left action of the group keeps the order relation between any two elements. A group is LO if and only if it is a subgroup in $\mathrm{Homeo}^+(\mathbb{R})$. We believe that the following mysterious relationship:

Conjecture 2.11 *Let Y be a rational homology sphere. Then Y is not LO, if and only if Y is an L-space.*

It is well-known that LO group is a non-torsion group. Then any elliptic manifold is not LO. Furthermore, Boyer, Gordon, and Watson [2] proved that for any Seifert manifold or Sol manifold this conjecture is true. Any other hyperbolic manifold is not so known. For example for not all double covers of (quasi-)alternating link this relationship is verified.

2.5 Surgery exact triangle

In general Floer theory an exact triangle among three homologies is inherent. In Heegaard Floer theory, the three homologies correspond to replacements of α or β-curves. In the context of a low-dimensional topology, the replacements are 0-surgery, and p-surgery. Let Y_n be a n-surgery of a 3-manifold Y along a null-homologous knot K. Then the following exact triangle holds:

$$\cdots \to HF^+(Y) \to HF^+(Y_0,[i]) \to HF^+(Y_p,i) \to HF^+(Y) \to \cdots.$$

Read [26] for surgery exact sequence. The decategorification of this exact triangle is the Dehn surgery formula of Casson invariant as follows:

$$\lambda_{CW}(Y_p) = \lambda(Y) - s(1,p) + \frac{1}{2p}\Delta_K''(1),$$

where $s(1,p)$ is the Dedekind sum, which is equivalent to the Casson-Walker invariant of $L(p,1)$. Unlike Casson invariant, generally, for determining Floer homology of Y_p we need homologies of Y and Y_0, exact triangle and a more extra information. The other triads of surgery exact sequence are $\{Y, Y_n, Y_{n+1}\}$ or $\{Y, Y_0, Y_{1/q}\}$. These correspond to surgery formulas for Casson invariant. The surgery triangle is applied on the variable situation.

2.6 Contact structure

Let (Y,ξ) be a contact 3-manifold, which the 2-plane filed ξ is nowhere integrable. Any contact 3-manifold is decomposed into two types of non-isotopic contact structures: *tight* and *overtwisted*. The classification of tight contact structures is more difficult than the one of overtwisted structures. The latter structures agrees with the classification of homotopy types of 2-plane fields due to Eliashberg [4]. The situations which a tight contact structure naturally appear is the cases where it is the boundary of a symplectic 4-manifold with some suitable boundary condition. This contact structure is called *symplectic fillable.* The boundary of Stein manifold is also a tight contact structure, which is called *Stein fillable.*

Ozsváth and Szabó define in [36] an isotopy invariant of contact structure. The class $c(\xi)$ is an element in $\widehat{HF}(-Y, \mathfrak{s}(\xi))/\pm 1$ and has the following property:

Theorem 2.12 *If ξ is an overtwisted contact structure, then $c(\xi) = 0$. If ξ is Stein fillable contact structure, then $c(\xi) \neq 0$.*

The converse of this theorem does not holds. This theorem implies that the invariant $c(\xi)$ detects the tight-ness and Stein fillability. Since the detection of tight-ness by using contact topology is difficult in general, after this theorem, the computation of $c(\xi)$ has been a reasonable method. The most simple example in [15] is the contact $+1$-surgery of a Legendrian right-handed trefoil, which is topologically $-\Sigma(2,3,4)$. It admits no symplectic fillable but a tight contact structure. The former is due to an application of the Seiberg-Witten invariant and the latter part is due to the computation of $c(\xi)$.

In general, invariants of some tight contact structures are vanishing (structures with some symplectic fillable or positive Giroux torsion etc.) However if one take a suitable twisted system, one can sometimes give a non-vanishing class [29]. For other examples related to contact structures, symplectic topology, Lefschetz fibration, and Stein manifold readers should take a look at [22].

2.7 Graph manifolds

In [34], Ozsváth and Szabó gave a method to compute HF^+ for plumbed 3-manifolds. In short, this method is a machinery put surgery exact sequence together. Némethi [19] gives a systematical algorithm (the tau function and the graded root) to compute Heegaard Floer homology for any plumbed 3-manifold with at most one bad vertex. This computation gives the module structure of HF^+ completely, and it is used as a first useful trial to explore Heegaard Floer behavior for some topological phenomenon.

