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Quandle cocycle invariants of cabled surface knots

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1 Introduction

A *surface knot* is a closed connected surface smoothly embedded in the 4-dimensional Euclidean space $\mathbb{R}^4$. For a surface knot F, we consider a *cabling* $F^{(m,\nu)}$ of F, i.e. a (not necessarily connected) surface link in a neighborhood of F, which is an m-fold covering of F; it is a kind of satellites (e.g. see [13]). A formula representing the value of an invariant of a cabled knot by means of invariants of the original knot is called a *cabling formula*. In some cases, e.g. in [11], a cabling formula has information which the original invariant does not, and here is an interesting problem: find an invariant which describes a cabling formula.

In this note, we give a solution to this problem on quandle cocycle invariants for surface knots. Quandle cocycle invariants (3-cocycle invariants) were defined in [1] and have been effective tools to examine isotopy classes and other geometrical properties of surface knots. In [9], the author showed that cabling formulae are generally described by what he calls kink cocycle invariants there and that in specific cases these were decomposed into 3-cocycle invariants and 2-cocycle invariants, which were defined in [4], and furthermore gave explicit cabling formulae on some popular quandles. However, in general cases kink cocycle invariants are not decomposed, and they are difficult to deal with; e.g. a kink cocycle is not defined as a cocycle of some topological space (a kink cocycle consists of three maps satisfying some conditions). Then, in this note, we define modified homotopy invariants, which are different from those of [14], and then explain that 3-cocycle invariants of cablings are deduced from them. In this context, a kink cocycle is a cocycle on a principal $U(1)$-bundle over a quandle space. Also, we show that modified homotopy invariants have universality among 2-cocycle invariants (original quandle homotopy invariants do not have universality among 2-cocycle invariants) as well as 3-cocycle invariants.

2 Preliminaries

2.1 Surface knots, diagrams, and cablings

Here we review basic definitions of surface links and diagrams. For details, see e.g. [3].

A *surface link* F is a closed surface smoothly embedded in the 4-dimensional Euclidean space $\mathbb{R}^4$ as a submanifold. If F is connected, it is also called a *surface knot*. In this note we assume that any surface link is oriented. If an ambient isotopy of $\mathbb{R}^4$ takes one surface link to another, with the orientation preserved, then they are said to be *equivalent*.

Let $F \subset \mathbb{R}^4$ be a surface link. Let $p : \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ be the projection. Perturbing F if necessary, we may assume that the image $p(F) \subset \mathbb{R}^3$ is composed of *non-singular points*, *double points*, *triple points*, and *branch points*, as is shown in Figure 1. Then, the image $p(F)$ with the information of the orientation and the relative height is called a *diagram* of a surface link F. We represent the relative height by eliminating lower surfaces along the double point curves, as in Figure 1. Then each connected component of a diagram D is called a *sheet* and we denote the set of the sheets of D by $S(D)$.

Next we review framing of surface links, following [12]. For a surface link L, a nowhere-vanishing section of the normal bundle νF of F in $\mathbb{R}^4$ is defined to be a *framing* of F. It is known, see e.g. [3], that the normal Euler number on an oriented surface link is zero, and then there exists a framing for any surface link. A pair $\mathcal{F} = (F, s)$ of a surface link and a framing of it is called a *framed surface link*. If the homomorphism $s_* : H_1(F) \to H_1(\mathbb{R}^4 - F) \cong \mathbb{Z}$ induced by a framing s is the zero map, s is called a *zero-framing*, or a *canonical framing*. This is determined uniquely, up to equivalence.

Cabling of surface knots

Let F be a surface knot. We here define cabling of F.

