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Groups Genarated by Prefix Reversals on Two Dimensional Arrays

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Abstract

We consider a two dimensional variant of the pancake problem which asks whether an arbitrary $n\times m$ array can be obtained from an initial array by prefix reversals. We shall show subgroups of symmetric groups generated by prefix reversals on $n\times m$ dimensional arrays. The alternating group is generated if both n and m are multiples of 4, and the symmetric group is generated otherwise.

1 Introduction

A two dimensional variant of the *pancake problem*, which asks if it is possible to sort randomly piled pancakes of different size by *prefix reversals*, is introduced by [4]. The reader is referred to [1, 2, 3] for the pancake sort problem. Given an $n\times m$ array filled with integers of different size, we ask whether every permutation of the integers on the array is obtained by prefix reversals. A prefix reversal in this setting consists in inserting a spatula vertically or horizontally into the array and rotating a left-hand part or an upper part 180 degree. We study subgroups of symmetric groups generated by permutations realized in terms of prefix reversals on two dimensional arrays.

2 Rearrangement of Two Dimensional Arrays

2.1 Prefix Reversals

We formulate a rearrangement of a two dimensional array by prefix reversals as follows. Suppose A is an $n\times m$ array. Then A comprises of $n\times m$ cells. We

a_{11}	a_{12}	$\cdots$	a_{1m-1}	a_{1m}
a_{21}	a_{22}	$\cdots$	a_{2m-1}	a_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{n1}	a_{n2}	$\cdots$	a_{nm-1}	a_{nm}

Figure 1: $n \times m$ array A

denote the entry in the (i, j) position of A by a_{ij} (see Fig 1). We employ the standard matrix representation (a_{ij}) to denote an array A. The *standard array* $E_{n\times m}$ of size $n \times m$ is defined to be (e_{ij}) where e_{ij} is the integer $(i-1) \times m + j$ (see Fig. 2). We denote $A = \frac{A_1}{A_2}$ if A comprises of an upper block A_1 and a lower block A_2, or $A = A_3|A_4$ if A comprises of a left block A_3 and a right block A_4. For an $n \times m$ array $A = (a_{ij})$, the *reversal* of A is the $n \times m$ array (b_{ij}) such that $b_{ij} = a_{n-i+1,m-j+1}$ for every (i, j). We denote it by $R(A)$ (see Fig. 3).

Suppose A is an $n \times m$ array. The transformation $\frac{A_1}{A_2} \Rightarrow \frac{R(A_1)}{A_2}$ is called a *horizontal prefix reversal* and denoted by $hr(1, i)$ if A_1 has i rows. The transformation $A_3|A_4 \Rightarrow R(A_3)|A_4$ is called a *vertical prefix reversal* and denoted by $vr(1, j)$ if A_3 has j columns. The *horizontal suffix reversal* $\frac{A_1}{A_2} \Rightarrow \frac{A_1}{R(A_2)}$ and the *vertical suffix reversal* $A_3|A_4 \Rightarrow A_3|R(A_4)$ are obtained by compositions of prefix reversals $hr(1,n) \circ hr(1, n-i) \circ hr(1,n)$ and $vr(1,m) \circ vr(1, m-j) \circ vr(1,m)$, respectively. We denote them by $hr(i+1, m)$ and $vr(j+1, n)$.

1	2	$\cdots$	m
$m+1$	$m+2$	$\cdots$	$2m$
$2m+1$	$2m+2$	$\cdots$	$3m$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(n-1)m+1$	$\cdots$	$\cdots$	nm

Figure 2: $n \times m$ standard array $E_{n\times m}$

a_{nm}	a_{nm-1}	$\cdots$	a_{n2}	a_{n1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{2m}	a_{2m-1}	$\cdots$	a_{22}	a_{21}
a_{1m}	a_{1m-1}	$\cdots$	a_{12}	a_{11}

Figure 3: R(A)

2.2 Groups generated by prefix reversals

Let A be an $n \times m$ array in which each integer in $\{1, 2, 3, \ldots, nm\}$ is placed on exactly one position. Suppose that σ is a permutation on the set $\{1, 2, 3, \ldots, nm\}$. The $n \times m$ array obtained from A by operating σ on each number placed on A is called a *rearrangement* of A by σ, and denoted by $\sigma(A)$.

Suppose that σ is a permutation of the set $\{1, 2, 3, \ldots, nm\}$. We say that *σ is generated by prefix reversals* if an $n \times m$ array $\sigma(E_{n\times m})$ is obtained from the standard array $E_{n\times m}$ by operating prefix reversals. Subgroups H of a symmetric group *is generated by prefix reversals* if all the element of H is generated by prefix reversals. The following theorem is given in [4]. In this paper, we give another proof of the theorem.

