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Some global well-posedness results for the compressible barotropic viscous fluid flow

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1 Introduction

This article is a brief survey of the paper [3], which is a joint work with Yoshihiro Shibata, Professor of Waseda University.

We consider the compressible Navier-Stokes equations:

$$\begin{cases} \rho_t + \mathrm{div}\,((\rho+\rho_*)\mathbf{u}) = 0 & \text{in } \Omega\times(0,T), \\ (\rho+\rho_*)(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \nabla\mathfrak{p} = 0 & \text{in } \Omega\times(0,T), \\ \mathbf{u} = 0 & \text{on } \Gamma\times(0,T), \\ (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega \end{cases} \tag{1.1}$$

where Ω is an exterior domain of the 3-dimensional Euclidean space $\mathbb{R}^3$, that is $\Omega = \mathbb{R}^3\setminus\mathcal{O}$ with some bounded domain $\mathcal{O}$ in $\mathbb{R}^3$ and Γ the boundary of Ω which is assume to be a smooth compact hyper-surface. Here, ρ_* is a positive constant describing the mass density of the reference body, and μ and ν are positive constants describing the first and second viscosity coefficients, respectively. Moreover, $\rho=\rho(x,t)$ with $x=(x_1,x_2,x_3)\in\Omega$, is a unknown function, $\rho_*+\rho$ being the density field, $\mathbf{u}=(u_1(x,t),u_2(x,t),u_3(x,t))$ the unknown velocity field, $\mathfrak{p}$ the unknown pressure field, $(\rho_0,\mathbf{u}_0)$ prescribed initial data, and $\mathbf{u}|_\Gamma=0$ is non-slip boundary condition. We consider only the barotropic case, that is

$$\mathfrak{p} = P(\rho_*+\rho),$$

where $P(s)$ is a C^∞ function defined for $s>0$ satisfying the condition:

$$\rho_1 \le P'(\rho+\rho_*) \le \rho_2 \quad \text{for } |\rho|\le\rho_*/2$$

with some positive constants ρ_1 and ρ_2.

The purpose of this article is to prove the global well-posedness of (1.1). Moreover, it is proved the optimal decay property.

When $\Omega=\mathbb{R}^3$ or Ω is a 3-dimensional exterior domain, Matsumura and Nishida [6, 7] proved the global well-posedness of problem (1.1) under the assumption that the H^3 norm of initial data ρ_0 and $\mathbf{u}_0$ are small enough. And also, Matsumura and Nishida [6, 7] and later on Deckelnick [1, 2] proved some convergence rate of solutions to the stationary solutions by using the energy method. The optimal decay properties were obtained by

Ponce [8] when $\Omega = \mathbb{R}^N$ and by Kobayashi and Shibata [5] when Ω is a 3-dimensional exterior domain with the help of the L_p-L_q decay estimate for the linearized equations.

In 2002, Kawashita [4] proved the global well-posedness of (1.1) when $\Omega = \mathbb{R}^N$ ($N \geq 2$) under the assumption that the H^s norm of initial data with $s = [N/2]+1$ are small enough, in particular, $s = 2$ when $N = 2$ and 3. Later, Wang and Tan [10] proved the optimal decay estimate of Kawashita's solution when $\Omega = \mathbb{R}^3$ with the help of the L_p-L_q decay estimate due to Ponce [8]. The purpose of this paper is to prove the same results for the initial boundary value problem in 3-dimensional exterior domains as that for the Cauchy problem obtained by Kawashita [4] and Wang and Tan [10].

The following theorem shows the global well-posedness of (1.1).

Theorem 1.1. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^2 compact hyper-surface. Then, there exists a small positive number δ such that if initial data $(\rho_0, \mathbf{u}_0) \in H^2(\Omega)^4$ satisfy the smallness assumption: $\|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)} \leq \delta$ and the compatibility condition: $\mathbf{u}_0|_\Gamma = 0$, then problem (1.1) with $T = \infty$ admits unique solutions ρ and $\mathbf{u}$ with*

$$
\begin{aligned}
&\rho \in C^0([(0,\infty), H^2(\Omega)) \cap C^1([0,\infty), H^1(\Omega)),\\
&\rho_t \in L_2((0,\infty), H^1(\Omega)), \quad \nabla\rho \in L_2((0,\infty), H^1(\Omega)^3)\\
&\mathbf{u} \in C^0([0,\infty), H^2(\Omega)^3) \cap C^1([0,\infty), L_2(\Omega)^3),\\
&\mathbf{u}_t \in L_2((0,\infty), H^1(\Omega)^3), \quad \nabla\mathbf{u} \in L_2((0,\infty), H^2(\Omega)^9)
\end{aligned}
\tag{1.2}
$$

possessing the estimate:

$$
\begin{aligned}
\sup_{0<s<t} &\|(\rho, \mathbf{u})(\cdot, s)\|^2 + \sup_{0<s<t} \|(\nabla\rho, \rho_s, \nabla\mathbf{u})(\cdot, s)\|^2 + \sup_{0<s<t} \|(\nabla^2\rho, \nabla\rho_s, \nabla^2\mathbf{u})(\cdot, s)\|^2\\
&+ \int_0^t (\|\nabla\mathbf{u}(\cdot, s)\|^2 + \|\rho_s(\cdot, s)\|^2 + \|\rho(\cdot, s)\|^2_{L_2(\Omega_R)})\, ds\\
&+ \int_0^t (\|\nabla^2\mathbf{u}(\cdot, s)\|^2 + \|\mathbf{u}_s(\cdot, s)\|^2 + \|\nabla\rho(\cdot, s)\|^2 + \|\nabla\rho_s(\cdot, s)\|^2)\, ds\\
&+ \int_0^t (\|\nabla^3\mathbf{u}(\cdot, s)\|^2 + \|\nabla\mathbf{u}_s(\cdot, s)\|^2 + \|\nabla^2\rho(\cdot, s)\|^2)\, ds\\
\leq\ & C\|(\rho_0, \mathbf{u}_0)\|^2_{H^2(\Omega)}
\end{aligned}
\tag{1.3}
$$

for any $t > 0$ with some constant C independent of δ.