We take a tubular neighborhood N of F in S^4 and we identify ∂N with the normal sphere bundle of F. We define a surface link $\tilde{F}$ to be an *m-cabling* of F if $\tilde{F}$ is a submanifold of ∂N and the composition $\tilde{F} \hookrightarrow \partial N \xrightarrow{p_{\partial N}} F$, where $p_{\partial N} : \partial N \to F$ is the projection of the normal sphere bundle ∂N, is an m-fold covering map preserving the orientation. We do

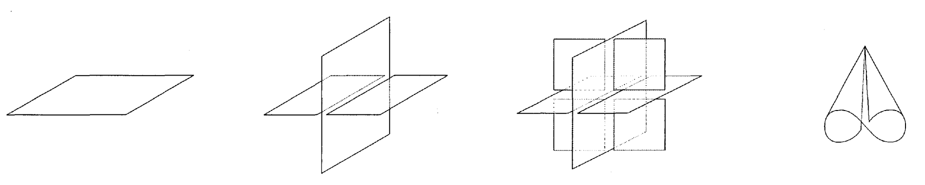


Figure 1: A non-singular point, a double point, a triple point, a branch point

not assume $\tilde{F}$ to be connected, and even when it is connected, the genus of $\tilde{F}$ does not coincide with that of F, in general.

An m-cabling $\tilde{F}$ of F corresponds to a cohomology class $\nu \in H^1(F;\mathbb{Z})$, as follows. We fix a canonical framing $s : F \to \partial N$ so that $s(P) \in \tilde{F}$ for a fixed point $P \in F$. Next, we consider the trivial principal $\mathbb{R}$-bundle $\mathbb{R} \times F$ and a covering map $\pi : \mathbb{R} \times F \to \partial N$ such that, $p_{\partial N} \circ \pi$ equals the projection of the bundle $\mathbb{R} \times F$, $\pi^{-1}(s(F)) = m\mathbb{Z} \times F \subset \mathbb{R} \times F$, and $\pi^{-1}(\tilde{F} \cap p_{\partial N}^{-1}(P)) = \mathbb{Z} \times \{P\}$. Then, as $\mathbb{Z} \times F$ is a principal $\mathbb{Z}$-bundle embedded in $\mathbb{R} \times F$, we obtain a monodromy representation $\pi_1(F, P) \to \mathbb{Z}$. Let $\nu \in H^1(F;\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(F, P), \mathbb{Z})$ be the obtained cohomology class. Then we define $\tilde{F}$ to be an (m, ν)-*cabling* of F. Conversely for any $\nu \in H^1(F;\mathbb{Z})$, we can easily construct an (m, ν)-cabling $F^{(m,\nu)}$ of F uniquely up to ambient isotopy in ∂N.

2.2 Quandles and cocycle invariants

Here we recall definitions of quandles and invariants induced by them. For details, see e.g. [3] or [1].

Let X be a set and let $*$ be a binary operator on X ($X \times X \ni (x, y) \mapsto x * y \in X$). The pair $(X, *)$ is defined to be a *quandle* if the following three conditions hold:

(Q1) For any $x \in X$, $x * x = x$.

(Q2) For any $y \in X$, the map $X \ni x \mapsto x * y \in X$ is a bijection.

(Q3) For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Sometimes we use a notation by Fenn-Rourke [6]: we denote $x * y$ by x^y. Also we presume that $x^{yz} = (x * y) * z$.

Here are some examples of quandles:

- For $k \in \mathbb{N}$, $\mathbb{Z}/k\mathbb{Z}$, with a binary operation $*$ defined as $x * y = 2y - x$, is a quandle. This is called a *dihedral quandle*, denoted as R_k.
- We put $Q_4 = \mathbb{Z}[t^{\pm 1}]$ and $x * y = tx + (1 - t)y$ for $x, y \in Q_4$. Q_4 is a quandle of order 4 and is called the *tetrahedral quandle.*
- Let M be a $\mathbb{Z}[T^{\pm}]$-module. M, with a binary operation $*$ defined as $x * y = Tx + (1 - T)y$, is a quandle and is called an *Alexander quandle.* The quandles of the two examples above are Alexander quandles.

Next, review the definition of quandle cocycles. Let X be a quandle and let A be an abelian group.