The more general 3-manifolds are hyperbolic 3-manifolds. Does Heegaard Floer homology capture hyperbolic structure? This is a natural and challenging question.

3 Knot Floer homology

3.1 Definition and an example

The most attractable point of Heegaard Floer theory is what it is able to build some variations of Floer homology. One of variation is knot Floer homology. Knot Floer homology was defined by Ozsváth and Szabó [27] and independently by Rasmussen [42].

Let K be a knot in S^3. We take a $(g,1)$-decomposition $S^3 = H_0 \cup_{\Sigma_g} H_1$ of (S^3, K). In other words K is transversal about Σ_g and the union of the two arcs $A_i = K \cap H_i$ $(i = 0, 1)$. Furthermore, we assume that A_i does not intersect compressing disks of α-curves and β-curves. Such a Heegaard splitting always exists for any knot K. From the assumption K intersects two points $\partial A_0 = \partial A_1$ in Σ_g with α-curves and β-curves disjoint. We denote the points by z, w. We call such a diagram *double pointed Heegaard diagram*

Then the chain complex of knot Floer homology $CFK^\infty(S^3, K, i)$ is isomorphic to $CF^\infty(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ as a $\mathbb{Z}[U]$-module. We denote by $Mas(\mathbf{x})$ the absolute grading on $CF^\infty(Y)$ which is defined in the previous section. The grading satisfies:

$$Mas(\mathbf{x}) - Mas(\mathbf{y}) = \mu(\phi) - 2n_w(\phi),$$

where $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

We will introduce a filtration (Alexander filtration $Alex(\mathbf{x})$) on the module by using a knot in the S^3. The filtration is defined as follows. For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$

$$Alex(\mathbf{x}) - Alex(\mathbf{y}) = n_z(\phi) - n_w(\phi),$$

where ϕ is a disk with $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. This definition is a relative $\mathbb{Z}$-filtration only. To make an absolute filtration we impose a symmetry.

$$\#\{\mathbf{x} | Alex(\mathbf{x}) = i\} = \#\{\mathbf{x} | Alex(\mathbf{x}) = -i\} \bmod 2.$$

We denote such a filtered chain complex of $CF^\infty(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ by $CFK^\infty(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$.

Here we give an example of double pointed Heegaard diagram and the filtered chain complex. Consider the trefoil $K = 3_1$. In Figure 1 we describe two digging arcs and two

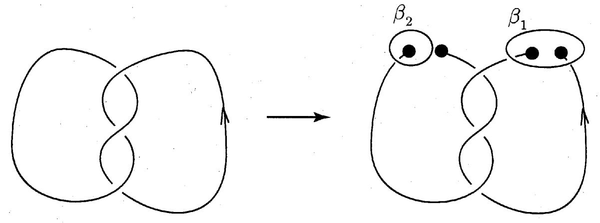

Figure 1: The trefoil knot and the Heegaard decomposition of S^3.

2-handles on a 3-ball. The circles β_1, β_2 are the attaching circles. The circle β_2 is the meridian disk of K. Moving the circles and holes on the 3-ball we obtain the first picture in Figure 2. The dashed arc presents a longitude of a knot K. In this case the double points z, w are two points close to the meridian circle β_2. The position of curves α_2, β_2 can be made cancel the two critical points in terms of Morse theory. Hence, the diagram moves to the next picture and it is isotopic to the last diagram in Figure 2.

From the last diagram we compute the Alexander filtration of the trefoil. The intersection points are 3 points: $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. The non-trivial disks on the diagram are D_1, D_2. D_1 is a disk connecting from $\mathbf{x}_2$ to $\mathbf{x}_1$ and D_2 is a disk connecting from $\mathbf{x}_2$ to $\mathbf{x}_3$. Thus, the filtration definition implies

$$Alex(\mathbf{x}_2) - Alex(\mathbf{x}_1) = -1, \ Alex(\mathbf{x}_2) - Alex(\mathbf{x}_3) = 1.$$

