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ASYMPTOTIC EXPANSIONS FOR THE LAPLACE-MELLIN AND RIEMANN-LIOUVILLE TRANSFORMS OF LERCH ZETA-FUNCTIONS (PRE-ANNOUNCEMENT VERSION)

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ABSTRACT. This article summarizes the results appearing in the forthcoming paper [13]. For a complex variable s, and real parameters a and λ with $a>0$, the Lerch zeta-function $\phi(s,a,\lambda)$ is defined by the Dirichlet series $\sum_{l=0}^{\infty}e(\lambda l)(a+l)^{-s}$ ($\operatorname{Re}s>1$), and its meromorphic continuation over the whole s-plane, where $e(\lambda)=e^{2\pi i\lambda}$, and the domain of the parameter a can be extended to the whole sector $|\arg z|<\pi$. It is treated in the present article several asymptotic aspects of the Laplace-Mellin and Riemann-Liouville (or Erdély-Köber) transforms of $\phi(s,a,\lambda)$, together with its slight modification $\phi^{*}(s,a,\lambda)$, both applied with respect to the (first) variable s and the (second) parameter a. We shall show that complete asymptotic expansions exist for these objects when the 'pivotal parameter' z of the transforms tends to both 0 and ∞ through the sector $|\arg z|<\pi$ (Theorems 1–8). It is further shown that our main formulae can be applied to deduce certain asymptotic expansions for the weighted mean values of $\phi^{*}(s,a,\lambda)$ through arbitrary vertical half-lines in the s-plane (Corollaries 2.1 and 4.1), as well as to derive several variants of the power series and asymptotic series for Euler's gamma and psi functions (Corollaries 8.1–8.8).

1. INTRODUCTION

Throughout the article, s is a complex variable, z a complex parameter, a and λ real parameters with $a>0$, and the notation $e(z)=e^{2\pi iz}$ is frequently used. The Lerch zeta-function $\phi(s,a,\lambda)$ is defined by the Dirichlet series

$$\phi(s,a,\lambda)=\sum_{l=0}^{\infty}e(\lambda l)(a+l)^{-s}\qquad(\operatorname{Re}s>1),\tag{1.1}$$

and its meromorphic continuation over the whole s-plane; this reduces if $\lambda\in\mathbb{Z}$ to the Hurwitz zeta-function $\zeta(s,a)$, and so $\zeta(s,1)=\zeta(s)$ is the Riemann zeta-function. The domain of the parameter a in (1.1) can be extended to the whole sector $|\arg z|<\pi$ through the procedure in [10], where it was shown for $\phi(s,a+z,\lambda)$ (with $a>0$) that the unified treatment of its power series expansion (in the disk $|z|<a$), and of its asymptotic series expansion (as $z\to\infty$ through the sector $|\arg z|<\pi$) is possible by means of Mellin-Barnes type integrals.

Let $\Gamma(s)$ denote the gamma function, α and β be complex numbers with $\operatorname{Re}\alpha>0$ and $\operatorname{Re}\beta>0$, $f(z)$ a function holomorphic in the sector $|\arg z|<\pi$, and write $X_{+}=$

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$\max(0, X)$ for any $X \in \mathbb{R}$. We introduce here the Laplace-Mellin and Riemann-Liouville (or Erdélyi-Köber) transforms of $f(z)$, in the forms

$$\mathcal{LM}^{\alpha}_{z;\tau} f(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} f(z\tau)\tau^{\alpha-1} e^{-\tau} d\tau \tag{1.2}$$

and

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau} f(\tau) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} f(z\tau)\tau^{\alpha-1}(1-\tau)_+^{\beta-1} d\tau \tag{1.3}$$

with the normalization gamma multiples, provided that the integrals converge; the factor $\tau^{\alpha-1}$ is inserted to secure the convergence of the integrals as $\tau \to 0^+$, while $e^{-\tau}$ and $(1-\tau)_+^{\beta-1}$ have effects to extract the portions of $f(\tau)$ corresponding to $\tau = O(z)$. Let $\delta(\lambda)$ denote the symbol which equals 0 or 1 according to $\lambda \notin \mathbb{Z}$ or $\lambda \in \mathbb{Z}$. We further introduce a slight modification $\phi^*(s, a, \lambda)$ of $\phi(s, z, \lambda)$, defined by

$$\phi^*(s, z, \lambda) = \phi(s, z, \lambda) - \frac{\delta(\lambda) z^{1-s}}{s-1} = \begin{cases} \zeta(s, z) - \dfrac{z^{1-s}}{s-1} & \text{if } \lambda \in \mathbb{Z}, \\ \phi(s, z, \lambda) & \text{otherwise,} \end{cases} \tag{1.4}$$

in which the only (possible) singularity at $s = 1$ can be removed. Let $f^{(m)}(s)$ for any entire function $f(s)$ denote its mth derivative if $m = 0, 1, 2, \ldots$, and further the nth primitive if $m = -n$ with $n = 1, 2, \ldots$, defined inductively by

$$f^{(-n)}(s) = \int_{s+\infty}^{s} f^{(-n+1)}(w) dw = -\int_0^{0+\infty} f^{(-n+1)}(s+z) dz \qquad (n = 1, 2, \ldots), \tag{1.5}$$

provided that the integral converges, where the path of integration is the horizontal line segment.