- A map $\phi : X \times X \to A$ is a *quandle 2-cocycle* if $\phi(x, x) = 0$ and

$$\phi(x, y) - \phi(x, z) - \phi(x * y, z) + \phi(x * z, y * z) = 0$$

  for any $x, y, z \in X$.

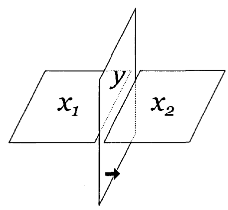
- A map $\psi : X \times X \times X \to A$ is a *quandle 3-cocycle* if $\psi(x, x, y) = \psi(x, y, y) = 0$ and

$$\psi(w, y, z) - \psi(w, x, z) + \psi(w, x, y) - \psi(w*x, y, z) + \psi(w*y, x*y, z) - \psi(w*z, x*z, y*z) = 0$$

  for any $w, x, y, z \in X$.

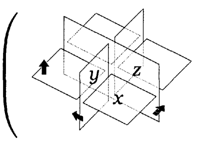
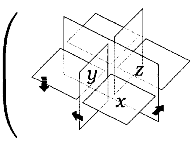
Quandle (co)homology groups $H_*^Q(X, A)$ $(H_Q^*(X; A))$ are defined generally, though we omit the definition here.

Quandle colorings and quandle 3-cocycle invariants



Let X be a quandle. For a diagram D of a surface link F, an X*-coloring* of D is a map $\mathcal{C} : S(D) \to X$ such that we have $\mathcal{C}(x_1) * \mathcal{C}(y) = \mathcal{C}(x_2)$ along each double point curve, as shown in the figure. We put $\mathrm{Col}_X(D)$ to be the set of the X-colorings of D. It is known that for a finite quandle X the number of the X-colorings is an invariant for surface links and called the (X-)*coloring number.* Quandle cocycle invariants defined below are refinements of coloring numbers.

Let X be a finite quandle and let ψ be a quandle 3-cocycle valued on an abelian group A (here we adopt the multiplicative notation for the multiplication in A). For a diagram D of a surface link F and an X-coloring $\mathcal{C}$, we associate a weight $W_\psi(\mathcal{C}, p)$ on each triple point p as

W_ψ () $= \psi(x, y, z)$, W_ψ () $= \psi(x, y, z)^{-1}$.

Then we put

$$\Psi_\psi(\mathcal{C}) = \prod_p W_\psi(\mathcal{C}, p) \in A,$$

where p is taken over the triple points in D, and put

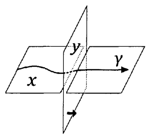
$$\Psi_\psi(D) = \sum_{\mathcal{C} \in \mathrm{Col}_X(D)} \Psi_\psi(\mathcal{C}) \in \mathbb{Z}[A].$$

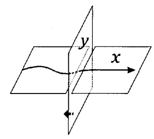
It is known [1] that this is an invariant for surface links, called a *quandle cocycle invariant* and denoted by $\Psi_\psi(F)$.

Quandle 2-cocycle invariants for surface links

We recall surface-knot (link) invariants using quandle 2-cocycles, which are also refinements of coloring numbers. These are introduced in [4], though we adopt different notation from that of [4].

Let X be a finite quandle and $\varphi : X \times X \to A$ be a quandle 2-cocycle of X valued on an abelian group A. Let F be a surface link and D be a diagram of it. For a moment we fix a coloring $\mathcal{C} \in \mathrm{Col}_X(D)$. For a generic loop γ smoothly immersed in F, we define weights as:

W_φ (

) $= \varphi(x, y)$, W_φ (

) $= \varphi(x, y)^{-1}$.