From the symmetric condition we have

$$Alex(\mathbf{x}_1) = 1, \ Alex(\mathbf{x}_2) = 0, \ Alex(\mathbf{x}_3) = -1.$$

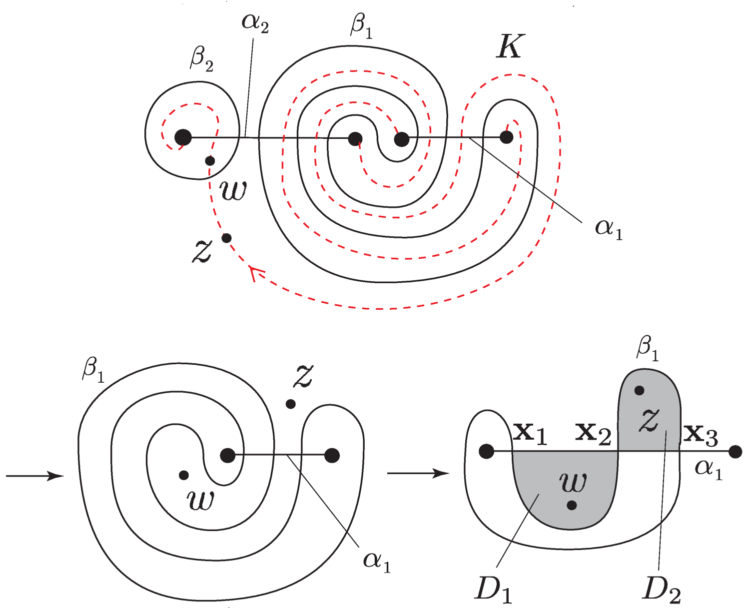

Figure 2: A double pointed Heegaard diagram of the trefoil knot and a move.

The Maslov grading is

$$Mas(\mathbf{x}_2) - Mas(\mathbf{x}_1) = -1, \ Mas(\mathbf{x}_2) - Mas(\mathbf{x}_3) = 1$$

The differential ∂^∞ works as follows:

$$\partial^\infty \mathbf{x}_1 = 0, \ \partial^\infty \mathbf{x}_2 = U \cdot \mathbf{x}_1 + \mathbf{x}_3, \ \partial^\infty \mathbf{x}_3 = 0$$

Here we compute hat Heegaard Floer homology. Since the version is embedded in the 0-level in CF^∞ with $U = 0$, then we have

$$\hat{\partial}\mathbf{x}_1 = 0, \ \hat{\partial}\mathbf{x}_2 = \mathbf{x}_3, \ \hat{\partial}\mathbf{x}_3 = 0,$$

$$\widehat{HF}(S^3) \cong \mathbb{Z} \cdot \mathbf{x}_1.$$

By the definition of the absolute grading, $\mathrm{gr}(\mathbf{x}_1) = 0$. Since U-action decreases by 2, the Maslov gradings are computed as $Mas(\mathbf{x}_1) = 0, Mas(\mathbf{x}_2) = -1, Mas(\mathbf{x}_3) = -2$.

Here we define differential $\hat{\partial}^K$ of knot Floer homology $\widehat{CFK}(S^3, K)$ as follows:

$$\hat{\partial}^K \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \ \sum_{\phi \in \pi_2(\mathbf{x},\mathbf{y}), n_z(\phi) = n_w(\phi) = 0} \#\widehat{\mathcal{M}}(\phi)\mathbf{y}.$$

The chain complex $\widehat{CKF}(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ is isomorphic to $\widehat{CF}(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ and the differentials are $\widehat{\partial}^K$.

Theorem 3.1 (Knot Floer homology) *The homology of* $(\widehat{CKF}(\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z), \hat{\partial}^K)$ *is an isotopy invariant of a knot* K*. This homology decomposes as follows:*

$$\widehat{HFK}(S^3, K) = \oplus_i \widehat{HFK}(S^3, K, i),$$

where $\widehat{HFK}(S^3, K, i)$ *is the homology with the Alexander grading* i.

