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Arithmetical properties of the values of power series

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Abstract

In this paper, we give a new criterion for the algebraic independence of the values of power series. In particular, we deduce the algebraic independence of the values $\sum_{m=0}^{\infty}\beta^{-w_m}$, where β is a Pisot or Salem number and $(w_m)_{m=0}^{\infty}$ is a certain increasing sequence of nonnegative integers satisfying $\lim_{m\to\infty} w_{m+1}/w_m = 1$.

1 Review of the β-expansions of real numbers

Criteria for the transcendence of real numbers are deduced from the properties of Diophantine approximations for algebraic numbers. In this paper, we study criteria for the transcendence of real numbers, using the β-expansions of algebraic numbers in the case where β is a Pisot or Salem number. Moreover, we also give a new criterion for the algebraic independence of real numbers, using the approximation properties of iterated sumsets.

We recall the definition of Pisot and Salem numbers. Let β be an algebraic integer greater than 1. We denote the conjugates of β by $\beta_1=\beta,\beta_2,\ldots,\beta_d$. Then β is a Pisot number if $|\beta_i|<1$ for any $i=2,\ldots,d$. For instance, any rational integer $b\geq 2$ and the golden ratio $(1+\sqrt{5})/2$ are Pisot numbers. Moreover, β is called a Salem number if $|\beta_i|\leq 1$ for any $i=2,\ldots,d$ and if there exists $j\geq 2$ such that $|\beta_j|=1$. For instance, the unique zero $\tau>1$ of $X^4-X^3-X^2-X+1$ is a Salem number. We denote the integral and fractional parts of a real number x by $\lfloor x\rfloor$ and $\{x\}$, respectively. Moreover, $\lceil x\rceil$ denotes the minimal integer not less than x.

For any real number $\beta>1$, the β-transformation $T_\beta:[0,1)\to[0,1)$ is defined by $T_\beta(x)=\{\beta x\}$ for $x\in[0,1)$. Then the β-expansion of x, introduced by Rényi [15], is denoted as

$$x=\sum_{n=1}^{\infty}t_n(\beta,x)\beta^{-n}, \tag{1.1}$$

where $t_n(\beta,x)=\lfloor\beta T_\beta^{n-1}(x)\rfloor$ for $n=1,2,\ldots$. If $\beta=b$ is an integer greater than 1, then (1.1) coincides with the ordinary base-b expansion of x.

Bertrand [4] and Schmidt [16] independently proved for each Pisot number β that any $x \in [0,1) \cap \mathbb{Q}$ has an ultimately periodic β-expansion $(t_n(\beta,x))_{n=1}^{\infty}$. Conversely, Schmidt [16] showed for each real number $\beta > 1$ that if any rational number $x \in [0,1)$ has an ultimately periodic β-expansion, then β is a Pisot or Salem number. Schmidt also conjectured for each Salem number β that any rational number has an ultimately periodic β-expansion, which is still an open problem.

Moreover, it is generally difficult to investigate the digits $(t_n(\beta,x))_{n=1}^{\infty}$ in the β-expansion of x in the case of $x \notin \mathbb{Q}(\beta)$. For instance, if $\beta = b$ is an integer, then Borel [6] conjectured that any algebraic irrational number x is normal in base-b. In particular, if this conjecture is true, then we see that any digit $h \in \{0,1,\ldots,b-1\}$ occurs with average frequency tending to $1/b$, which is still unproved.

We introduce known results for Borel's conjecture, that is, lower bounds for the numbers of nonzero digits in β-expansions. Moreover, we consider the numbers of nonzero digits in the case where β is a Pisot or Salem number. Let $\beta > 1$ and $x \in [0,1)$ be real numbers. For any positive integer N, put

$$\lambda_N(\beta,x) := \mathrm{Card}\{n \in \mathbb{Z} \mid 1 \leq n \leq N,\ t_n(\beta,x) \neq 0\},$$

where Card denotes the cardinality.