Then we put

$$\Phi_\varphi(\mathcal{C})(\gamma) = \prod_p W_\varphi(p) \in A,$$

where the product is taken over all of the crossing points p of γ and the lower decker curves. It is shown in [4] that if cycles γ and γ' are homologous, $\Phi_\varphi(\mathcal{C})(\gamma) = \Phi_\varphi(\mathcal{C})(\gamma')$. Then $\Phi_\varphi(D, \mathcal{C})$ defines a group homomorphism $H_1(F) \to A$. Thus, using the same notation, we obtain $\Phi_\varphi(\mathcal{C}) \in H^1(F; A)$ for each coloring $\mathcal{C}$. We put

$$\Phi_\varphi(D) = \sum_{\mathcal{C} \in \mathrm{Col}_X(D)} \Phi_\varphi(\mathcal{C}) \in \mathbb{Z}[H^1(F; A)],$$

which is an invariant for surface links and is denoted by $\Phi_\varphi(F)$.

3 Modified homotopy invariants for surface knots

3.1 Definition of modified homotopy invariants

In this section, we introduce a modified homotopy invariant Ξ'_X.

To begin with, we define a modified quandle space. Let X be a quandle and let $B_X X$ be the action rack space (see [7]) of the primitive X-set. Roughly speaking, we add the n-cells bounding the $(n-1)$-cells labeled by $(x_1, \cdots, x_n)$ $(x_i = x_{i+1}$ for some $i)$ to $B_X X$ and then we define the obtained CW-complex to be the modified quandle space $B'X$. More strictly, we now describe the 3-skeleton:

- The 0-cells are $\#X$ discrete points labeled by the elements of X.
- The 1-skeleton is obtained by attaching 1-cells labeled by $(x, y) \in X^2$ to the 0-cells x and $x * y$, as Figure 2 shows.

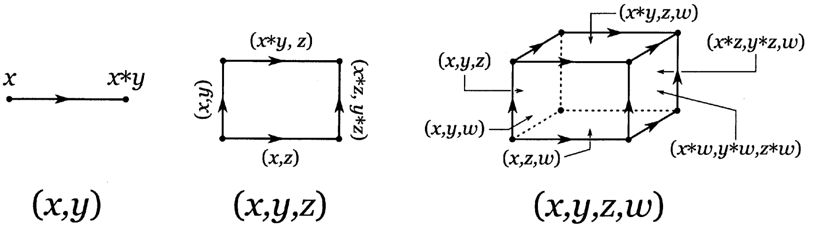


Figure 2: Cells of $B'X$

- The 2-skeleton is obtained by attaching 2-cells labeled by $(x, y, z) \in X^3$ to the 1-skeleton as shown in Figure 2 and adding 2-cells bounding the 1-cells (x, x).

- To describe the 3-skeleton, we first attach 3-cells labeled by $(x, y, z, w) \in X^4$ as shown in Figure 2 and then adding ones bounding the 2-cells (x, y, y). For $x, y \in X$ $(x \neq y)$ we further add a 3-cell which bounds three 2-cells: (x, x, y) and those bounding (x, x) and $(x * y, x * y)$ (this 3-cell is like a cylinder). This is how we obtain the 3-skeleton of $B'X$.

We need only the 3-skeleton to define the modified homotopy invariant.

We now define the modified homotopy invariant. Let F be a surface link and let D be a diagram of F. For $\mathcal{C} \in \mathrm{Col}_X(D)$, we construct a continuous map $\Xi'_X(\mathcal{C}) : F \to B'X$ as follows. First, we remark the lower decker curves on F form a 1,4-valent graph (the lower graph) G on F, where a monovalent vertex corresponds to a branch point and a 4-valent one to a triple point. The color $\mathcal{C}$ induces a shadow coloring on (F, G). Here we regard G as a generalized link diagram of [2] on F. We consider the dual decomposition of F as to G and let $\Xi'_X(\mathcal{C})$ map the 0-cells of the dual decomposition to the 0-cell of $B'X$ labeled by the corresponding shadow color $x \in X$. Further, we define the map $\Xi'_X(\mathcal{C}) : F \to B'X$ so that the 1- and the 2-cells of F are mapped to the corresponding 1- and 2-cells of