Moreover, we proved the optimal decay property.

Theorem 1.2. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^3 compact hyper-surface. Then, there exists a small positive number δ such that if initial data $(\rho_0, \mathbf{u}_0) \in L_1(\Omega)^4 \cap H^2(\Omega)^4$ satisfy the smallness assumption: $\|(\rho_0, \mathbf{u}_0)\|_{L_1(\Omega)} + \|(\rho_0, \mathbf{u}_0)\|_{H^2(\Omega)} \leq \delta$ as well as the compatibility condition, then problem (1.1) with $T = \infty$ admits unique solutions ρ and $\mathbf{u}$ satisfying the same regularity conditions (1.2) and (1.3)* *in Theorem 1.1 and possessing the decay estimate:*

$$\begin{aligned}\sup_{0<s<t}(1+s)^{3/4}\|(\rho,\mathbf{u})(\cdot,s)\| + \sup_{0<s<t}(1+s)^{5/4}\|(\nabla\rho,\nabla\mathbf{u})(\cdot,s)\| \\ + \sup_{0<s<t}(1+s)^{5/4}\|(\nabla^2\rho,\nabla^2\mathbf{u})(\cdot,s)\| \\ \leq C(\|(\rho_0,\mathbf{u}_0)\|_{L_1(\Omega)} + \|(\rho_0,\mathbf{u}_0)\|_{H^2(\Omega)})\end{aligned}$$

for any $t>0$ with some constant C independent of δ.

2 Outline of proof of Theorem 1.1

In this section we consider the following equation:

$$\begin{cases}\rho_t + \mathbf{u}\cdot\nabla\rho + \rho_*\mathrm{div}\,\mathbf{u} = f_n & \text{in } \Omega\times(0,T),\\ \mathbf{u}_t - \mu_*\Delta\mathbf{u} - \nu_*\nabla\mathrm{div}\,\mathbf{u} + \gamma_*\nabla\rho = \mathbf{g}_n & \text{in } \Omega\times(0,T),\\ \mathbf{u}|_\Gamma = 0, \quad (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega,\end{cases} \tag{2.1}$$

where

$$\begin{aligned}&\mu_* = \mu/\rho_*, \quad \nu_* = \nu/\rho_*, \quad \gamma_* = P'(\rho_*)/\rho_*,\\ &f_n = \rho\mathrm{div}\,\mathbf{u},\\ &\mathbf{g}_n = \Big(\frac{1}{\rho+\rho_*} - \frac{1}{\rho_*}\Big)(\mu\Delta\mathbf{u} + \nu\nabla\mathrm{div}\,\mathbf{u}) - \Big(\frac{P'(\rho+\rho_*)}{\rho+\rho_*} - \frac{P'(\rho_*)}{\rho_*}\Big)\nabla\rho.\end{aligned}$$

For the sake of simplicity, we use the abbreviation: $\|\cdot\|_{L_2(\Omega)} = \|\cdot\|$, $\|\cdot\|_{L_q(\Omega)} = \|\cdot\|_q$ $(q\neq 2)$ and $\|\cdot\|_{H^s(\Omega)} = \|\cdot\|_{H^s}$ $(s=1,2)$. We set

$$\begin{aligned}\mathbf{I}_1(t) &= \sup_{0<s<t}\|(\rho,\mathbf{u})(\cdot,s)\|^2 + \int_0^t(\|\nabla\mathbf{u}(\cdot,s)\|^2 + \|\rho_s(\cdot,s)\|^2 + \|\rho(\cdot,s)\|^2_{L_2(\Omega_R)})\,ds,\\ \mathbf{I}_2(t) &= \sup_{0<s<t}\|(\nabla\rho,\rho_s,\nabla\mathbf{u})(\cdot,s)\|^2\\ &\quad + \int_0^t(\|\nabla^2\mathbf{u}(\cdot,s)\|^2 + \|\mathbf{u}_s(\cdot,s)\|^2 + \|\nabla\rho(\cdot,s)\|^2 + \|\nabla\rho_s(\cdot,s)\|^2)\,ds,\\ \mathbf{I}_3(t) &= \sup_{0<s<t}\|(\nabla^2\rho,\nabla\rho_s,\nabla^2\mathbf{u})(\cdot,s)\|^2\\ &\quad + \int_0^t(\|\nabla^3\mathbf{u}(\cdot,s)\|^2 + \|\nabla\mathbf{u}_s(\cdot,s)\|^2 + \|\nabla^2\rho(\cdot,s)\|^2)\,ds,\\ \mathbf{I}(t) &= \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}_3(t).\end{aligned}$$

To prove Theorem 1.1, it suffices to prove that

$$\mathbf{I}(t) \leqq K_1(\|(\rho_0,\mathbf{u}_0)\|^2_{H^2} + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2) \tag{2.2}$$

provided that $\|(\rho_0,\mathbf{u}_0)\|_{H^2} \leq 1$.

