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INVERSE PROBLEM ON ISOMORPHISM THEOREM OF
\( A^p(G) \)-ALGEBRAS \( 1 \leq p \leq 2 \)

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Abstract

Let \( G_1 \) and \( G_2 \) be locally compact Hausdorff groups, \( E(G_1) \) and \( E(G_2) \) the function spaces (Banach algebras or Banach spaces) on \( G_1 \) and \( G_2 \) respectively. Then it is known that if \( G_1 \simeq G_2 \), implies \( E(G_1) \) and \( E(G_2) \) are isomorphic. Naturally, an inverse problem arises that

\( \Phi : E(G_1) \rightarrow E(G_2) \)

whether an algebraic isomorphism could deduce \( G_1 \simeq G_2 \)?

In this paper, we would solve Problem (P) for \( A^p(G) \)-algebras, \( 1 \leq p \leq 2 \).

1. PRELIMINARIES

(1) 1948, Y. Kawada [7] solved this problem under bipositive isomorphism

\( \Phi : L^1(G_1) \rightarrow L^1(G_2) \).

(2) 1952, Wendel [13] proved (P) under the isomorphism \( \Phi \) from the algebra \( L^1(G_1) \)

onto \( L^1(G_2) \) by assuming \( \Phi \) is a norm nonincreasing.

(3) 1965, Edwards [2] considered the groups \( G_i(i = 1, 2) \) are compact, and if there exists a bipositive isomorphism of \( L^p(G_1) \) onto \( L^p(G_2) \) to get, then \( G_1 \simeq G_2 \). He asked whether the compact groups \( G_1 \) and \( G_2 \) are necessarily homeomorphic, if bipositive is replaced by isometry?

(4) 1966, The affirmative answer to this question in [2] by positive replaced isometry was given by Strichartz [12].

(5) 1968, Further, Parrott [11] proved the question in Edwards [2] for general locally compact groups \( G_1 \) and \( G_2 \) if there is an isometric transformation of \( L^p(G_1) \) onto \( L^p(G_2) \) (\( 1 \leq p < \infty, \ p \neq 2 \)).

Remark: The Lebesgue space \( L^p(G) \) need not be an algebra if \( G \) is not compact.
(6) 1973, Lai/Lien [10] solved the problem (P) by assume that if there exists an injective bipositive linear mapping from the Banach space $L^p(G_1)$ onto Banach space $L^p(G_2)$, then $G_1 \simeq G_2$ is deduced.

(7) Some other isomorphism problems were solved by Johnson [6], Gaudry [4] and Figa-Talamanca [3] in different view points.

(8) In this article we would solve problem (P) on the Banach algebra $A^p(G)$, $1 \leq p \leq 2$.

2. $A^p(G)$-ALGEBRAS, $1 \leq p < \infty$

In this paper, we would consider the isomorphism theorem for $A^p(G)$-algebras.

Let $G$ be a LCA group with dual group $\hat{G}$. The space $A^p(G)$ is defined by

$$A^p(G) = \{ f \in L^1(G) \; ; \; \text{Fourier transform } \hat{f} \in L^p(\hat{G}) \}, \quad 1 \leq p < \infty. \quad (1)$$

Then $A^p(G)$ is a commutative Banach algebra under convolution product with the norm given by

$$\| f \|_p = \| f \|_1 + \| \hat{f} \|_p, \text{ for each } p, \quad 1 \leq p < \infty \text{ for } f \in A^p(G). \quad (2)$$

The norm $\| \|_p$ is equivalent to max($\| f \|_1, \| \hat{f} \|_p$).

Since $f \in A^p(G) \Rightarrow \hat{f} \in L^p(\hat{G}) \cap C_0(\hat{G})$, thus $\hat{f} \in L^r(\hat{G})$ for $r > p > 1$, but such $f \not\in A^r$ for $1 \leq p \leq 2 \leq q < r < \infty$.

By this fact, we know that $A^p(G)$ can not include all Fourier transforms of $C_c(G) \cap A^r$. And $A^1(G) \supset A^p(G) \supset A^2(G) \supset A^q(G) \supset C_0(G)$, where $A^1(G) = \bigcup_{1 < p \leq 2} A^p(G)$ is the closure of such union sets.

We then conclude that

$$1 \leq p \leq 2, \; C_c \cap A^p(G) \text{ is dense in } A^p(G) \text{ with respect to the } A^p \text{-norm.}$$

Thus,

if $f \in A^p(G)$, then $\hat{f} \in L^p(\hat{G})$ and $\hat{f} \in L^q(\hat{G})$ for $p \leq 2 < q$, $f \not\in A^q(G)$,

and so $\forall p, \; 1 \leq p \leq 2 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $A^p(G) \cap A^q(G) = \emptyset$.

Hence the index $p$, only taken in the interval $1 \leq p \leq 2$ could get $T(A^p) \subset A^p$ by a continuous linear operator $T$. So we can discuss the multipliers $T$ on $A^p(G)$ only taken
1 \leq p < \infty \) which could get \( T(A^p) \subset A^p \). Therefore in later part, all \( A^p(G) \) we discuss will take \( 1 \leq p \leq 2 \).

3. **Multipliers of \( A^p(G) \)**

A multiplier \( T \) of \( A^p(G) \) is a continuous linear mapping of \( A^p(G), 1 \leq p \leq 2 \) into itself, such that

\[
T(f * g) = T(f) * g = f * T(g), \text{ for all } f, g \in A^p(G).
\]

In order to solve problem (P) on \( A^p(G) \)-algebras. We use a technique by passing the multiplier of \( A^p(G) \), thus we subscript the definition of \( A^p(G) \), as follows. Let \( \mathcal{L}(A^p) \) be the space of all bounded linear operator of \( A^p(G), 1 \leq p \leq 2 \) into itself.