First, we consider the case where $\beta = b > 1$ is an integer. If Borel's conjecture is true, then, for any algebraic irrational number x, we have

$$\lim_{N\to\infty} \frac{\lambda_N(b,x)}{N} = \frac{b-1}{b} > 0.$$

However, there is no algebraic irrational number x such that the inequality

$$\limsup_{N\to\infty} \frac{\lambda_N(b,x)}{N} > 0$$

was proved. Suppose that x is an algebraic irrational number of degree D. If $b = 2$, then Bailey, Borwein, Crandall, and Pomerance [3] showed that there exist positive constants $C_1(x)$ and $C_2(x)$, depending only on x, such that

$$\lambda_N(2,x) \geq C_1(x) N^{1/D} \tag{1.2}$$

for any integer $N \geq C_2(x)$. Note that $C_1(x)$ is effectively computable but $C_2(x)$ is not. Adamczewski, Faverjon [2], and Bugeaud [7] proved an effective version of (1.2) for general integral base $b \geq 2$ as follows: there exist effectively computable positive constants $C_3(b,x)$ and $C_4(b,x)$, depending only on b and x, such that

$$\lambda_N(b,x) \geq C_3(b,x) N^{1/D}$$

for any integer $N \geq C_4(b,x)$.

Next, we consider the case where β is a general Pisot or Salem number. In the rest of this section, suppose that x is an algebraic number with $D = [\mathbb{Q}(\beta,x) : \mathbb{Q}(\beta)]$, where $[L;K]$ denotes the degree of field extension L/K. Moreover, we denote $f \gg g$ if there exists an effectively computable positive constants C, depending only on β and x, such that $f \geq Cg$. Bugeaud [9] showed that if there exist infinitely many integers $n \geq 1$ with $t_n(\beta, x) \neq t_{n+1}(\beta, x)$, then

$$\lambda_N(\beta, x) \gg \gamma_N(\beta, x) \gg \frac{(\log N)^{3/2}}{(\log\log N)^{1/2}} \tag{1.3}$$

for any integer $N \gg 1$, where

$$\gamma_N(\beta, x) = \mathrm{Card}\{n \in \mathbb{Z} \mid 1 \leq n \leq N,\ t_n(\beta, x) \neq t_{n+1}(\beta, x)\}.$$

Under the assumption that there exist infinitely many integers $n \geq 1$ with $t_n(\beta, x) \neq 0$, (1.3) was improved as follows [13]:

$$\lambda_N(\beta, x) \gg \left(\frac{N}{\log N}\right)^{1/(2D-1)} \tag{1.4}$$

for any integer $N \gg 1$. In this paper, we introduce further improvement of (1.4):

THEOREM 1.1 ([14]). *Let β be a Pisot or Salem number and x an algebraic number with $D = [\mathbb{Q}(\beta, x) : \mathbb{Q}(\beta)]$. Let A be a positive integer and $(t_n)_{n=0}^{\infty}$ a sequence of nonnegative integers with $t_n \leq A$ for any $n \geq 0$. Assume that*

$$x = \sum_{n=0}^{\infty} t_n \beta^{-n}$$

and that there exist infinitely many integers n with $t_n \neq 0$. Then, there exist effectively computable positive constants $C_5(\beta, x, A)$ and $C_6(\beta, x, A)$, depending only on $\beta, x,$ and A, such that

$$\mathit{Card}\{n \in \mathbb{Z} \mid n \leq N, t_n \neq 0\} \geq C_5(\beta, x, A) \left(\frac{N}{\log N}\right)^{1/D}$$

for any integer $N \geq C_6(\beta, x, A)$.

In particular, applying Theorem 1.1 to the numbers of nonzero digits in the β-expansions, we sharpen (1.4) as follows:

$$\lambda_N(\beta, x) \gg \left(\frac{N}{\log N}\right)^{1/D}$$

for any integer $N \gg 1$.