It is the principal aim of the present article to treat asymptotic aspects of the Laplace-Mellin and Riemann-Liouville transforms of the (modified) Lerch zeta-function, given by

$$\mathcal{LM}^{\alpha}_{z;\tau} (\phi^*)^{(m)}(s+\tau, a, \lambda) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} (\phi^*)^{(m)}(s+z\tau, a, \lambda)\tau^{\alpha-1} e^{-\tau} d\tau, \tag{1.6}$$

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau} (\phi^*)^{(m)}(s+\tau, a, \lambda) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} (\phi^*)^{(m)}(s+z\tau, a, \lambda)\tau^{\alpha-1}(1-\tau)_+^{\beta-1} d\tau, \tag{1.7}$$

for any $m \in \mathbb{Z}$, and

$$\mathcal{LM}^{\alpha}_{z;\tau} \phi(s, a+\tau, \lambda) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \phi(s, a+z\tau, \lambda)\tau^{\alpha-1} e^{-\tau} d\tau, \tag{1.8}$$

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau} \phi(s, a+\tau, \lambda) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} \phi(s, a+z\tau, \lambda)\tau^{\alpha-1}(1-\tau)_+^{\beta-1} d\tau, \tag{1.9}$$

where the conditions $a > 1$ and $|\arg z| \leq \pi/2$ are required in (1.6) for convergence of the integral, while $a > 0$ and $|\arg z| < \pi/2$ in (1.8). We shall present here that complete asymptotic expansions exist for (1.6)–(1.9) when both $z \to 0$ and $z \to \infty$ through the sector $|\arg z| < \pi$.

We give here a brief overview of history of research relevant to asymptotic aspects of the integral transforms of zeta-functions.

The study of Laplace transforms for (the mean square of) $\zeta(s)$ seems to be initiated by Hardy-Littlewood [5], who obtained the asymptotic relation

$$\mathcal{L}_{1/2}(\delta)=\int_0^\infty\left|\zeta\left(\frac{1}{2}+it\right)\right|^2 e^{-\delta t}dt\sim\frac{1}{\delta}\log\frac{1}{\delta}\qquad(\text{as }\delta\to 0^+),$$

say, in connection with the research of asymptotic aspects of the (upper-truncated) mean square of $\zeta(s)$, in the form $\int_0^T|\zeta(1/2+it)|^2dt$ (as $T\to+\infty$). Wilton [16] then refined the result above to

$$\mathcal{L}_{1/2}(\delta)=\frac{1}{\delta}\log\frac{1}{\delta}-\frac{\log 2\pi-\gamma_0}{\delta}+O\left(\frac{1}{\sqrt{\delta}}\log^{3/2}\frac{1}{\delta}\right)\qquad(\text{as }\delta\to 0^+),$$

where γ_0 is the 0th Euler-Stieljes constant (cf. [4, p.34, 1.12(17)]); the last O-term was in fact replaced by a complete asymptotic expansion by Köber [14], who showed, for any integer $N\geq 0$,

$$\mathcal{L}_{1/2}(\delta)=\frac{1}{\delta}\log\frac{1}{\delta}-\frac{\log 2\pi-\gamma_0}{\delta}+a_0+\sum_{n=1}^{N}\delta^n\left(a_n+b_n\log\frac{1}{\delta}+c_n\log^2\frac{1}{\delta}\right)$$
$$+O\left(\delta^{N+1}\log^2\frac{1}{\delta}\right)\qquad(\text{as }\delta\to 0^+),$$

where a_0, a_n, b_n and c_n ($n=1,2,\ldots$) are some constants. It was finally succeeded, through rather more elementary arguments, by Atkinson [2] (among other things) in dropping the terms with $\log^2(1/\delta)$ in the asymptotic series above (i.e. $c_n=0$), and in improving the error term to $O\{\delta^{N+1}\log(1/\delta)\}$.