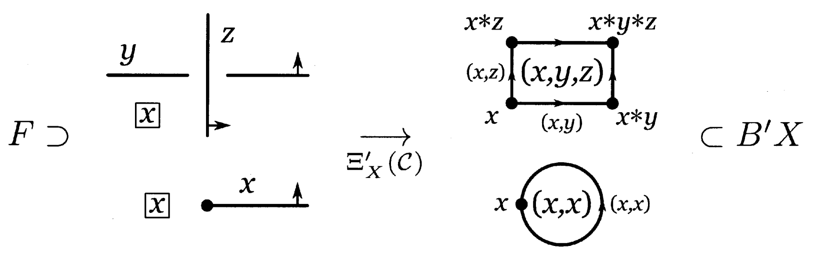


Figure 3: Definition of $\Xi'_X(\mathcal{C})$

$B'X$, respectively; see Figure 3. For example, a branch point colored x is send to the 2-cell bounding the 1-cell (x, x). Thus we obtain a continuous map $\Xi'_X(\mathcal{C})$. Of course deformation of the diagram changes the map $\Xi'_X(\mathcal{C})$, but the homotopy class of $\Xi'_X(\mathcal{C})$ is unchanged. Precisely,

Proposition 3.1. *Let D and D' be diagrams of surface links and suppose that there exists a sequence of Roseman moves which takes D to D'. Let $\varphi : \mathrm{Col}_X(D) \to \mathrm{Col}_X(D')$ be the induced bijection between the X-colorings. Then, for $\mathcal{C} \in \mathrm{Col}_X(D)$, $\Xi'_X(\mathcal{C})$ is equal to $\Xi'_X(\varphi(\mathcal{C}))$ up to homotopy.*

Thus the multi-set $\{\Xi'_X(\mathcal{C}) \in [F; B'X] \,|\, \mathcal{C} \in \mathrm{Col}_X(D)\}$ is an invariant of surface links.

3.2 Basic properties of $B'X$

On the rack homology theory, the (co)homology groups of $B_X X$ are isomorphic to the rack (co)homology groups with a shift of degree (see [8]). Similarly, the (co)homology of $B'X$ is described by the quandle (co)homology:

Proposition 3.2. *Let X be a quandle and let A be an abelian group. Then we have*

$$H_n(B'X, A) \cong H^Q_{n+1}(X, A), \qquad H^n(B'X; A) \cong H_Q^{n+1}(X; A).$$

In deed, we can identify the cellular n-(co)chains with the quandle $(n+1)$-(co)chains.

Next, we assume that X is connected ($\Leftrightarrow B'X$ is connected).

Proposition 3.3. (1) *$\pi_1(B'X)$ is isomorphic to the "fundamental group" of X in the sense of* [5].
(2) $\pi_2(B'X) \cong \pi_2^Q(X)$, *which is the second quandle homotopy group of X.*

Corollary 3.4. *If X is finite and connected, $\#[F; B'X] < \infty$.*

Remark 3.5. Let X be a connected quandle and let $\tilde{X}$ be the universal covering of X in the sense of [5]. We find from Proposition 3.3 that $\pi_2^Q(X) \cong H_2^Q(\widetilde{B'X})$, where $\widetilde{B'X}$ is the universal covering of $B'X$. Especially if X is finite, we can compute $\pi_2^Q(X)$ from the homology of a finite CW-complex. Also, this implies that the quandle homotopy invariant for 1-links is equivalent to a shadow cocycle invariant on the X-set $\tilde{X}$.

3.3 Universality among 2- and 3-cocycle invariants

The original quandle homotopy invariants of [14] have the universality among the (generalized, shadow) 3-cocycle invariants, but do not among 2-cocycle invariants. On the other hand, the modified homotopy invariants have the universality among 2- and 3-cocycle invariants.