Multiplying the first equation in (2.1) by $\rho_*^{-1}\rho$ and the second one in (2.1) by $\gamma_*^{-1}\mathbf{u}$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\rho_*^{-1}\|\rho(\cdot,t)\|^2+\gamma_*^{-1}\|\mathbf{u}(\cdot,t)\|^2\}+\mu_*\|\nabla\mathbf{u}(\cdot,t)\|^2+\nu_*\|\operatorname{div}\mathbf{u}(\cdot,t)\|^2\\&\quad=\frac{1}{\rho_*}(f_n,\rho)+\frac{1}{\gamma_*}(\mathbf{g}_n,\mathbf{u})+((\operatorname{div}\mathbf{u})\rho,\rho).\end{aligned}\tag{2.3}$$

In what follows, to estimate the non-linear term we use the following estimates:

$$\begin{aligned}\|u\|_6\le C\|\nabla u\|,\qquad \|u\|_3\le C\|u\|_{H^1},\\ \|u\|_{L_2(\Omega_R)}\le C_R\|\nabla u\|,\quad \|u\|_\infty\le C\|\nabla u\|_{H^1}.\end{aligned}\tag{2.4}$$

Integrating (2.3) and using (2.4), we have

$$\|(\rho,\mathbf{u})(\cdot,t)\|^2+\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|^2\,ds\le C(\|(\rho_0,\mathbf{u}_0)\|^2+\mathbf{I}(t)^{3/2}).\tag{2.5}$$

By the first equation in (2.1) and (2.4), we have

$$\|\rho_t\|\le C(1+\|\nabla\rho\|_{H^1})\|\nabla\mathbf{u}\|,\tag{2.6}$$

so that

$$\int_0^t\|\rho_s(\cdot,s)\|^2\,ds\le C(\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|^2\,ds+\mathbf{I}(t)^2).\tag{2.7}$$

To estimate $\|\rho\|_{L_2(\Omega_R)}$, we use the following lemma:

Lemma 2.1. *Let ρ and $\mathbf{u}$ be solutions to problem* (1.1) *with*

$$\begin{aligned}\rho&\in C^1([0,T),H^1(\Omega))\cap C^0([0,T),H^2(\Omega)),\\ \mathbf{u}&\in C^1([0,T),L_2(\Omega)^N)\cap C^0([0,T),H^2(\Omega)^N).\end{aligned}$$

Then, we have

$$\begin{aligned}\int_0^t\|\rho(\cdot,s)\|^2_{L_2(\Omega_R)}\,ds\le C\int_0^t&\|\nabla\rho\|\{(1+\|\nabla\rho\|_{H^1})\|\nabla\mathbf{u}\|\\&+\|\mathbf{u}_s\|+\|\nabla\mathbf{u}\|\|\nabla\mathbf{u}\|_{H^1}+\|\nabla\mathbf{u}\|^{1/2}\|\nabla^2\mathbf{u}\|^{1/2}\}\,ds.\end{aligned}\tag{2.8}$$

By (2.8), we see that for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ depending on ε such that

$$\begin{aligned}\int_0^t\|\rho(\cdot,s)\|^2_{L_2(\Omega_R)}\,ds\le\varepsilon&\int_0^t(\|\nabla\rho(\cdot,s)\|^2+\|\nabla^2\mathbf{u}(\cdot,s)\|^2)\,ds\\&+C_\varepsilon\int_0^t(\|\nabla\mathbf{u}(\cdot,s)\|^2+\|\mathbf{u}_s(\cdot,s)\|^2)\,ds\\&+C_\varepsilon(\sup_{0,s<t}\|\nabla\mathbf{u}(\cdot,s)\|)^2\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|^2\,ds.\end{aligned}\tag{2.9}$$

Next, we estimate $\|\mathbf{u}_t(\cdot,t)\|^2$. Multiplying the second equation in (2.1) by $\mathbf{u}_t$, we have

$$\|\mathbf{u}_t(\cdot,t)\|^2 + \frac{1}{2}\frac{d}{dt}\{\mu_*\|\nabla\mathbf{u}(\cdot,t)\|^2 + \nu_*\|\mathrm{div}\,\mathbf{u}(\cdot,t)\|^2\} + \gamma_*(\nabla\rho,\mathbf{u}_t) = (\mathbf{g}_n,\mathbf{u}_t). \tag{2.10}$$

By the first equation in (2.1), we have

$$(\nabla\rho,\mathbf{u}_t) = -\frac{d}{dt}(\rho,\mathrm{div}\,\mathbf{u}) + (\rho_t,\mathrm{div}\,\mathbf{u}),$$

which, combined with (2.10), furnishes that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\mu_*\|\nabla\mathbf{u}(\cdot,t)\|^2 + \nu_*\|\mathrm{div}\,\mathbf{u}(\cdot,t)\|^2 - 2(\rho,\mathrm{div}\,\mathbf{u})\} + \frac{1}{2}\|\mathbf{u}_t(\cdot,t)\|^2\\ &\quad\le \frac{1}{2}(\|\mathbf{g}_n(\cdot,t)\|^2 + \|\rho_t(\cdot,t)\|^2 + \|\nabla\mathbf{u}(\cdot,t)\|^2).\end{aligned} \tag{2.11}$$

By (2.4) and (2.6), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\mu_*\|\nabla\mathbf{u}(\cdot,t)\|^2 + \nu_*\|\mathrm{div}\,\mathbf{u}(\cdot,t)\|^2 - 2(\rho,\mathrm{div}\,\mathbf{u})\} + \frac{1}{2}\|\mathbf{u}_t(\cdot,t)\|^2\\ &\quad\le C\{\|\nabla\rho\|_{H^1}^2(\|\nabla\mathbf{u}\|_{H^1}^2 + \|\nabla\rho\|^2) + \|\nabla\mathbf{u}\|^2\}.\end{aligned} \tag{2.12}$$