In the mean time, a more general Laplace transform

$$\mathcal{L}_\rho(s)=\int_0^\infty|\zeta(\rho+ix)|^2e^{-sx}dx$$

was treated in the late 1990's by Jutila [8], who made a detailed study of $\mathcal{L}_\rho(s)$ especially on the critical line $\rho=1/2$, while obtaining its asymptotic formula as $s\to 0$ through the sector $|\arg s|<\pi/2$, and applied it to re-derive the classical (so-called) Atkinson's formula for the error term of the (upper-truncated) mean square of $\zeta(s)$ (cf. [3]). A further study of $\mathcal{L}_\rho(s)$ has been carried out by Kačinskaitė-Laurinčikas [9]. On the other hand, the (lower-truncated) Mellin transform

$$\mathcal{M}_{k,\rho}(s)=\int_1^\infty|\zeta(\rho+ix)|^{2k}x^{-s}dx\qquad(k=1,2,\ldots),$$

was explored by Ivić-Jutila-Motohashi [7], who applied it to investigate the higher power moments (and in particular the eighth power moment) of $\zeta(s)$. A research subsequent to [7] was due to Ivić [6], while Laurinčikas [15] made a detailed study of the case $k=1$ (i.e. the mean square case) of $\mathcal{M}_{k,\rho}(s)$.

Next let a, b, μ and ν be arbitrary real parameters, and $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ denote the generalized Epstein zeta-function defined for $\operatorname{Im}z>0$ by

(1.10)
$$\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)=\sum_{m,n=-\infty}^{\infty}{}' e((a+m)\mu+(b+n)\nu)$$
$$\times|a+m+(b+n)z|^{-2s}\qquad(\operatorname{Re}s>1),$$

and its meromorphic continuation over the whole s-plane, where the (possibly emerging) singular term 0^{-2s} is to be excluded; the particular case $(a,b)\in\mathbb{Z}^2$ and $(\mu,\nu)=(0,0)$

reduces to the classical Epstein zeta-function $\zeta_{\mathbb{Z}^2}(s;z)$. The author [11] has shown that complete asymptotic expansions exist for $\zeta_{\mathbb{Z}^2}(s;z)$ when $\operatorname{Im} z = y \to +\infty$, and also for the Laplace-Mellin transform $\mathcal{LM}^{\alpha}_{Y;y}\zeta_{\mathbb{Z}^2}(s;x+iy)$ when $Y \to +\infty$. The method developed in [11] could be extended to show in (the subsequent paper) [12] that similar expansions exist further for $\psi_{\mathbb{Z}^2}(s;a,b;\mu,\nu;z)$ when $y \to +\infty$, as well as for the Riemann-Liouville transform $\mathcal{RL}^{\alpha,\beta}_{Y;y}\zeta_{\mathbb{Z}^2}(s;x+iy)$ when $Y \to +\infty$.

The present article is organized as follows. Various complete asymptotic expansions, together with their applications, for the transforms (1.6) and (1.7) are presented in the next section, while those for (1.8) and (1.9) are given in Section 3. The final section is devoted to stating several applications of our results to Euler's gamma and psi functions.

2. Statement of results: the first variable

We first introduce the Riemann-Liouville type operators with the initial point at ∞, defined for any $(r,s) \in \mathbb{C}^2$ by

$$\mathcal{I}^{r}_{\infty,s}f(s) = \frac{1}{\Gamma(r)\{e(r)-1\}}\int_{\infty}^{(0+)} f(s+z)z^{r-1}dz, \tag{2.1}$$

provided that the integral converges, where the path of integration is a contour which starts from ∞, proceeds along the real axis to a small $\varepsilon > 0$, encircles the origin counter-clockwise, and returns to ∞ along the real axis; $\arg z$ varies from 0 to 2π along the contour.

The auxiliary zeta-function $\phi^*_r(s,a,\lambda)$ is defined for any $(r,s) \in \mathbb{C}^2$, for any real $a > 1$ and for any $\lambda \in \mathbb{R}$ by

$$\phi^*_r(s,a,\lambda) = \mathcal{I}^{r}_{\infty,s}\phi^*(s,a,\lambda), \tag{2.2}$$

which is crucial in describing the assertions on (1.6) and (1.7). We further let $(s)_n = \Gamma(s+n)/\Gamma(s)$ for any integer n denote the shifted factorial of s, and write

$$\Gamma\binom{\alpha_1,\ldots,\alpha_m}{\beta_1,\ldots,\beta_n} = \frac{\prod_{h=1}^{m}\Gamma(\alpha_h)}{\prod_{k=1}^{n}\Gamma(\beta_k)}$$

for complex numbers α_h and β_k $(h = 1,\ldots,m;\ k = 1,\ldots,n)$.

We now state our results on the Laplace-Mellin transform (1.6) of $\phi(s,a,\lambda)$ with respect to the first variable s.

Theorem 1. *Let α and s be any complex numbers with $\operatorname{Re}\alpha > 0$, a and λ any real parameters with $a > 1$, and m any integer. Then for any integer $N \geq 0$ the formula*

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = \sum_{n=0}^{N-1}\frac{(-1)^{n+m}(\alpha)_n}{n!}\phi^*_{-n-m}(s,a,\lambda)z^n + R^{+}_{\alpha,m,N}(s,a,\lambda;z) \tag{2.3}$$

holds in the sector $|\arg z| < \pi$. Here $R^{+}_{\alpha,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{+}_{\alpha,m,N}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^N\} \tag{2.4}$$

as $z \to 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on α, a, $\operatorname{Re} s$, m, N and δ.