2 Transcendence of the power series at certain algebraic points

In what follows, $\boldsymbol{w} = (w_m)_{m=0}^{\infty}$ denotes a sequence of nonnegative integers satisfying $w_{m+1} > w_m$ for any sufficiently large m. For such a sequence $\boldsymbol{w}$, put

$$f(\boldsymbol{w}, X) := \sum_{m=0}^{\infty} X^{w_m}.$$

Bugeaud [8] posed the problem following: If $\boldsymbol{w}$ increases sufficiently rapidly, then $f(\boldsymbol{w},\alpha)$ is transcendental for any algebraic number α with $0<|\alpha|<1$. Corvaja and Zannier [10] verified for any algebraic number α with $0<|\alpha|<1$ that if

$$\liminf_{m\to\infty}\frac{w_{m+1}}{w_m}>1,$$

then $f(\boldsymbol{w},\alpha)$ is transcendental. For instance, the numbers

$$\sum_{m=0}^{\infty}\alpha^{m!},\sum_{m=0}^{\infty}\alpha^{h^m}$$

are transcendental, where h is any integer greater than 1. Moreover, Adamczewski [1] showed for any Pisot or Salem number β that if

$$\limsup_{m\to\infty}\frac{w_{m+1}}{w_m}>1,$$

then $f(\boldsymbol{w},\beta^{-1})$ is transcendental. The purpose of this section is to consider the transcendence of $f(\boldsymbol{w},\beta^{-1})$ under the assumption that

$$\lim_{m\to\infty}\frac{w_{m+1}}{w_m}=1 \tag{2.1}$$

Using Theorem 1.1, we get the following:

THEOREM 2.1. *Let $R>1$ be a real number. Suppose that*

$$\limsup_{m\to\infty}\frac{v_m}{m^R}>0. \tag{2.2}$$

Then, for any Pisot or Salem number β, we have

$$\left[\mathbb{Q}\Big(\beta,f\left(\boldsymbol{w};\beta^{-1}\right)\Big):\mathbb{Q}(\beta)\right]\geq\lceil R\rceil.$$

Note that Theorem 2.1 was essentially proved by Bailey, Borwein, Crandall, and Pomerance [3] in the case where $\beta=b$ is an integer.

Proof. We may assume that $f(\boldsymbol{w},\beta^{-1})$ is an algebraic number, namely,

$$D:=\left[\mathbb{Q}\Big(\beta,f\left(\boldsymbol{w};\beta^{-1}\right)\Big):\mathbb{Q}(\beta)\right]<\infty.$$

Put $x:=f(\boldsymbol{w},\beta^{-1})$. Then, for an arbitrary positive real number ε, (2.2) implies that

$$\lambda_N(\beta,x)\leq N^{\varepsilon+1/R} \tag{2.3}$$

for any sufficiently large N. On the other hand, Theorem 1.1 implies that there exist infinitely many integers N satisfying

$$N^{-\varepsilon+1/R}\leq\lambda_N(\beta,x). \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\frac{1}{D}-\varepsilon\leq\frac{1}{R}+\varepsilon.$$

Since ε is arbitrary, we obtain $R\leq D$. $\square$

For instance, put

$$\zeta_y(X) := \sum_{m=0}^{\infty} X^{\lfloor m^y \rfloor}$$

for any real number $y > 1$. Note that

$$\lim_{m\to\infty} \frac{\lfloor (m+1)^y \rfloor}{\lfloor m^y \rfloor} = 1.$$

Duverney, Nishioka, Nishioka, Shiokawa [11], and Bertrand [5] independently showed that if $y = 2$, then $\zeta_2(\alpha)$ is transcendental for any algebraic α with $0 < |\alpha| < 1$. However, if $y \neq 2$, then the transcendence of $\zeta_y(\alpha)$ is not known. Applying Theorem 2.1, we see for any Pisot or Salem number β that

$$\left[\mathbb{Q}\Big(\beta, \zeta_y\left(\beta^{-1}\right)\Big) : \mathbb{Q}(\beta)\right] \geq \lceil y \rceil.$$