Theorem 2.1 *Let n and m be positive integers.*
(1) Suppose either $n \not\equiv 0 \pmod 4$ or $m \not\equiv 0 \pmod 4$ holds. The symmetric group S_{nm} is generated by prefix reversals.
(2) Suppose $n \equiv m \equiv 0 \pmod 4$. The alternating group A_{nm} is generated by prefix reversals.

First, we shall show that A_{nm} is generated by prefix reversals in any case in Section2. Then, we show that a certain transposition is generated by prefix reversals if either $n \not\equiv 0 \pmod 4$ or $m \not\equiv 0 \pmod 4$ holds in Sections 4 and 5.

2.3 Elementary Operations

We introduce several operations realized by prefix reversals. Suppose A is an $n \times m$ array throughout this section. It is easy to see that if the composition $hr(1, i) \circ hr(1, 1) \circ hr(1, i)$ is operated to A, we obtain an array that coincides with A except for the ith row is reversed. Such an operation is called a *row reversal* and denoted by $rr(i)$. Similarly, the composition $vr(1,j) \circ vr(1,1) \circ vr(1,j)$ reverses the jth column of A and called the *column reversal* and denoted by $cr(j)$.

Now we operate the composition $hr(n,n) \circ vr(1,1) \circ hr(n,n) \circ hr(1,1) \circ vr(1,1) \circ hr(1,1)$ to A. The resulting array is obtained from A by exchanging the four corners diagonally. We call this operation a *corner exchanging* and denote it by ce.

For example, we see the composition $hr(4,4) \circ vr(1,1) \circ hr(4,4) \circ hr(1,1) \circ vr(1,1) \circ hr(1,1)$ exchanges the four corners of $E_{4,4}$ diagonally, that is, 1 and 16 are exchanged and 4 and 13 are exchanged, respectively.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \to \begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \to \begin{bmatrix} 13 & 3 & 2 & 1 \\ 9 & 6 & 7 & 8 \\ 5 & 10 & 11 & 12 \\ 4 & 14 & 15 & 16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 3 & 13 \\ 9 & 6 & 7 & 8 \\ 5 & 10 & 11 & 12 \\ 4 & 14 & 15 & 16 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 13 \\ 9 & 6 & 7 & 8 \\ 5 & 10 & 11 & 12 \\ 16 & 15 & 14 & 4 \end{bmatrix} \to \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 1 & 15 & 14 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

We consider even permutations on the entries of A in a special form. We consider entries on the (i,k), (i,l), (j,k), and (j,l) positions of A ($i \neq j$ and $k \neq l$). These four positions are the intersections of the ith and jth rows and kth and lth columns. The permutations $(a_{i,k}, a_{j,k})(a_{i,l}, a_{j,l})$, $(a_{i,k}, a_{i,l})(a_{j,k}, a_{j,l})$ and $(a_{i,k}, a_{j,l})(a_{i,l}, a_{j,k})$ of entries of A are called a *twin transposition along column*, a *twin transposition along row* and a *twin transposition along diagonal*, respectively.

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(Twin transposition along column)

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,l} & \cdots & a_{i,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,l} & \cdots & a_{j,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(Twin transposition along row)

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,k} & \cdots & a_{i,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,k} & \cdots & a_{j,l} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \to \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{j,l} & \cdots & a_{j,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{i,l} & \cdots & a_{i,k} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

(Twin transposition along diagonal)

Note that the twin transposition $(a_{i,k}, a_{j,k})(a_{i,l}, a_{j,l})$ along column is realized by $hr(j,n) \circ hr(1,i) \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ hr(1,1) \circ ce \circ hr(1,1) \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ hr(j,n) \circ hr(1,i)$. It is denoted by $tt((i,k) \leftrightarrow (j,k),(i,l) \leftrightarrow (j,l))$.

Similarly, the twin transposition along row $(a_{i,k}, a_{i,l})(a_{j,k}, a_{j,l})$ is realized by $hr(j,n) \circ hr(1,i) \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ vr(1,1) \circ ce \circ vr(1,1) \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ hr(j,n) \circ hr(1,i)$, and the twin transposition along diagonal $(a_{i,k}, a_{j,l})(a_{i,l}, a_{j,k})$ is realized by $hr(j,n) \circ hr(1,i) \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ ce \circ vr(m-k+1,m) \circ vr(1,m-l+1) \circ hr(j,n) \circ hr(1,i)$. The twin transpositions along row and diagonal are denoted by $tt((i,k) \leftrightarrow (i,l),(j,k) \leftrightarrow (j,l))$ and $tt((i,k) \leftrightarrow (j,l),(i,l) \leftrightarrow (j,k))$, respectively.