Let X be a quandle, and let ϕ and ψ be quandle 2- and 3-cocycles on an abelian group A, respectively. As in the previous section, we regard ϕ (ψ) as a 1(2)-cocycle of $B'X$. Here we recall that the modified homotopy invariant Ξ'_X defines continuous maps $F \to B'X$ up to homotopy. Cocycle invariants are in fact pullbacks of the cocycle by the maps; Ξ'_X has universality among cocycle invariants. Precisely,

Proposition 3.6. *For a coloring $\mathcal{C} \in \mathrm{Col}_X(D)$ of a diagram D of F, we have*

(1) $\Phi_\phi(\mathcal{C}) = (\Xi'_X(\mathcal{C}))^*\phi \qquad \in H^1(F; A)$,

(2) $\Psi_\psi(\mathcal{C}) = \langle(\Xi'_X(\mathcal{C}))^*\psi, [F]\rangle \qquad \in A$,

where, in the right-hand side of the last equation, $\langle\cdot,\cdot\rangle$ is the Kronecker product and $[F]$ is the fundamental homology class of F.

Remark 3.7. There is a component-wise version of a 3-cocycle invariant; we sum the weights over the triple points whose bottom sheets belong to a fixed component of F. We can recover that version from Ξ'_X by substituting the fundamental homology class of the component for that of F in Proposition 3.6(2).

4 Quandle cocycle invariants of cabled surface knots

4.1 The main theorem

Let X be a quandle. To calculate quandle cocycle invariants of cablings, we regard an X-coloring on a cabling $F^{(m,\nu)}$ as an X^m-coloring on the original surface knot. Hence we introduce a binary operation $*$ on X^m as follows:

$$(x_1, \cdots, x_m) * (y_1, \cdots, y_m) = (x_1^{\overline{x_m}\cdots\overline{x_1}y_1\cdots y_m}, \cdots, x_m^{\overline{x_m}\cdots\overline{x_1}y_1\cdots y_m}).$$

Then $X^m = (X^m, *)$ is a quandle (see [9]). Further we define a map $\tau : X^m \to X^m$ as

$$\tau(x_1, \cdots, x_m) = (x_m, x_1 * x_m, \cdots, x_{m-1} * x_m).$$

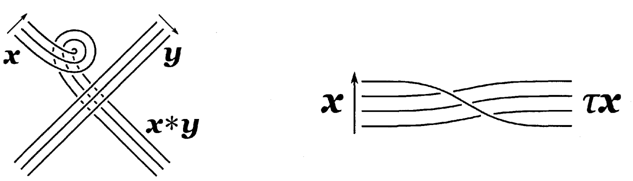


Figure 4: The quandle X^m and the kink map τ

τ is a quandle automorphism on X^m and we have $\boldsymbol{x} * (\tau \boldsymbol{y}) = \boldsymbol{x} * \boldsymbol{y}$ for any $\boldsymbol{x}, \boldsymbol{y} \in X^m$; i.e. τ is a kink map on X^m. In terms of a cabling, an X^m-coloring corresponds to a coloring on an m-cabling without twist and τ to a $1/m$-twist, as illustrated in Figure 4.

We put X^m_τ to be the quotient quandle of (X^m, τ). That is,

$$X^m_\tau := X^m / \sim, \qquad \boldsymbol{x} \sim \boldsymbol{y} \text{ iff } \boldsymbol{y} = \tau^k \boldsymbol{x} \text{ for some } k \in \mathbb{Z}.$$

We have the quandle operation on X^m_τ induced from that of X^m. The main theorem below says that quandle cocycle invariants of cablings are deduced from the modified homotopy invariant on X^m_τ.

Theorem 4.1. *We assume X to be finite and let $\psi : X \times X \times X \to A$ be a quandle 3-cocycle valued on an abelian group A. For a surface knot F, we denote the (m, ν)-cabling of F by $F^{(m,\nu)}$. Then, there exists a map $f^{(m,\nu)}_\psi : [F; B'X^m_\tau] \to \mathbb{Z}[A]$ such that*

$$\Psi_\psi(F^{(m,\nu)}) = \sum_{\mathcal{C}} f^{(m,\nu)}_\psi(\Xi'_{X^m_\tau}(\mathcal{C})) \in \mathbb{Z}[A],$$

where $\mathcal{C}$ is taken over the X^m_τ-colorings on (a diagram of) F.