Integrating (2.12), we have

$$\begin{aligned}\|\nabla\mathbf{u}(\cdot,t)\|^2 + \int_0^t \|\mathbf{u}_s(\cdot,s)\|^2\,ds &\le C\{\|\nabla\mathbf{u}_0\|^2 + \|\rho_0\|^2 + \|\rho(\cdot,t)\|^2 + \int_0^t \|\nabla\mathbf{u}(\cdot,s)\|^2\,ds\\ &+ (\sup_{0<s<t}\|\nabla\rho(\cdot,s)\|_{H^1})^2\int_0^t(\|\nabla\mathbf{u}(\cdot,s)\|_{H^1}^2 + \|\nabla\rho(\cdot,s)\|^2)\,ds\}.\end{aligned} \tag{2.13}$$

By (2.7), (2.9) and (2.13), we have

$$\begin{aligned}\int_0^t(\|\rho_s(\cdot,s)\|^2 + \|\rho(\cdot,s)\|_{L_2(\Omega_R)}^2)\,ds &\le \varepsilon\int_0^t(\|\nabla^2\mathbf{u}(\cdot,s)\|^2 + \|\nabla\rho(\cdot,s)\|^2)\,ds\\ &+ C_\varepsilon(\|(\nabla\mathbf{u}_0,\rho_0)\|^2 + \|\rho(\cdot,t)\|^2 + \int_0^t\|\nabla\mathbf{u}(\cdot,s)\|^2\,ds + \mathbf{I}(t)^2),\end{aligned}$$

which, combined with (2.5), furnishes that

$$\mathbf{I}_1(t) \le C_\varepsilon(\|(\rho_0,\mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2) + \varepsilon\mathbf{I}_2(t). \tag{2.14}$$

We estimate ρ_t, $\mathbf{u}_t$ and $\nabla\mathbf{u}_t$. Differentiating the equations in (2.1) once with respect to t, we have

$$\begin{cases}\partial_t(\rho_t) + \mathbf{u}\cdot\nabla(\rho_t) + \rho_*\mathrm{div}\,\mathbf{u}_t = \partial_t f_n - \mathbf{u}_t\cdot\nabla\rho & \text{in } \Omega\times(0,T),\\ \partial_t(\mathbf{u}_t) - \mu_*\Delta\mathbf{u}_t - \nu_*\nabla\mathrm{div}\,\mathbf{u}_t + \gamma_*\nabla\rho_t = \partial_t\mathbf{g}_n & \text{in } \Omega\times(0,T),\\ \mathbf{u}_t|_\Gamma = 0, \quad (\rho_t,\mathbf{u}_t)|_{t=0} = (\rho_1,\mathbf{u}_1) & \text{in } \Omega,\end{cases}$$

where we have set

$$\rho_1 = -\mathrm{div}\,((\rho_* + \rho_0)\mathbf{u}_0),$$
$$\mathbf{u}_1 = -\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 + \frac{1}{\rho_* + \rho_0}(\mu \Delta \mathbf{u}_0 + \nu \nabla \mathrm{div}\, \mathbf{u}_0 - P'(\rho_* + \rho)\nabla \rho).$$

Analogously to (2.3), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\rho_*^{-1}\|\rho_t(\cdot,t)\|^2 + \gamma_*^{-1}\|\mathbf{u}_t(\cdot,t)\|^2\} + \mu_*\|\nabla \mathbf{u}_t(\cdot,t)\|^2 + \nu_*\|\mathrm{div}\,\mathbf{u}_t(\cdot,t)\|^2 \\ &\quad = \frac{1}{\rho_*}(\partial_t f_n - \mathbf{u}_t \cdot \nabla \rho, \rho_t) + \frac{1}{\gamma_*}(\partial_t \mathbf{g}_n, \mathbf{u}_t) + ((\mathrm{div}\,\mathbf{u})\rho_t, \rho_t). \end{aligned} \tag{2.15}$$

By (2.4),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}&\{\rho_*^{-1}\|\rho_t(\cdot,t)\|^2 + \gamma_*^{-1}\|\mathbf{u}_t(\cdot,t)\|^2\} + \mu_*\|\nabla \mathbf{u}_t(\cdot,t)\|^2 + \nu_*\|\mathrm{div}\,\mathbf{u}_t(\cdot,t)\|^2 \\ &\le C(\|\nabla \mathbf{u}_t\|^2\|\nabla\rho\|_{H^1} + \|\nabla \mathbf{u}_t\|\|\nabla \rho\|\|\rho_t\|_{H^1} \\ &\qquad + \|\mathbf{u}_t\|\|\nabla\rho\|\|\nabla\rho_t\| + \|\nabla \mathbf{u}\|\|\rho_t\|_{H^1}^2). \end{aligned} \tag{2.16}$$

Integrating (2.16), we have

$$\begin{aligned} &\|(\rho_t, \mathbf{u}_t)(\cdot,t)\|^2 + \int_0^t \|\nabla \mathbf{u}_s(\cdot,s)\|^2\,ds \\ &\le C\{\|(\rho_1, \mathbf{u}_1)\|^2 + (\sup_{0<s<t}\|\nabla\rho(\cdot,s)\|_{H^1})\int_0^t (\|\mathbf{u}_s(\cdot,s)\|_{H^1}^2 + \|\nabla\rho_s(\cdot,s)\|^2)\,ds \\ &\quad + (\sup_{0<s<t}\|\nabla \mathbf{u}(\cdot,s)\|)\int_0^t \|\rho_s(\cdot,s)\|_{H^1}^2\,ds\}. \end{aligned} \tag{2.17}$$

To prove the estimate of the higher order derivatives, we localize the problem in the whole space and near the boundary. Then, we consider the whole space problem:

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho + \rho_*\mathrm{div}\,\mathbf{v} = f & \text{in } \mathbb{R}^3, \\ \mathbf{v}_t - \mu_*\Delta\mathbf{v} - \nu_*\nabla\mathrm{div}\,\mathbf{v} + \gamma_*\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}^3 \end{cases} \tag{2.18}$$

and the half space problem:

$$\begin{cases} \rho_t + \mathbf{u}\cdot\nabla\rho + \rho_*\mathrm{div}\,\mathbf{v} = f & \text{in } \mathbb{R}^3_+, \\ \mathbf{v}_t - \mu_*\Delta\mathbf{v} - \nu_*\nabla\mathrm{div}\,\mathbf{v} + \gamma_*\nabla\rho = \mathbf{g} & \text{in } \mathbb{R}^3_+, \\ \mathbf{v}|_{x_3=0} = 0 \end{cases} \tag{2.19}$$

where $\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$.