Moreover, using Theorem 2.1, we deduce the following criterion for transcendence:

COROLLARY 2.2. *Assume for any real number R that*

$$\limsup_{m\to\infty} \frac{v_m}{m^R} > 0.$$

Then, for any Pisot or Salem number β, we have $f(\boldsymbol{w}, \beta^{-1})$ is transcendental.

For instance, let

$$\varphi(p; m) := m^{(\log m)^p} = \exp\left((\log m)^{1+p}\right)$$

for a real number $p > 1$ and an integer $m \geq 1$. Set

$$\Phi_p(X) := \sum_{m=1}^{\infty} X^{\lfloor \varphi(p;m) \rfloor}.$$

Using the mean value theorem, we see that

$$\lim_{m\to\infty} \frac{\lfloor \varphi(p; m+1) \rfloor}{\lfloor \varphi(p; m) \rfloor} = 1.$$

It is easily checked that the sequence $\lfloor \varphi(p;m) \rfloor (m = 1, 2, \ldots)$ satisfies the assumption of Corollary 2.2. Thus, Corollary 2.2 implies for any Pisot or Salem number β that $\Phi_p(\beta^{-1})$ is transcendental.

3 Algebraic independence of the power series at certain algebraic points

In this section, we consider the algebraic independence of real numbers including the values $\zeta_y(\beta^{-1})$ and $\Phi_p(\beta^{-1})$ defined in the previous section. Results in this section are deduced from Theorem 3.4, a criterion for linear independence.

First we investigate the algebraic independence of $\Phi_p(\beta^{-1})$ for distinct real numbers p. Recall that a nonempty set S of complex numbers is algebraically independent if any distinct elements $x_1, \ldots, x_r$ with any number r are algebraically independent.

THEOREM 3.1. *Let β be a Pisot or Salem number.*
(1) *The set*

$$\{\Phi_p(\beta^{-1}) \mid p \geq 1,\ p \in \mathbb{R}\}$$

is algebraically independent.
(2) *If p and p' are distinct positive real numbers, then the numbers $\Phi_p(\beta^{-1})$ and $\Phi_{p'}(\beta^{-1})$ are algebraically independent.*

Using our criterion for linear independence, we also deduce the following:

THEOREM 3.2. *For any Pisot or Salem number β, the two numbers*

$$\sum_{m=1}^{\infty} \beta^{-\lfloor m^{\log m} \rfloor}, \sum_{m=3}^{\infty} \beta^{-\lfloor m^{\log \log m} \rfloor}$$

are algebraically independent.

Next, we introduce partial results for the algebraic independence of real numbers including the values $\zeta_y(\beta^{-1})$. For any integer $k \geq 4$, it is easily seen that there exists a unique real number σ_k satisfying $0 < \sigma_k < 1$ and

$$(1-\sigma_k)^k + (k-1)\sigma_k - 1 = 0.$$

THEOREM 3.3. *Let A be a positive integer and y a real number. Suppose that*

$$\begin{cases} y > A & \text{if } A = 1, 2, 3, \\ y > \sigma_A^{-1} & \text{if } A \geq 4. \end{cases}$$

Moreover, let β be a Pisot or Salem number.
(1) *For any positive real number p, the set*

$$\left\{ \zeta_y \left(\beta^{-1}\right)^{k_1} \Phi_p \left(\beta^{-1}\right)^{k_2} \middle| k_1, k_2 \in \mathbb{Z}, 0 \leq k_1 \leq A, 0 \leq k_2 \right\}$$

is linearly independent over $\mathbb{Q}(\beta)$.
(2) *For any integer $h \geq 2$, the set*

$$\left\{ \zeta_y \left(\beta^{-1}\right)^{k_1} \left(\sum_{m=0}^{\infty} \beta^{-h^m} \right)^{k_2} \middle| k_1, k_2 \in \mathbb{Z}, 0 \leq k_1 \leq A, 0 \leq k_2 \right\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