3 Even Permutations Are Generated

We show that any even permutation is generated by prefix reversals. Suppose σ is factored as $\tau_1\tau_2 \cdots \tau_{2r}$ where τ_i $(1 \le i \le 2r)$ is a transposition. If we could prove that ρ is generated by prefix reversals, where $\rho = \tau_1\tau_2$ and τ_1 and τ_2 are transpositions, then the general case can be obtained by induction. Therefore, we consider the case that $\sigma = \tau_1\tau_2$, where τ_1 and τ_2 are transpositions. There are two cases to be considered: (1) τ_1 and τ_2 are disjoint and (2) τ_1 and τ_2 are not disjoint.

Suppose τ_1 and τ_2 are not disjoint, that is the case (2). We have $\rho = \tau_1\tau_2 = \tau_1\delta\delta\tau_2$ for any transposition δ because the inverse of a transposition is itself. In particular, we may take any transposition δ that is disjoint both from τ_1 and from τ_2. Then both $\tau_1\delta$ and $\delta\tau_2$ fall in the case (1). Therefore, it suffices to consider only the case (1). Consequently, we shall show that if a permutation ρ factors as $\tau_1\tau_2$, where τ_1 and τ_2 are disjoint transpositions then it is generated by prefix reversals.

Suppose that $\rho = \tau_1\tau_2$, $\tau_1 = (a, b)$, $\tau_2 = (c, d)$ and a, b, c, d are distinct from each other (see the array transition below). We suppose that x is placed on the same column as a and the same row as d and that y is placed on the same column as d and on the same row as a. Then we operate two twin transpositions along row, column or diagonal to exchange b and y, and c and x, respectively. Note that since we use twin transpositions, there are other entries replaced. Next we operate twin transpositions $(a\ b)(c\ d)$ to exchange a and b and c and d, respectively. After that we operate the same twin transpositions above in the reverse order. Note that any entries except for a, b, c, d stay the same position because the operations are carried out twice. Therefore, ρ is realized by prefix reversals.

$$\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a & \cdots & \cdots & \cdots & y & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & b & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & c & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & x & \cdots & \cdots & \cdots & d & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a & \cdots & \cdots & \cdots & b & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & y & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & x & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & c & \cdots & \cdots & \cdots & d & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\rightarrow$$

$$\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & b & \cdots & \cdots & \cdots & a & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & y & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & x & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & d & \cdots & \cdots & \cdots & c & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & b & \cdots & \cdots & \cdots & y & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & a & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & d & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & x & \cdots & \cdots & \cdots & c & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{bmatrix}$$

We only considered the case that a, b, c, d are on different rows and columns. Similarly we can show that any composition of disjoint transpositions is realized by twin transpositions as well. It follows that prefix reversals realize all the even permutations.

4 Symmetric Group S_{nm} Is Generated

We show that the symmetric group S_{nm} is generated by prefix reversals if either $n \not\equiv 0 \pmod 4$ or $m \not\equiv 0 \pmod 4$ holds. Note that we have already shown that all the even permutations are generated by prefix reversals. Thus, the alternating group is generated by prefix reversals. Since the symmetric group is generated by the alternating group and any transposition, it is sufficient to show that one of transpositions is generated by prefix reversals. If $n = 1$, it is easy to see that any permutation is generated by vertical reversal. It is also easy to see that any permutation is generated by vertical reversal if $n = 2$ (or $m = 2$) (see [4]). Thus, we assume $n > 2$ and $m > 2$.

4.1 $n \equiv 1, 3 \pmod 4$

Suppose $n = 2k+1$ for some positive integer k. We shall show that a certain transposition is realized by prefix reversals.

We shall show how to realize a horizontal transposition of entries on the $(k+1, 1)$ and $(k+1, 2)$ positions in a $(2k+1) \times m$ array A. There are $2k$ rows except for the $k+1$ row in A. First, we operate k twin transpositions along diagonal $tt((i, 1) \leftrightarrow (n-i+1, 2), (i, 2) \leftrightarrow (n-i+1, 1))$ for every $i = 1, 2, 3, \ldots, k$. Second, we operate $vr(1, 2)$. The resulting array is obtained from A by transposing the entries of $(k+1, 1)$ and $(k+1, 2)$ positions.

4.2 $n \equiv 2 \pmod 4$

Suppose $n = 4k+2$ for some positive integer k. We shall show that a certain transposition is realized by prefix reversals.

We show how to realize a vertical transposition of entries on the $(2k+1, 1)$ and $(2k+2, 1)$ positions in a $(4k+2, m)$ array.

We operate $vr(1, 1)$ to exchange entries on the $(2k+1, 1)$ and $(2k+2, 1)$ positions, however, this operation also moves the other entries on the first column. We restore the entries on the first column except for $(2k+1, 1)$ and $(2k+2, 1)$ positions by iterating the following operations for every $1 \leq i \leq k$. The basic idea is to exchange the entries on the $(2i, 1)$ and $(4k-2i+3, 1)$ positions and $(2i-1, 1)$ and $(4k-2i+4, 1)$ positions, respectively by moving these four entries in positions for a twin transposition and then exchange couples of these entries, simultaneously.