In the half space problem, the points are estimations of $\partial_3\rho$ and $\partial_3^2\rho$. To do this, we use the idea due to Matsumura-Nishida [7]. From the third component in the sedond equation in (2.19) we have

$$\partial_3^2 v_3 = \frac{1}{\mu_* + \nu_*}\left\{(v_3)_t - \mu_*\sum_{i=1}^2 \partial_i^2 v_3 - \nu_*\sum_{i=1}^2 \partial_i\partial_3 v_i - g_3\right\} + \frac{\gamma_*}{\mu_* + \nu_*}\partial_3\rho. \tag{2.20}$$

Differentiating the first equation in (2.19) with respact to x_3 and inserting the formula (2.20), we have

$$
\begin{aligned}
&\partial_3(\rho_t + \mathbf{u}\cdot\nabla\rho) + \delta_*\partial_3\rho \\
&\quad = \partial_3 f - \frac{\rho_*\nu_*}{\mu_*+\nu_*}\sum_{i=1}^{2}\partial_i\partial_3 v_i + \frac{\rho_*\mu_*}{\mu_*+\nu_*}\sum_{i=1}^{2}\partial_i^2 v_3 - \frac{\rho_*}{\mu_*+\nu_*}(v_3)_t + \frac{\rho_*}{\mu_*+\nu_*}g_3 \qquad (2.21)
\end{aligned}
$$

where $\delta_* = \rho_*\gamma_*/(\mu_*+\nu_*)$. Multiplying (2.21) by $\partial_3\rho$, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_3\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \frac{\delta_*}{2}\|\partial_3\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} \\
&\quad \le C\left(\|\nabla f\|^2_{L_2(\mathbb{R}^3_+)} + \|\mathbf{g}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla\nabla'\mathbf{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\mathbf{v}_t\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla\mathbf{u}\cdot\nabla\rho\|^2_{L_2(\mathbb{R}^3_+)}\right)
\end{aligned}
$$

where $\nabla\nabla' w = (\partial_i\partial_j w \mid i=1,2,3,\ \ j=1,2)$. Differentiating (2.21) with respacet to x_3, we have

$$
\begin{aligned}
&\partial_3^2(\rho_t + \mathbf{u}\cdot\nabla\rho) + \delta_*\partial_3^2\rho \\
&= \partial_3^2 f - \partial_3\left(\frac{\rho_*\nu_*}{\mu_*+\nu_*}\sum_{i=1}^{2}\partial_i\partial_3 v_i + \frac{\rho_*\mu_*}{\mu_*+\nu_*}\sum_{i=1}^{2}\partial_i^2 v_3 - \frac{\rho_*}{\mu_*+\nu_*}(v_3)_t + \frac{\rho_*}{\mu_*+\nu_*}g_3\right) \qquad (2.22)
\end{aligned}
$$

and therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_3^2\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \delta_*\|\partial_3^2\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} \\
&\qquad\qquad \le C\Big(\|(\operatorname{div}\mathbf{u})\cdot\partial_3^2\rho\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla\mathbf{u}\cdot\nabla^2\rho\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla^2\mathbf{u}\cdot\nabla\rho\|^2_{L_2(\mathbb{R}^3_+)} \\
&\qquad\qquad\qquad + \|\nabla^2 f\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla^2\nabla'\mathbf{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla\mathbf{v}_t\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla\mathbf{g}\|^2_{L_2(\mathbb{R}^3_+)}\Big)
\end{aligned}
$$

where $\nabla^2\nabla'\mathbf{v} = (\nabla^2\partial_1\mathbf{v}, \nabla^2\partial_2\mathbf{v})$.

Using Sobolev's embedding theorem and (2.4), we have

$$
\begin{aligned}
\int_0^t \|\nabla^2\rho(\cdot,s)\nabla\mathbf{u}(\cdot,s)\|^2\,ds &\le \int_0^t \|\nabla^2\rho(\cdot,s)\|^2\|\nabla\mathbf{u}(\cdot,s)\|^2_\infty\,ds \\
&\le C\int_0^t \|\nabla^2\rho(\cdot,s)\|^2\|\nabla\mathbf{u}(\cdot,s)\|^2_{H^2}\,ds \\
&\le C\sup_{0<s<t}\|\nabla^2\rho(\cdot,s)\|^2\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|^2_{H^2}\,ds.
\end{aligned}
$$

In this way, we can enclose the estimation as the gradient $\mathbf{u}$ belongs to $L_2((0,\infty),H^2(\Omega))$ and ρ belongs to $L_\infty((0,\infty),H^2(\Omega))$.

