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GENERALIZATIONS OF SCHEIN THEOREM *

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In the paper [], B.M. Schein proved that any finte cyclic subsemigroup of the full transformation semigroup $\mathcal{T}(X)$ on a finite set X is covered by an inverse subsemigroup of $\mathcal{T}(X)$. In this paper, we study a generalization of Schein theorem.

1 Schein theorem

Let X be a finite set and $\mathcal{T}_X$ the full transformation semigroup on X with composition being from the right to the left.

For any $a \in \mathcal{T}_X$, We say that $a(X)$ is the *range* of a and $\cap_{k=0}^{\infty} a^k(X)$ is called *the stable range* of a and denoted by $SR(a)$.

(1) Define the *depth* $\mathrm{d}(x)$ of x by an integer k such that $x \in a^k(X)$, but $x \notin (X)a^{k+1}$ for $x \in X - SR(a)$. $\mathrm{d}(x) = \infty$ if $x \in SR(a)$.

(2) Define the *height* of x by the least non-negative integer k such that $\mathrm{d}(a^k(x)) > k+\mathrm{d}(x)$ if $\mathrm{d}(x) < \infty$. Denote it by $\mathrm{h}(x)$. Further, $\mathrm{h}(x)=\infty$ if $x \in SR(a)$.

Given a partial map b of $a(X)$ to X such that for $x \in a(X)$, $b(x) \in a^{-1}(x)$ and $\mathrm{d}(b(x)) \geq \mathrm{d}(y)$ forall $y \in a^{-1}(x)$.

(3) Define the *gap* $\mathrm{g}(x)$ as the greatest integer k such that $b^k a^k(x) = x$.

(4) Define the *reach* $\mathrm{r}(x)$ by $\mathrm{r}(x) = \mathrm{g}(x) + 1$.

Thus, $\mathrm{h}(x) \geq \mathrm{r}(x) > \mathrm{g}(x)$. Hence $(b^h a^h)(x) \neq x$ and $(b^r a^r)(x) \neq x$.

Schein theorem

For an element $a \in \mathcal{T}_X$,

there exists an element $b \in \mathcal{T}_X$ such that $y = bx$ $(x, y \in X)$

if and only if (1) *if $d(x) > 0$ $(x \in X)$ then $y \in a^{-1}(x)$ has a maximal depth.*

(2) *$d(x) = 0$ $(x \in X)$ then $a^h(x) = a^{h+1}(y)$ and $(b^r a^r)(ab)(x) = (ab)(b^r a^r)(x)$, where $h = h(x)$, $r = r(x)$, $g(y) \geq r$.*

In this case, the subsemigroup $< a, b >$ is an inverse semigroup generated by a and b in $\mathcal{T}_X$.

*This is an absrtact and the paper will appear elsewhere.

Acutally, take $y = b^{h+1}a^h(x)$ the colt is possible because of definition of h (that is, $\mathrm{d}(a^h(x)) > h + \mathrm{d}(x)$).

Remark, By L.M. Gluskin theorem[1], inverse semigroups $< a, b >$ and $< a', b' >$ are isomorphic to each other if $< a >$ and $< a' >$ isomorphic. In [2]T.E. Hall showed that Schein theorem is applicable to amalgamation problem.

2 Generalizations of Schein theorem

We pose generalizations of Schein theorem from cyclic semigroups to extensions of cyclic semigroups by groups in $\mathcal{T}(X)$ and several questions as follows :

Let a semigroup $S = G \cup \bigcup_{i=1}^{n} Ga^i$, where G is the group of units in S and $Ga = aG$.

Question 1 Suppose that S is a subsemigroup of $\mathcal{T}(X)$. The does there exist an elemenet $b \in \mathcal{T}(X)$ such that $< S, b >$ is an inverse semigroup?
(the first genralization of Schein theorem)

Let X be a finite set and $a \in \mathcal{T}(X)$ such that there exists distinct $x, y \in X - a(X)$ with $a(x) = a(y)$. Let $g \in \mathcal{T}(X)$ such that $g(x) = y, g(y) = x$ and $g(z) = z$ for all $z \in X - \{x, y\}$.
Let $b \in \mathcal{T}(X)$ such that $bx = b^{h(x)+1}a^{h(x)}(x)$ Then $ga = ag = a$ and $bg = b$ (since $h(x) = h(y)$) but $gb \neq b$. Both gb and b are an inverse element of a.
$< S, b >$ is a left inverse semigroup.

Question 1 has a negative answer.

Let a semigroup $S = G \cup \bigcup_{i=1}^{n} Ga^i$, where G is the group of units in S and $Ga = aG$.

Question 2 Suppose that S is a subsemigroup of $\mathcal{T}(X)$. The does there exist an elemenet $b \in \mathcal{T}(X)$ such that $< S, b >$ is an orthodox semigroup? (That is, is the set of idempotents in a regular semigroup $< S, b >$ a subsemigroup?)
(the second genralization of Schein theorem)

The answer of Question 2 would be negative.

Let a semigroup$S = G \cup \bigcup_{i=1}^{n} Ga^i$, where G is the group of units in S and $ga = ag$ for any $a \in G$.
When $S \subseteq \mathcal{T}(X)$, X is a directed graph with edges labelled by a. For $x \in \mathrm{SR}(X)$, the subgraph $Gr(a; x) = \{y \in X \mid ya^k = x, \mathrm{d}(x) < \infty\}$ is a tree, that is, a directed graph without cycle. Each $g \in G$ indeuces an automorphism of a directed graph X.

Question 3 Let a semigroup$S = G \cup \bigcup_{i=1}^{n} Ga^i$, where G is the group of units in S and $ga = ag$ for any $a \in G$.

Suppose that S is a subsemigroup of $\mathcal{T}(X)$. The does there exist an elemenet $b \in \mathcal{T}(X)$ such that $< S, b >$ is an inverse semigroup?
(the third genralization of Schein theorem)

References

[1] L. M. Gluskin, *Elementary Generalized Groups* Mat. Sb. **41**(83)(1957), 23-36.

[2] T. E. Hall, *Representation extension and amalgamation for semigroups.* Quart. J. Math. Oxford (2) **29**(1978), 309-334. 489-496.

[3] B. M. Schein, *A symmetric semigroup of transformations is covered by its inverse subsemigroups*, Acta Math. Acad. Sc. Hungar. **22**(1971), 163-171.