Furthermore, the Euler characteristic of $\widehat{HKF}(S^3, K)$ *is the* i*-th Alexander polynomial*

$$\chi(\widehat{HFK}(S^3, K, j)) = a_i.$$

In the case of our trefoil, the differentials $\hat{\partial}^K$ are computed as follows:

$$\hat{\partial}^K \mathbf{x}_1 = 0,\ \hat{\partial}^K \mathbf{x}_2 = 0,\ \hat{\partial}^K \mathbf{x}_3 = 0.$$

Hence, we obtain

$$\widehat{HFK}(S^3, K, j) = \begin{cases} \mathbb{Z}_{(0)} & j = 1 \\ \mathbb{Z}_{(-1)} & j = 0 \\ \mathbb{Z}_{(-2)} & j = -2. \end{cases}$$

Taking the Euler characteristic of these complexes, we have

$$\sum_{j=-g(K)}^{g(K)} \chi(\widehat{HFK}(S^3, K, j)) \cdot t^j = t - 1 + t^{-1} = \Delta_K(t).$$

We denote for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ the map $CFK^\infty(S^3, K) \ni U^n \cdot \mathbf{x} \mapsto -n \in \mathbb{Z}$ by Alg. Then one can visualize the double complex $Alex$-filtration and Alg-filtration on the plane. In the case of $K = 3_1$, the double complex is described as Figure 3.

Generally, due to [40] if K is alternating, then $\widehat{HFK}(K, j) \cong \mathbb{Z}^{|a_i|}_{(s+\frac{\sigma}{2})}$. In this case the homology is determined by the Alexander polynomial and the knot signature. However, for non-alternating knot, it is different in general, see Theorem 3.7.

Other important variations of definitions of knot Floer homology are the ones by grid diagram [17] and Kauffman state [40]. By the former, knot Floer homology is computed from a purely combinatorial information of a knot. By the latter, on the knot Floer homology, a geometric meaning of a state sum is given.

3.2 Genus and fiberness.

In this section we give the characterizations of Seifert genus and fiberness on knot Floer homology. The knot Floer homology detects the Seifert genus $g(K)$.

Theorem 3.2 ([29]) *Let* K *be a knot in* S^3*. Then we have*

$$g(K) = \max\{s | \widehat{HFK}(S^3, K, s) \neq \{0\}\}.$$

Furthermore, Heegaard Floer homology detects the fibered-ness of knot.

Theorem 3.3 ([20]) *Let* K *be a knot in* S^3*. Then* K *is a fibered knot if and only if* $\widehat{HFK}(S^3, K, g(K)) \cong \mathbb{Z}$.

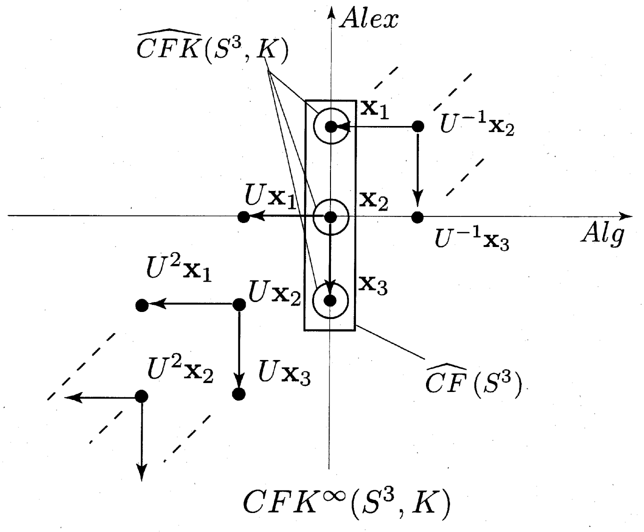

Figure 3: Differentials of $CFK^\infty(S^3, K)$ and $\widehat{CFK}(S^3, K)$ in it.

3.3 Legendrian and transverse knot.

Legendrian knot (or transverse knot) in a contact 3-manifold is a curve tangent (or transverse) on any plane of the standard contact S^3 at any point on the knot. Isotopy invariants for those knots are classically the Thurston-Bennequin invariant (tb), and the rotation number (rot) for Legendrian knots, and the self-linking number (sl) for transverse knots. These and smooth isotopy type are called *classical invariants.* Classical invariants distinguish these knots but not complete. For example Chekanov [3] and Eliashberg found non-isotopic Legendrian knots (CE-pairs) with the same classical invariants.