Applying the same argument as in the above and using the cut-off technique, we have

$$
\begin{aligned}
&\|(\nabla\rho,\nabla\mathbf{u})(\cdot,t)\|^2 + \int_0^t \|(\nabla^2\mathbf{u},\mathbf{u}_s,\nabla\rho,\nabla\rho_s)(\cdot,s)\|^2\,ds \\
&\qquad\qquad\qquad\qquad \le C\{\|(\rho_0,\mathbf{u}_0)\|^2_{H^1} + \mathbf{I}_1(t) + \mathbf{I}(t)^2\},
\end{aligned}
$$

which, combined with (2.17), furnishes that

$$\mathbf{I}_2(t) \le C\{\|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}_1(t) + \mathbf{I}(t)^2\}. \tag{2.23}$$

Therefore, choosing $\varepsilon > 0$ small enough in (2.14), by (2.23) we have

$$\mathbf{I}_1(t) + \mathbf{I}_2(t) \le C\{\|(\rho_0, \mathbf{u}_0)\|_{H^1}^2 + \mathbf{I}(t)^{3/2} + \mathbf{I}(t)^2\}. \tag{2.24}$$

Analogously to (2.24), we have

$$\mathbf{I}_3(t) \le C\{\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 + \mathbf{I}_1(t) + \mathbf{I}_2(t) + \mathbf{I}(t)^2\} \tag{2.25}$$

provided that $\|(\rho_0, \mathbf{u}_0)\|_{H^2} \le 1$. Combining (2.24) and (2.25), we have (2.2). This completes the proof of Theorem 1.1.

3 Outline of proof of Theorem 1.2

We set

$$\begin{aligned}
\mathbf{D}_0(t) &= \sup_{0<s<t} (1+s)^{3/4}\|(\rho, \mathbf{u})(\cdot, s)\|, \\
\mathbf{D}_1(t) &= \sup_{0<s<t} (1+s)^{5/4}\|(\nabla\rho, \nabla\mathbf{u})(\cdot, s)\|, \\
\mathbf{D}_2(t) &= \sup_{0<s<t} (1+s)^{5/4}\|(\nabla^2\rho, \nabla^2\mathbf{u})(\cdot, s)\|, \\
\mathbf{D}(t) &= \mathbf{D}_0(t) + \mathbf{D}_1(t) + \mathbf{D}_2(t).
\end{aligned}$$

To prove Theorem 1.2, it suffices to prove that

$$\mathbf{D}(t) \le K_2(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}(t)^2) \tag{3.1}$$

with some constant K_2 with the help of L_p-L_q decay estimate for the linearized problem.

First of all, we introduce L_p-L_q decay estimate. To do this, we consider the following linearized problem:

$$\begin{cases}
\rho_t + \gamma \operatorname{div} \mathbf{v} = 0 & \text{in } \Omega \times (0, \infty), \\
\mathbf{v}_t - \alpha\Delta\mathbf{v} - \beta\nabla \operatorname{div} \mathbf{v} + \gamma\nabla\rho = 0 & \text{in } \Omega \times (0, \infty), \\
\mathbf{v}|_\Gamma = 0, \quad (\rho, \mathbf{v})|_{t=0} = (\rho_0, \mathbf{v}_0) \quad \text{in } \Omega,
\end{cases} \tag{3.2}$$

where α, β and γ are positive constants. Let m be a non-negative integer. We assume that Γ is a compact $C^{m+1,1}$ hyper-surface. Let A be an operator defined by the formula:

$$A(\rho, \mathbf{v}) = (\gamma \operatorname{div} \mathbf{v}, -\alpha\Delta\mathbf{v} - \beta\nabla \operatorname{div} \mathbf{v} + \gamma\nabla\rho)$$

for any element $(\rho, \mathbf{v})$ of the domain $W_{p,0}^{m+1,m+2}(\Omega) = \{(\rho, \mathbf{v}) \in W_p^{m+1}(\Omega) \times W_p^{m+2}(\Omega) \mid \mathbf{v}|_\Gamma = 0\}$. By Shibata and Tanaka [9] we know the generation of C_0 semigroup $\{T(t)\}_{t\ge 0}$ on $W_p^{m+1,m}(\Omega) = W_p^{m+1}(\Omega) \times W_p^m(\Omega)$, which is analytic. For the solution ρ and $\mathbf{v}$ of the equation (3.2), the following L_p-L_q decay estimate holds.

Theorem 3.1. *Let Ω be a 3-dimensional exterior domain whose boundary Γ is a C^3 compact hyper-surface. Let p and q be indices such that $1 \le q \le 2 \le p < \infty$. Let*

$$[(f, \mathbf{g})]_{p,q} = \|(f, \mathbf{g})\|_{L_q(\Omega)} + \|(f, \mathbf{g})\|_{W_p^{1,0}(\Omega)}.$$

Then, for any $(f, \mathbf{g}) \in W_p^{1,0}(\Omega) \cap L_q(\Omega)^4$ and $t \ge 1$ we have

$$\begin{aligned}
\|T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\le Ct^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}[(f, \mathbf{g})]_{p,q}, \\
\|\nabla T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\le C \begin{cases} t^{-\frac{3}{2}\left(\frac{1}{q}-\frac{1}{p}\right)-\frac{1}{2}}[(f, \mathbf{g})]_{p,q} & p \le 3, \\ t^{-\frac{3}{2q}}[(f, \mathbf{g})]_{p,q} & p \ge 3, \end{cases} \\
\|\nabla^2 P_{\mathbf{v}} T(t)(f, \mathbf{g})\|_{L_p(\Omega)} &\le Ct^{-\frac{3}{2q}}[(f, \mathbf{g})]_{p,q}
\end{aligned}$$

where $P_{\mathbf{v}}$ is the projection acting on $(\rho, \mathbf{v})$ defined by $P_{\mathbf{v}}(\rho, \mathbf{v}) = \mathbf{v}$.