4.2 Outline of the proof

In this section we give outline of the proof of Theorem 4.1. The proof consists of two parts: one is to determine the colorings and the other is to compute the weights.

Colorings of the cabling

First of all, we remark that there exists an obvious map $\mathrm{Col}_X(F^{(m,\nu)}) \to \mathrm{Col}_{X^m_\tau}(F)$ induced by the projection $X^m \to X^m_\tau$. Conversely, we here consider whether or not a coloring $\mathcal{C} \in \mathrm{Col}_{X^m_\tau}(F)$ is lifted to a coloring of $F^{(m,\nu)}$.

To see it, we construct a covering $\widetilde{B'X^m_\tau} \to B'X^m_\tau$. Although τ acts on $B'X^m$ and the action of τ on X^m is a deck transformation of the covering $X^m \to X^m_\tau$ in the sense of [5], $B'X^m \to B'X^m_\tau$ is not a covering. Hence we reduce $B'X^m$ (roughly speaking, we identify a cell $(\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n)$ with cells in a form $(\boldsymbol{x}_1, \tau^{k_2}\boldsymbol{x}_2, \cdots, \tau^{k_n}\boldsymbol{x}_n)$) to $\widetilde{B'X^m_\tau}$, and then we find τ to act on $\widetilde{B'X^m_\tau}$ as a generator of the deck transformation group of a cyclic covering $\widetilde{B'X^m_\tau} \to B'X^m_\tau$. For each connected component B_i of $B'X^m_\tau$, we obtain the monodromy representation $\rho_i : \pi_1(B_i) \to \mathbb{Z}/N_i\mathbb{Z}$. We think of ρ_i as a cohomology class: $\rho_i \in H^1(B_i; \mathbb{Z}/N_i\mathbb{Z})$.

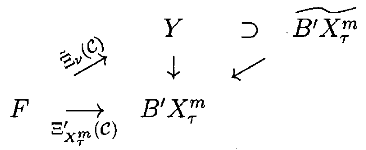
We take a coloring $\mathcal{C} \in \mathrm{Col}_{X^m_\tau}(F)$. We have a continuous map $\Xi'_{X^m_\tau}(\mathcal{C}) : F \to B'X^m_\tau$ and we assume that the image is in B_i. As we (try to) lift $\Xi'_{X^m_\tau}(\mathcal{C})$ to $\widetilde{B'X^m_\tau}$, there are two obstruction: one is the monodromy ρ_i and the other is the "twist" by ν. Summing up them, we find that $\mathcal{C}$ is lifted to $\tilde{\mathcal{C}} \in \mathrm{Col}_X(F^{(m,\nu)})$ if and only if $(\Xi'_{X^m_\tau}(\mathcal{C}))^*\rho_i + \nu = 0 \in H^1(F;\mathbb{Z}/N_i\mathbb{Z})$. This is a condition dependent only on $\Xi'_{X^m_\tau}(\mathcal{C}) \in [F; B'X^m_\tau]$ (we could express this condition by means of a 2-cocycle invariant on X^m_τ).

If $\mathcal{C}$ lifts, there are N_i lifts, each of which is obtained by setting a color of a sheet. We calculate the weights $W \in A$ of them below (in fact the N_i lifts have the same weight) and put $f^{(m,\nu)}_\psi(\Xi'_{X^m_\tau}(\mathcal{C})) = N_i \cdot W \in \mathbb{Z}[A]$. If $\mathcal{C}$ does not lift, we put $f^{(m,\nu)}_\psi(\Xi'_{X^m_\tau}(\mathcal{C})) = 0$.