In the Heegaard Floer theory, (Legendrian or transverse) (unclassical) isotopy invariants in (S^3, ξ_{std}) are included. Let L be a Legendrian knot in the standard contact S^3. Ozsváth, Szabó and Thurston defined in [41] defined Legendrian isotopy invariant $\lambda(L) \in HFK^-(S^3, L)$ using the grid diagram. After that, in [14] Lisca, Ozsváth, Stipsicz, and Szabó defined another Legendrian invariant $\mathfrak{L}(L) \in HFK^-(S^3, L)$ using open book decomposition compatible with the contact structure, which it is proved these invariants are equivalent to each other in [1]. For the transverse knot invariant $\mathfrak{T}$, which is defined by Legendrian approximation, the same history has been traced. This equivalence is so fruitful in terms of what some deeper information may be appeared. The image of the hat version homology $\widehat{HFK}(L)$ is referred to as $\widehat{\mathfrak{L}}(L)$. The invariants reprove CE-pair is not isotopic and a link 6^2_3 is not simple (i.e., the knot is determined uniquely from classical invariants), and transverse CE-pair, which is trivial DGA invariant are also not (transverse) isotopic [23]. The DGA invariant is sometimes not useful for Legendrian approximation of transverse knot because it vanishes for a stabilized knot. However the invariant $\mathfrak{L}$ is sensitive for stabilized knots (it is multiplied by U or identity depending on stabilization). By using $\mathfrak{T}$, it is proven that the CE's twist knots are transverse not simple. The Alexander and Maslov gradings of $\mathfrak{L}$ and $\mathfrak{T}$ compute as follows:

$$Alex(\mathfrak{L}(L)) = \frac{1}{2}(\mathrm{tb}(L) - \mathrm{rot}(L) + 1)$$

$$2Alex(\mathfrak{L}(L)) - Mas(\mathfrak{L}(L)) = d_3(\xi).$$

$$M(\mathfrak{T}(T)) = 2A(\mathfrak{T}(T)) = \mathrm{sl}(T) + 1.$$

Here $d_3(\xi)$ is the invariant of 2-plane field on 3-dimensional obstruction defined in [7].

3.4 L-space surgery

The *lens space knot* K is a knot yielding a lens space by an integral Dehn surgery. This research field is the one which was remarkably developed by Heegaard Floer homology. Ozsváth and Szabó obtained the Alexander polynomial restriction for lens space knot (or for L-space knot more strongly).

Theorem 3.4 *Let K be an L-space knot. Then the Alexander polynomial is of form:*

$$\Delta_K(t) = (-1)^m + \sum_{j=1}^{m}(-1)^{m-j}(t^{n_j} + t^{-n_j}),$$

where the exponents n_j give an increasing sequence:

$$0 < n_1 < n_2 < \cdots < n_m = d.$$

As a corollary, this theorem and Theorem 3.3 imply that any L-space knot must be a fibered knot. By this theorem many knots are ruled out from L-space knot, in particular lens space knot. Besides, if K yields a lens space $L(p,q)$, then due to [12], we have

$$2g(K) - 1 \leq p. \tag{1}$$

This inequality follows from $HF^+(S^3_0(K), i) \cong T^+(t_i)$ $(i \neq 0)$ and $g(K) = d$ (degree of $\Delta_K(t)$). Greene in [8] improved the inequality (1) in the case of lens space knot in S^3 as follows:

$$2g(K) - 1 \leq p - 2\sqrt{(4p+1)/5}$$

by using Theorem 3.4 essentially. [8] classifies all lens spaces obtained by Dehn surgeries in S^3 by extracting some information of a lattice embedding coming from embedding of the resolution of a lens space in a definite 4-manifold. This implies this property in Theorem 3.4 is so strong among the other L-space knot restrictions.

3.5 Dehn surgery formula

We introduce the Heegaard Floer homology of a Dehn surgery $S^3_{p/q}(K)$ in [27], [38], [39]. Briefly speaking, the computation of the homology is the mapping cone technique of the chain complex. Let K be a knot in S^3. We denote $Alex$ and Alg by j- and i-coordinate.