We go back to the proof of the inequality (3.1). First, we estimate $\mathbf{D}_0(t)$ and $\mathbf{D}_1(t)$ with the help of L_p-L_q decay estimate for the linearized problem. Let $\{T(t)\}_{t\ge 0}$ be the analytic semigroup associated with the linearized problem:

$$\begin{cases} \rho_t + \rho_* \operatorname{div} \mathbf{u} = f_n - \mathbf{u} \cdot \nabla \rho & \text{in } \Omega \times (0, T), \\ \mathbf{u}_t - \mu_* \Delta \mathbf{u} - \nu_* \nabla \operatorname{div} \mathbf{u} + \gamma_* \nabla \rho = \mathbf{g}_n & \text{in } \Omega \times (0, T), \\ \mathbf{u}|_\Gamma = 0, \quad (\rho, \mathbf{u})|_{t=0} = (\rho_0, \mathbf{u}_0) & \text{in } \Omega. \end{cases} \tag{3.3}$$

Then, we have $(\rho, \mathbf{u})(\cdot, t) = T(t)(\rho_0, \mathbf{u}_0) + \mathbf{U}(t)$ with

$$\mathbf{U}(t) = \int_0^t T(t-s)((f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s), \mathbf{g}_n(\cdot, s))\, ds.$$

Here, we write $H^{1,0} = W_2^{1,0}(\Omega)$ and $\|\cdot\|_{H^s} = \|\cdot\|_{H^s(\Omega)}$ for $s = 1, 2$. By Theorem 3.1, we have

$$\begin{aligned}
\|T(t)(\rho_0, \mathbf{u}_0)\| &\le C(1+t)^{-\frac{3}{4}}(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^{1,0}}), \\
\|\nabla T(t)(\rho_0, \mathbf{u}_0)\| &\le C(1+t)^{-\frac{5}{4}}(\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^1})
\end{aligned} \tag{3.4}$$

for any $t > 0$. To estimate $\mathbf{U}(t)$, we observe that

$$\begin{aligned}
\|(f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s)\|_1 + \|(f_n - \mathbf{u} \cdot \nabla \rho)(\cdot, s)\|_{H^1} &\le C(1+s)^{-2}\mathbf{D}(s)^2, \\
\|\mathbf{g}_n(s)\|_1 + \|\mathbf{g}_n(s)\| &\le C(1+s)^{-2}\mathbf{D}(s)^2.
\end{aligned}$$

Thus, applying the L_p-L_q decay estimate and the usual analytic semi-group estimate, we have

$$\begin{aligned}
\|\mathbf{U}(t)\| &\le C\Big\{\int_0^{t-1} (t-s)^{-\frac{3}{4}}(1+s)^{-2}\, ds + \int_{t-1}^t (1+s)^{-2}\, ds\Big\}\mathbf{D}(t)^2 \\
&\le Ct^{-\frac{3}{4}}\mathbf{D}(t)^2, \\
\|\nabla \mathbf{U}(t)\| &\le C\Big\{\int_0^{t-1} (t-s)^{-\frac{5}{4}}(1+s)^{-2}\, ds + \int_{t-1}^t (t-s)^{-\frac{1}{2}}(1+s)^{-2}\, ds\Big\}\mathbf{D}(t)^2 \\
&\le Ct^{-\frac{5}{4}}\mathbf{D}(t)^2
\end{aligned} \tag{3.5}$$

for $t \geq 1$. Since $\|(\rho, \mathbf{u})(\cdot, t)\|_{H^2} \leq C\|(\rho_0, \mathbf{u}_0)\|_{H^2}$ as follows from (1.3), by (3.4) and (3.5)

$$\mathbf{D}_0(t) + \mathbf{D}_1(t) \leq C\{\|(\rho_0, \mathbf{u}_0)\|_1 + \|(\rho_0, \mathbf{u}_0)\|_{H^2} + \mathbf{D}(t)^2\}. \tag{3.6}$$

Next, we estimate $\|(\rho_t, \mathbf{u}_t)(\cdot, t)\|$. For this purpose, we use

Lemma 3.2. *Let $f(t)$ be a non-negative $C^1([0, \infty))$ function and let $g_i(t)$ $(i = 1, 2, 3, 4)$ be non-negative functions such that $g_i \in C^0((0, \infty))$ $(i = 1, 2, 3)$ and $g_4 \in L_2((0, \infty))$. Assume that*

$$\frac{d}{dt} f(t) + cf(t) \leq g_1(t) + g_2(t) + g_3(t) g_4(t)$$

for any $t > T_0$ with some constant $c > 0$. Then, for any $\alpha > 0$ there exists a $T_1 \geq T_0$ such that

$$\begin{aligned}(1+t)^\alpha f(t) &\leq (1+T_1)^\alpha f(T_1) + (2/c)(\sup_{T_1 < s < t} (1+s)^\alpha g_1(s)) \\ &\quad + \int_{T_1}^t (1+s)^\alpha g_2(s)\, ds + \sqrt{1/c}\,(\sup_{T_1 < s < t} (1+s)^\alpha g_3(s)) \Big(\int_{T_1}^t g_4(s)^2\, ds\Big)^{1/2}.\end{aligned}$$