**Definition 1.** An operator \( T \in \mathcal{L}(A^p(G)) \) is said to be a multiplier of \( A^p(G) \) if

\[
T(f * g) = T(f) * g = f * T(g) \text{ for } f, g \in A^p(G).
\]

The concept of multiplier \( T \), one can consult Lai/ Lee / Liu [9, Theorem 1.1]. It deduces the space \( \mathfrak{M}(A^p) \) of multipliers of \( A^p(G) \) is isometrically isomorphic to \( M(G) \), the space of all regular measures of \( G \), that is

\[
\mathfrak{M}(A^p) \cong \mathfrak{M}(L^1) \cong M(G), 1 \leq p \leq 2.
\]

On the other hand, it is known that \( A^p(G) \) is essential \( L^1(G) \)-module, since \( L^1(G) \) has bounded approximate identity of norm 1 [9, Theorem 2.1]. It is remarkable that \( A^p(G) \) has no \( A^p \)-uniform bounded approximate identity [8, p.574].

\[
A^p * L^1 = A^p, \text{ and } ||f * g||^p \leq ||f||^p ||g||, \text{ for } f \in A^p, g \in L^1.
\]

Thus the space \( \mathfrak{M}(A^p, L^1) \) of multiplier \( A^p \) into \( L^1 \) is identical to \( \mathfrak{M}(A^p) \). Hence there exists a unique \( \mu \in M(G) \) such that

\[
Tf = \mu * f \text{ for all } f \in A^p(G)
\]

for any \( T \in \mathfrak{M}(A^p, L^1) \cong \mathfrak{M}(A^p) \). By the property of \( A^p(G) \)-algebras, we will show the Isomorphism Theorem of \( A^p(G) \)-algebras can be stated as the following:
Theorem 2. Let $G_1$ and $G_2$ be locally compact abelian groups and $\Phi$ an algebraic isomorphism of $A^p(G_1)$ onto $A^p(G_2)$, $1 \leq p \leq 2$. Suppose that one of $\hat{G}_1$ and $\hat{G}_2$ is connected, then $\Phi$ induces a topological isomorphism $\tau$ carrying $G_2$ onto $G_1$. Furthermore,

$$\Phi f(x) = c\hat{x}(x)f(\tau x) \text{ for } f \in A^p(G_1), \text{ and } x \in G_2,$$

where $\hat{x}(x)$ is a fixed character on $G_2$ and $c$ a constant depending only on the choice of Haar measure in $G_2$.

Outline of the proof for the main Theorem is given as follows:

Since the isomorphism

$$\Phi : A^p(G_1) \onto A^p(G_2),$$

implies $\Phi$ maps the Maximal ideal spaces $\mathfrak{M}ax(A^p(G_1))$ of $A^p(G_1)$

on to $\mathfrak{M}ax(A^p(G_2))$ of $A^p(G_2)$,

$$\Rightarrow \Phi : \mathfrak{M}ax(A^p(G_1)) \rightarrow \mathfrak{M}ax(A^p(G_2))$$

$$\Rightarrow \Phi : \hat{G}_1 \onto \hat{G}_2$$

The reason of (7) is that since $A^p(G)$ is a semisimple commutative Banach algebra, then the space $\mathfrak{M}ax(A^p(G))$ is characterized by $\hat{G}$.

Hence if one of $\hat{G}_1$ and $\hat{G}_2$ is connected, then both of $\hat{G}_1$ and $\hat{G}_2$ are connected. Therefore $G_1$ and $G_2$ are non-compact.

Since the theorem in [3] is applicable, we note that operator $T$ commutes with convolution on $A^p(G_1)$ is represented uniquely by $\mu \in M(G_1)$

$$Tf = \mu * f = 0 \text{ for all } f \in A^p(G_1)$$

$$\Rightarrow \mu = 0.$$  \hfill (8)

Thus we take $\nu \in M(G_2)$ for any $f \in A^p(G_1)$, it can define this operator

$$T : A^p(G_1) \rightarrow A^p(G_2)$$

by

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf.$$  \hfill (9)
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It is well-defined by (8) since $A^p(G_1)$ is semisimple, and by Loomis [book: p.76 Theorem], one sees that $\Phi$ is bicontinuous and hence $T$ is a multiplier of $A^p(G_1)$, thus $\exists! \mu \in M(G_1)$ such that

$$\mu * f = \Phi^{-1}(\nu * \Phi f) = Tf.$$ 

This $\mu$ is uniquely determined by $\nu$, we define a mapping $\Psi$ of $M(G_2)$ into $M(G_1)$ by

$$\Psi \nu * f = \Phi^{-1}(\nu * \Phi f).$$

It is not hard to prove that $\Psi$ is an isomorphism of $M(G_2)$ onto $M(G_1)$. Since both measure algebras $M(G_1)$ and $M(G_2)$ are semi-simple and commutative, $\Psi$ is bicontinuous and one can show that

$\Psi|_{A^p(G_2)}$ on the algebra $A^p(G_2)$ is dense in $L^1(G_2)$,

hence $\Psi|L^1(G_2)$ becomes an isomorphism of $L^1(G_2)$ onto $L^1(G_1)$ [See Rudin’s book Theorem 6.6.4]. Hence by Helsen [5], the theorem is complete. $\square$

The full paper about Isomorphism Theorem of $A^p(G)$-algebras will appear in elsewhere.

REFERENCES

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