Let C be $CFK^\infty(K)$. Let $A_s^+ = C\{\min\{i, j-s\} \geq 0\}$ and $B^+ = C\{i \geq 0\}$. We define $\mathbb{A}_i^+$, $\mathbb{B}^+$, v^+, h^+ as follows:

$$\mathbb{A}_i^+ = \oplus_{s\in\mathbb{Z}}(s, A^+_{\lfloor (i+ps)/q \rfloor}),\ \mathbb{B}_i^+ = \oplus_{s\in\mathbb{Z}}(s, B^+)$$

$$v^+ : (s, A^+_{\lfloor (i+ps)/q \rfloor}) \to (s, B^+),\ h^+ : (s, A^+_{\lfloor (i+ps)/q \rfloor}) \to (s+1, B^+).$$

The second components v_s^+ and h_s^+ of v^+ and h^+ on the level s are defined as follows: The map v_s^+ is defined to be the quotient to $i \geq 0$. The map h_s^+ is defined to be the composition of the quotient to $j \geq \lfloor \frac{i+ps}{q} \rfloor$ and the identification with $i \geq 0$. Here we define $\mathbb{A}_i^+ \to \mathbb{B}^+$ to be

$$D^+_{i,p/q}(s, a_s) = \{(s, v^+_{\lfloor \frac{i+ps}{q} \rfloor}(a_s) + h^+_{\lfloor \frac{i+p(s-1)}{q} \rfloor}(a_{s-1})\}.$$

Theorem 3.5 *$HF^+(S^3_{p/q}(K), i)$ is isomorphic to the homology of the mapping cone $\mathbb{X}^+_{i,p/q}$ with respect to $D^+_{i,p/q}$.*

The mapping cone with respect to $D^+_{i,p/q}$ is the chain complex which the generators are $\mathbb{A}_i^+ \oplus \mathbb{B}_i^+$ and the differential is

$$\begin{pmatrix} \partial_{\mathbb{A}_i^+} & 0 \\ D^+_{i,p/q} & \partial_{\mathbb{B}^+ i} \end{pmatrix}.$$

In the case where $q = 1$ and $p \geq 2g(K) - 1$, as in [27], the module of HF^+ of $S^3_p(K)$ or $S^3_{-p}(K)$ is isomorphic to

$$HF^+_*(S^3_p(K), [s]) \cong H_{*+\frac{(2s-p)^2-p}{4p}}(A_s^+, [s]),\quad HF^+_*(S^3_{-p}(K), [s]) \cong H_{*+\frac{p-(2s+p)^2}{4p}}({}^bA_s^+, [s])$$

where ${}^bA^+ = C\{\min\{i, j-s\} \geq 0\}$. Further the homology of $\hat{A}_s := C\{\max\{i, j-s\} = 0\}$ and ${}^b\hat{A}_s := C\{\min\{i, j-s\} = 0\}$ compute the hat version of $S^3_p(K)$ and $S^3_{-p}(K)$ in the same way.

Ni and Wu [21] gave the correction term formula of rational surgery of a knot. The spin^c structures of Dehn surgery of a knot are identified with $\mathbb{Z}/p\mathbb{Z}$ naturally.

Theorem 3.6 *Let K be a knot in S^3 and p, q positive integers. For $0 \leq i < p$ we have*

$$d(S^3_{p/q}(K), i) = d(L(p,q), i) - 2\max\{V_{\lfloor \frac{i}{q} \rfloor}, V_{\lfloor \frac{p+q-1-i}{q} \rfloor}\}.$$

Here V_i is a concordance invariant will be defined as below.

3.6 Satellite knots

Hedden computed $\widehat{HFK}$ of the $(+)$-Whitehead double $D_+(K, n)$ of a knot K. The coefficient is $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$. Let g be the genus of K.

Theorem 3.7 ([9]) *Let K be a knot in S^3. The case of $n \geq 2\tau(K)$:*

$$\widehat{HFK}(D_+(K,n), i) = \begin{cases} \mathbb{F}^{n-2g-2}_{(1)} \oplus_{j=-g}^{g} [H_{*-1}(\mathcal{F}(K,j))]^2 & i = 1 \\ \mathbb{F}^{2n-4g-3}_{(0)} \oplus_{j=-g}^{g} [H_{*}(\mathcal{F}(K,j))]^4 & i = 0 \\ \mathbb{F}^{n-2g-2}_{(-1)} \oplus_{j=-g}^{g} [H_{*+1}(\mathcal{F}(K,j))]^2 & i = -1. \end{cases}$$