In view of Theorem 1.1, we may assume that

$$\begin{aligned}&\int_0^t (\|\nabla \mathbf{u}(\cdot, s)\|_{H^2}^2 + \|\nabla \rho(\cdot, s)\|_{H^1}^2 + \|\rho_s(\cdot, s)\|_{H^1}^2 + \|\mathbf{u}_s(\cdot, s)\|_{H^1}^2)\, ds \\ &\qquad + \|\rho_t(\cdot, t)\|_{H^1}^2 + \|(\rho, \mathbf{u})(\cdot, t)\|_{H^2}^2 \leq C\|(\rho_0, \mathbf{u}_0)\|_{H^2}^2 \leq \varepsilon \end{aligned} \tag{3.7}$$

for any $t > 0$ with some small $\varepsilon > 0$ which is decided later. We choose $\varepsilon > 0$ small enough eventually, so that we may assume that $0 < \varepsilon < 1$. By (2.6) and (3.7), we have

$$\|\rho_t(\cdot, t)\| \leq C\|\nabla \mathbf{u}(\cdot, t)\|. \tag{3.8}$$

Let κ be a small positive number $> \varepsilon$ determined later. By (2.11) and (2.15),

$$\begin{aligned}&\frac{d}{dt}\{\mu_*\|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_*\|\mathrm{div}\, \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1}\|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1}\|\mathbf{u}_t(\cdot, t)\|^2\} \\ &\quad + \kappa\{\mu_*\|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_*\|\mathrm{div}\, \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1}\|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1}\|\mathbf{u}_t(\cdot, t)\|^2\} \\ &\quad \leq C\kappa(\|\nabla \mathbf{u}(\cdot, t)\|^2 + \|\rho_t(\cdot, t)\|^2 + \|\mathbf{u}_t(\cdot, t)\|^2) - \min(\frac{1}{2}, \mu_*)\|\mathbf{u}_t(\cdot, t)\|_{H^1}^2 \\ &\quad + C(\|\mathbf{g}_n(\cdot, t)\|^2 + \|\rho_t(\cdot, t)\|^2 + \|\nabla \mathbf{u}(\cdot, t)\|^2 + |(\partial_t f_n, \rho_t)| \\ &\quad + |(\mathbf{u}_t \cdot \nabla \rho, \rho_t)| + |(\partial_t \mathbf{g}_n, \mathbf{u}_t)| + |((\mathrm{div}\, \mathbf{u})\rho_t, \rho_t)|)\end{aligned} \tag{3.9}$$

Combining (3.7), (3.8), (3.9) and (2.4), choosing $\kappa > 0$ in such a way that $C\kappa < \min(1/2, \mu_*)$ with some constant independent of κ, and setting

$$\begin{aligned}f(t) &= \{\mu_*\|\nabla \mathbf{u}(\cdot, t)\|^2 + \nu_*\|\mathrm{div}\, \mathbf{u}(\cdot, t)\|^2 + \rho_*^{-1}\|\rho_t(\cdot, t)\|^2 + \gamma_*^{-1}\|\mathbf{u}_t(\cdot, t)\|^2\}, \\ g_1(t) &= C_\kappa(\|\nabla \mathbf{u}(\cdot, t)\|^2 + \|\nabla \rho(\cdot, t)\|^2), \\ g_2(t) &= C_\kappa\{(\|\nabla \mathbf{u}(\cdot, t)\|_{H^2}^2 + \|\nabla \rho(\cdot, t)\|_{H^1}^2)\|\nabla \rho(\cdot, t)\|^2 + \|\nabla \mathbf{u}(\cdot, t)\|_{H^2}^2\|\rho_t(\cdot, t)\|^2\},\end{aligned}$$

we have

$$\frac{d}{dt}f(t) + \kappa f(t) \le g_1(t) + g_2(t).$$

Thus, applying Lemma 3.2 and using Theorem 1.1 to estimate the term corresponding to $(1+T_1)^\alpha f(T_1)$, we have

$$\begin{aligned}(1+t)^{5/2}\|(\rho_t, \mathbf{u}_t)(\cdot,t)\|^2 \le C_\kappa\{(\|(\rho_0,\mathbf{u}_0)\|_{H^2}^2 + \sup_{0<s<t}(1+s)^{5/2}\|(\nabla\rho,\nabla\mathbf{u})(\cdot,s)\|^2 \\ + (\sup_{0<s<t}(1+s)^{5/2}\|\nabla\rho(\cdot,s)\|^2)\int_0^t(\|\nabla\mathbf{u}(\cdot,s)\|_{H^2}^2 + \|\nabla\rho(\cdot,s)\|_{H^1}^2)\,ds \\ + (\sup_{0<s<t}(1+s)^{5/2}\|\rho_s(\cdot,s)\|^2)\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|_{H^2}^2\,ds\}.\end{aligned}$$

Since

$$C_\kappa(\sup_{0<s<t}(1+s)^{5/2}\|\rho_s(\cdot,s)\|^2)\int_0^t\|\nabla\mathbf{u}(\cdot,s)\|_{H^2}^2\,ds \le C_\kappa\varepsilon(\sup_{0<s<t}(1+s)^{5/2}\|\rho_s(\cdot,s)\|^2)$$

as follows from (3.7), after fixing κ, we choose $\varepsilon < \kappa$ in such a way that $C_\kappa\varepsilon \le 1/2$, we have

$$(1+t)^{5/4}\|(\rho_t,\mathbf{u}_t)(\cdot,t)\| \le C(\|(\rho_0,\mathbf{u}_0)\| + \mathbf{D}_1(t)). \tag{3.10}$$

Applying the elliptic estimate to the second equation in (3.3) and using (3.10), we have

$$(1+t)^{5/4}\|\nabla^2\mathbf{u}(\cdot,t)\| \le C(\|(\rho_0,\mathbf{u}_0)\|_{H^2} + \mathbf{D}_1(t)). \tag{3.11}$$

Finally, we estimate $\|\nabla^2\rho\|$. To do this, we consider the whole space problem (2.18) and the half space problem (2.19) and analogously to (3.11), we have

$$(1+t)^{5/4}\|\nabla^2\rho(\cdot,t)\| \le C(\|(\rho_0,\mathbf{u}_0)\|_{H^2} + \mathbf{D}_1(t)). \tag{3.12}$$

Thus, by (3.6), (3.10), (3.11) and (3.12), we have (3.1), which completes the proof of Theorem 1.2.

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