First, we operate $rr(4k-2i+3) \circ rr(2i)$. Second, we operate $tt((2i, m-1) \leftrightarrow (2i-1, m-1), (2i, m) \leftrightarrow (2i-1, m)) \circ tt((4k-2i+3, m-1) \leftrightarrow (4k-2i+4, m-1), (4k-2i+3, m) \leftrightarrow (4k-2i+4, m))$. These two operations move the entries on the $(2i, 1)$ and $(4k - 2i + 3, 1)$ positions and $(2i - 1, 1)$ and $(4k - 2i + 4, 1)$ positions in positions for a twin transposition. Third, we operate $tt((2i - 1, 1) \leftrightarrow (4k - 2i + 4, 1), (2i - 1, m) \leftrightarrow (4k - 2i + 4, m))$. This operation realizes the exchange of targeted entries. Fourth, we carry out the same operations in the reverse order, that is, we operate $tt((2i, m - 1) \leftrightarrow (2i - 1, m - 1), (2i, m) \leftrightarrow (2i - 1, m)) \circ tt((4k - 2i + 3, m - 1) \leftrightarrow (4k - 2i + 4, m - 1), (4k - 2i + 3, m) \leftrightarrow (4k - 2i + 4, m))$. Lastly we operate $rr(4k-2i+3) \circ rr(2i)$. Then the transposition $(a_{2k+1,1}\ a_{2k+2,1})$ is realized. In the iteration process, entries other than the targeted entries are not replaced because we operate the same operation twice.

5 Alternating Group A_{nm} Is Generated

We shall show that prefix reversals generate the alternating group A_{nm} if $n \equiv 0 \pmod 4$ and $m \equiv 0 \pmod 4$. We have already shown that all the even permutations are generated by prefix reversals. Therefore we shall show that no transposition is generated by prefix reversals in this case. Suppose that A is an $n \times m$ array, where $n = 4h$ and $m = 4k$ for some positive integers h, k.

We consider the horizontal prefix reversal $hr(1, p)$ for an arbitrary $1 \leq p \leq 4k$. The sub-array operated by $hr(1, p)$ is shown below:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2h} & a_{1,2h+1} & \cdots & a_{1,4h-1} & a_{1,4h} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2h} & a_{2,2h+1} & \cdots & a_{2,4h-1} & a_{2,4h} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{p-1,1} & a_{p-1,2} & \cdots & a_{p-1,2h} & a_{p-1,2h+1} & \cdots & a_{p-1,4h-1} & a_{p-1,4h} \\ a_{p,1} & a_{p,2} & \cdots & a_{p,2h} & a_{p,2h+1} & \cdots & a_{p,4h-1} & a_{p,4h} \end{bmatrix}$$

After operating $hr(1, p)$ to A, the upper p rows of A turns out to be

$$\begin{bmatrix} a_{p,4h} & a_{p,4h-1} & \cdots & a_{p,2h+1} & a_{p,2h} & \cdots & a_{p,2} & a_{p,1} \\ a_{p-1,4h} & a_{p-1,4h-1} & \cdots & a_{p-1,2h+1} & a_{p-1,2h} & \cdots & a_{p-1,2} & a_{p-1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{2,4h} & a_{2,4h-1} & \cdots & a_{2,2h+1} & a_{2,2h} & \cdots & a_{2,2} & a_{2,1} \\ a_{1,4h} & a_{1,4h-1} & \cdots & a_{1,2h+1} & a_{1,2h} & \cdots & a_{1,2} & a_{1,1} \end{bmatrix}$$

It is a routine to see that $hr(1, p)$ generates a permutation which is a product of disjoint transpositions $(a_{w,x}, a_{y,z})$ where $1 \leq w, y \leq p$, $1 \leq x \leq 2h$, and $2h + 1 \leq z \leq 4h$ satisfying $w + y = p + 1$ and $x + z = 4h + 1$. Note that there are $p \times 2h$ elements $a_{w,x}$ satisfying $1 \leq w \leq p$ and $1 \leq x \leq 2h$, and also there are $p \times 2h$ elements $a_{y,z}$ satisfying $1 \leq y \leq p$ and $2h+1 \leq z \leq 4h$. Hence, the permutation realized by $hr(1,p)$ is a product of $p \times 2h$ disjoint transpositions and so it is even.

Similarly a permutation realized by any vertical prefix reversal $vr(1,q)$ is even. Therefore every permutation realized by prefix reversals is a product of even permutations, and so, no transposition is generated by prefix reversals. Consequently, prefix reversals generate the alternating group A_{nm}.

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