The case of $n < 2\tau(K)$:

$$\widehat{HFK}(D_+(K,n),i) = \begin{cases} \mathbb{F}_{(1)}^{2\tau(K)-2g-2} \oplus \mathbb{F}_{(0)}^{2\tau(K)-n} \oplus_{j=-g}^{g} [H_{*-1}(\mathcal{F}(K,j))]^2 & i=1 \\ \mathbb{F}_{(0)}^{4\tau(K)-4g-4} \oplus \mathbb{F}_{(-1)}^{4\tau(K)-2n-1} \oplus_{j=-g}^{g} [H_{*}(\mathcal{F}(K,j))]^4 & i=0 \\ \mathbb{F}_{(-1)}^{2\tau(K)-2g-2} \oplus \mathbb{F}_{(-2)}^{2\tau(K)-n} \oplus_{j=-g}^{g} [H_{*+1}(\mathcal{F}(K,j))]^2 & i=-1. \end{cases}$$

Here $\tau(K)$ (concordance invariant defined in the knot Floer theory) will be defined in the next section. Since the Alexander polynomial of $D_+(K,n)$ is $-nt+(2n+1)-nt^{-1}$, this theorem says that knot Floer homology is strictly stronger invariant than Alexander polynomial. For example, it follows immediately that if $D_+(K,n)$ is fibered non-trivial knot, then the double is figure-8 knot only ($K=$ the unknot and $n=1$) by using Theorem 3.3 and the reduced knot filtration formula of $\tau(K)$ in [44]. Thus, due to [9] the τ-invariant of $D_+(K,n)$ is following:

$$\tau(D_+(K,n)) = \begin{cases} 0 & n \geq 2\tau(K) \\ 1 & n < 2\tau(K). \end{cases}$$

Hence, untwisted ($n=0$) Whitehead double with positive $\tau(K)$ is not smoothly slice. On the other hand, Freedman's result [5] says that any knot with trivial Alexander polynomial is topologically slice. Then, for example for any positive torus knot K, $D_+(K,0)$ is topologically slice but not smoothly slice knot. Such a knot gives an *exotic* $\mathbb{R}^4$ by Freedman's result [5] and Gompf's result [6]. The result of 0-framed attachment of the knot on the 4-ball is embeddable in a 4-manifold homeomorphic to $\mathbb{R}^4$ but not in $\mathbb{R}^4$.

4 Concordance invariants

Two knots K_0, K_1 are defined to be *(knot) concordant* if there exists a smoothly embedded annulus $f: S^1 \times [0,1] \hookrightarrow S^3 \times [0,1]$ such that $f|_{S^1\times\{i\}} = K_i$. The set $\mathcal{C}$ of the equivalent classes of knots by knot concordance is an abelian group so that the connected-sum is the addition. The zero element in $\mathcal{C}$ corresponds to the equivalent class of slice knot. We say a knot K to be *slice*, if there exists a proper smoothly embedded disk in 4-ball such that the boundary is isotopic to K. In the Heegaard Floer package there exist some concordance invariants. We give the definitions of those. In this section we denote $CFK^\infty(K)$ by C.

4.1 The τ-invariant and variations.

Let $\mathcal{F}(K,j)$ be $C\{Alex \leq j, Alg = 0\}$. Then we have

$$\cdots \subset \mathcal{F}(K,i-1) \subset \mathcal{F}(K,i) \subset \mathcal{F}(K,i+1) \subset \cdots$$

and $\cup_{i\in\mathbb{Z}}\mathcal{F}(K,i) = \widehat{CF}(S^3)$. Knot Floer homology $\widehat{HFK}(S^3,K,j)$ is the graded homology $H_*(\mathcal{F}(K,j)/\mathcal{F}(K,j-1))$. The inclusion map $\iota^m : \mathcal{F}(K,m) \subset \widehat{CF}(S^3)$ induces a map $\iota^m_* : H_*(\mathcal{F}(K,m)) \to \widehat{HF}(S^3) \cong \mathbb{Z}$ on the homology. If m is sufficiently small, then $\mathcal{F}(K,m)$ is the 0-map. If m is sufficiently large, then $\mathcal{F}(K,m)$ is isomorphic to $\widehat{CF}(S^3)$. The τ-invariant is defined as the following minimal:

$$\tau(K) := \min\{m | \iota^m_* \text{ is non-trivial}\}.$$

Theorem 4.1 **([28])** *The τ-invariant gives a group homomorphism $\tau : \mathcal{C} \to \mathbb{Z}$.*

This invariant τ gives a bound of the 4-ball genus $g_4(K)$.