In the rest of this section, we introduce a general criterion for linear independence. For any nonzero power series $f(X) = \sum_{n=0}^{\infty} t_n X^n$, put

$$S(f) := \{n \in \mathbb{Z} \mid n \geq 0, t_n \neq 0\}.$$

We define the iterated sumsets of $S(f)$ as follows: For any nonnegative integer k, set

$$kS(f) := \begin{cases} \{0\} & \text{if } k = 0, \\ \{s_1 + \cdots + s_k \mid s_1, \ldots, s_k \in S(f)\} & \text{if } k \geq 1. \end{cases}$$

For any nonzero power series $f_1(X), \dots, f_r(X)$ and nonnegative integers $k_1, \dots, k_r$, put

$$\sum_{i=1}^{r} k_i S(f_i) := \{s_1 + \cdots + s_r \mid s_1 \in k_1 S(f_1), \dots, s_r \in k_r S(f_r)\}.$$

Moreover, for any nonempty set $\mathcal{A}$ of nonnegative integers, let

$$\theta(x; \mathcal{A}) := \max\{a \in \mathcal{A} \mid a < R\},$$
$$\lambda(R; \mathcal{A}) := \mathrm{Card}([0, R] \cap \mathcal{A})$$

for any real numbers x, R with $x > \min \mathcal{A}$ and $R \geq 0$. We say that $g(R) = o(h(R))$ if $g(R)/h(R)$ tends to zero.

THEOREM 3.4. *Let $A \geq 1$ and $r \geq 2$ be integers and $f_i(X) = \sum_{n=0}^{\infty} t_n^{(i)} X^n \in \mathbb{Z}[[X]] \setminus \mathbb{Z}[X]$ $(i = 1, \dots, r)$ be power series. Assume that $f_1(X), \dots, f_r(X)$ satisfy the following four assumptions:*

1. *There exists a positive constant C_7 such that*

$$0 \leq t_n^{(i)} \leq C_7$$

for any nonnegative integer $n \geq 0$ and $1 \leq i \leq n$.

2. *Let $k_1, k_2, \dots, k_r$ be nonnegative integers satisfying*

$$\begin{cases} k_1 \leq A - 1 & \text{if } r = 2, \\ k_1 \leq A & \text{if } r \geq 3. \end{cases}$$

Then

$$R - \theta\left(R; \sum_{i=1}^{r-2} k_i S(f_i) + (1 + k_{r-1}) S(f_{r-1})\right) = o\left(\frac{R}{\prod_{i=1}^{r} \lambda(R; S(f_i))^{k_i}}\right)$$

as R tends to infinity.

3. *There exists a positive real number δ satisfying*

$$\lambda(R; S(f_1)) = o\left(R^{-\delta + 1/A}\right)$$

as R tends to infinity. Moreover, for any $i = 2, \dots, r$ and any positive real number ε, we have

$$\lambda(R; S(f_i)) = o\left(\lambda\left(R; S(f_{i-1})\right)^{\varepsilon}\right)$$

as R tends to infinity.

4. *There exists a positive constant C_8 satisfying*

$$[R, C_8 R] \cap S(f_r) \neq \emptyset$$

for any sufficiently large R.

Then the set

$$\left\{ f_1(\beta^{-1})^{k_1} f_2(\beta^{-1})^{k_2} \cdots f_r(\beta^{-1})^{k_r} \;\middle|\; 0 \leq k_1 \leq A, 0 \leq k_2, k_3, \dots, k_r \right\}$$

is linearly independent over $\mathbb{Q}(\beta)$.

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