Computing the weights

For the calculation, we put Y to be the mapping torus of $(\widetilde{B'X^m_\tau}, \tau)$. Since $\widetilde{B'X^m_\tau} \to B'X^m_\tau$ is a cyclic covering and τ is a generator of the deck transformation group, we have the projection $Y \to B'X^m_\tau$ and it is a principal $U(1)$-bundle.

We assume that a coloring $\mathcal{C} \in \mathrm{Col}_{X^m_\tau}(F)$ is lifted to a coloring of the cabling $F^{(m,\nu)}$. Then we construct a lift $\tilde{\Xi}_\nu(\mathcal{C})$ of $\Xi'_{X^m_\tau}(\mathcal{C})$:



We lift $\Xi'_{X^m_\tau}(\mathcal{C})$ along $\widetilde{B'X^m_\tau}$ (this is like the holonomy representation of a flat connection), and where the cable twists, we bend the surface as if it fills in the gap of τ, as illustrated in Figure 5. By assumption, we obtain a lift $\tilde{\Xi}_\nu(\mathcal{C})$ defined over the whole surface. The homotopy class of $\tilde{\Xi}_\nu(\mathcal{C})$ is in fact determined by ν and the homotopy class $\Xi'_{X^m_\tau}(\mathcal{C}) \in [F; B'X^m_\tau]$. Then, Theorem 4.1 follows from a claim:

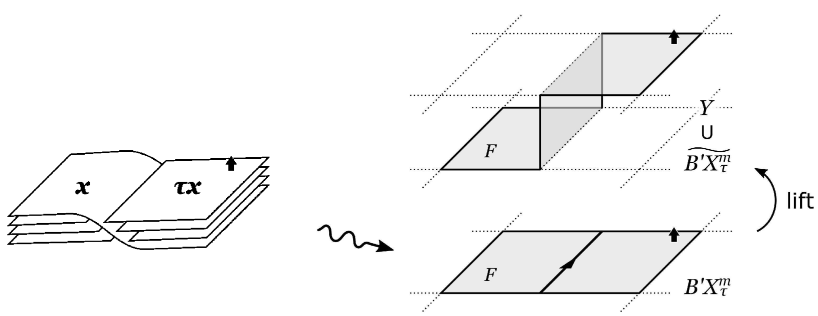


Figure 5: The lift $\tilde{\Xi}_\nu(\mathcal{C})$ of $\Xi'_{X^m_\tau}(\mathcal{C})$

Claim. *There exist an abelian group A' including A and a cohomology class $\tilde{\psi} \in H^2(Y; A')$ such that*

$$\Psi_\psi(\tilde{\mathcal{C}}) = \langle \tilde{\Xi}_\nu(\mathcal{C})^* \tilde{\psi}, [F] \rangle \in A,$$

where $\tilde{\mathcal{C}} \in \mathrm{Col}_X(F^{(m,\nu)})$ is a lift of $\mathcal{C}$.

We replace the coefficient group for a technical reason, which we do not explain in this note.

The claim is shown by a method used in [9]; a kink cocycle consists of three maps, but we can reduce the third map. Then the two maps represent the required cohomology class. We recall here that the 2-cells of Y are composed of the 2-cells of $\widetilde{B'X_\tau^m}$ and those in a form of (a 1-cell of $\widetilde{B'X_\tau^m}$) $\times [0,1]$ (we regard Y as a quotient of $\widetilde{B'X_\tau^m} \times [0,1]$). These are respectively mapped by the two reduced maps. Generally speaking, the former cells correspond to the triple points in the cabling which are generated near the triple points in the original surface knot, and the latter correspond to those appearing near the intersection of the twist and the original double point curves. We define $\tilde{\psi}$ to map a 2-cell to the weight-sum on the corresponding triple points.

Remark 4.2. Similarly, we can show that kink cocycle invariants are deduced from modified homotopy invariants. Especially, a rack cocycle invariant is represented by the modified homotopy invariant on the quotient quandle.

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