Theorem 4.2 **([28])** *Let K be a knot. Then the following inequality holds:*

$$|\tau(K)| \leq g_4(K).$$

The knot signature $\sigma(K)$ is also similar inequality $|\sigma(K)| \leq 2g_4(K)$. For any alternating knot, $\sigma(K) = -2\tau(K)$. Generally, this inequality on τ is sharper than the one on σ. For τ, various sharpener invariants exist. To compute τ of a cable knot, Hom in [10] introduced ϵ-invariant:

$$\epsilon = \begin{cases} 1 & \widehat{CF}(S^3) \to {}^b\hat{A}_{\tau(K)}\text{: trivial on the homology,} \\ -1 & \hat{A}_{\tau(K)} \to \widehat{CF}(S^3)\text{: trivial on the homology,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.3 **([10])** *Let $K_{p,q}$ be the (p,q)-cable knot of K with $p > 0$. Then, we have*

$$\tau(K_{p,q}) = \begin{cases} p\tau(K) + (p-1)(q - \epsilon(K))/2 & \epsilon(K) \neq 0 \\ (p-1)(q - sgn(q))/2 = \tau(T_{p,q}) & \epsilon(K) = 0 \end{cases}$$

Here $T_{p,q}$ is the (p,q)-torus knot.

Ni and Wu defined knot concordance invariant V_k to compute the correction term of Dehn surgery of a knot in [21]. Let v_k^+ be a map $A_k^+ \to B^+$ which is defined in the similar way to the previous section (note that the degree of the map is slightly different from the previous section). The restriction to the T^∞-part gives a U-power map. We define the exponent to be V_k. The same invariant H_k replacing v_k^+ with $h_k^+ : A_k^+ \to B^+$ is given. V_k is a decreasing $\mathbb{Z}_{\geq 0}$-valued function on $\mathbb{Z}$ and $V_{k+1} = V_k$ or $V_k - 1$. The following minimal value $\nu(K) := \min\{k | V_k = 0\}$ is a concordance invariant and $\tau(K) \leq \nu(K) \leq g_4(K)$ holds in [11]. This $\nu(K)$ is a sharpener invariant than $\tau(K)$.

4.2 Υ-invariant.

Recently, Ozsváth, Stipsicz and Szabó in [24] (also Livingston [16]) defined the concordance invariant $\Upsilon_K(t)$, whose value is a continuous function on the interval $[0,2]$. And Υ is a group homomorphism $\mathcal{C} \to Cont([0,2])$. $\Upsilon_K(t)$ is a piecewise linear function with finite non-smooth points, which is the number of the smooth points is also concordance invariant. We have $\Upsilon_K'(0) = -\tau(K)$.

Here we define Υ according to [16]. Let $\mathcal{F}_t$ be an s-filtered chain complex with $\mathcal{F}_{t,s} = C\{\frac{t}{2}Alex + (1 - \frac{t}{2})Alg \leq s\}$. Then Υ is defined to be $\Upsilon_K(t) = -2\nu(K, \mathcal{F}_t)$, where

$$\nu(K, \mathcal{F}_t) = \min\{s | \text{Image}(H_*(\mathcal{F}_{t,s}) \to H_*(C)) \text{ contains the non-trivial element of grading } 0\}.$$

$\Upsilon_K(t)$ give a 4-ball genus bound:

Theorem 4.4 **([24])** *For any $0 < t < 1$,*

$$|\Upsilon_K(t)|/t \leq g_4(K).$$

This invariant Υ is weaker than $\{V_k\}$ invariant but is useful for the linear independence on concordance group of a family of knots. For example, consider torus knots $\{T_{p,p+1}\}$. Let G_0 be generators in C with grading 0. Any element in G_0 lies on the boundary of the convex hull of G_0 in $\mathbb{R}^2$. Since $\Upsilon_{T_{p,p+1}}$ has a convex function with p singular points, these are linearly independent in $Cont([0,2])$ and the knots are also linearly independent.

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