Theorem 2. *Let α, s, a, λ and m be as in Theorem 1. Then for any integer $N \geq 0$ the formula*

$$\mathcal{LM}^{\alpha}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = \sum_{n=0}^{N-1} \frac{(-1)^{n+m}(\alpha)_n}{n!} \phi^*_{\alpha+n-m}(s,a,\lambda) z^{-\alpha-n} + R^-_{\alpha,m,N}(s,a,\lambda;z) \tag{2.5}$$

holds in the sector $|\arg z| < \pi$. Here $R^-_{\alpha,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^-_{\alpha,m,N}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^{-\operatorname{Re}\alpha-N}\} \tag{2.6}$$

as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on α, a, $\operatorname{Re} s$, m, N and δ.

We write $\operatorname{sgn} X = X/|X|$ for any real $X \neq 0$. Then the case $(s,z) = (\sigma, it)$ with $\sigma, t \in \mathbb{R}$ of Theorem 2 asserts the following result.

Corollary 2.1. *Let s, a, α, λ and m be as in Theorem 1, and σ any real number. Then for any integer $N \geq 0$ the asymptotic expansion*

$$\mathcal{LM}^{\alpha}_{t;\tau}(\phi^*)^{(m)}(\sigma+i\tau,a,\lambda) = \sum_{n=0}^{N-1} \frac{(-1)^{n+m}(\alpha)_n}{n!} \phi^*_{\alpha+n-m}(\sigma,a,\lambda)(e^{\pi i \operatorname{sgn} t/2}|t|)^{-\alpha-n} + O(|t|^{-\operatorname{Re}\alpha-N}) \tag{2.7}$$

holds as $t \to \pm\infty$, where the constant implied in the O-symbol depends at most on α, σ, a, m and N.

We proceed to state our results on the Riemann-Liouville transform (1.7) of $\phi(s,a,\lambda)$ with respect to the first variable s.

Theorem 3. *Let α, β and s be any complex numbers with $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$, a and λ any real parameters with $a > 1$, and m any integer. Then for any integer $N \geq 0$ the formula*

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}(\phi^*)^{(m)}(s+\tau,a,\lambda) = \sum_{n=0}^{N-1} \frac{(-1)^{n+m}(\alpha)_n}{(\alpha+\beta)_n n!} \phi^*_{-n-m}(s,a,\lambda) z^{n} + R^+_{\alpha,\beta,m,N}(s,a,\lambda;z) \tag{2.8}$$

holds in the sector $|\arg z| < \pi$. Here $R^+_{\alpha,\beta,m,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^+_{\alpha,\beta,m,N}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^{N}\} \tag{2.9}$$

as $z \to 0$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on α, β, a, $\operatorname{Re} s$, m, N and δ.

We write $\varepsilon(z) = \operatorname{sgn}(\arg z)$ for any complex z in the sectors $0 < |\arg z| < \pi$.

Theorem 4. *Let α, β, s, a, λ and m be as in Theorem 4. The for any integers $N_j \geq 0$ $(j = 1, 2)$ the formula*

$$\begin{aligned}\mathcal{RL}_{z;\tau}^{\alpha,\beta}(\phi^*)^{(m)}(s+\tau,a,\lambda) \\ &= \Gamma\binom{\alpha+\beta}{\beta} e^{-\varepsilon(z)\pi i\alpha}\left\{\sum_{n=0}^{N_1-1}\frac{(-1)^{n+m}(\alpha)_n(1-\beta)_n}{n!}\right. \\ &\quad\left.\times\phi^*_{\alpha+n-m}(s,a,\lambda)(e^{-\varepsilon(z)\pi i}z)^{-\alpha-n} + R^-_{1,\alpha,\beta,m,N_1}(s,a,\lambda;z)\right\} \\ &\quad + \Gamma\binom{\alpha+\beta}{\alpha} e^{\varepsilon(z)\pi i\beta}\left\{\sum_{n=0}^{N_2-1}\frac{(-1)^{n+m}(\beta)_n(1-\alpha)_n}{n!}\right. \\ &\quad\left.\times\phi^*_{\beta+n-m}(s+z,a,\lambda)z^{-\beta-n} + R^-_{2,\alpha,\beta,m,N_2}(s,a,\lambda;z)\right\}\end{aligned} \tag{2.10}$$

holds in the sectors $0 < |\arg z| < \pi$. Here $R^-_{j,\alpha,\beta,m,N_j}(s,a,\lambda;z)$ $(j = 1, 2)$ are the remainder terms expressed by certain Mellin-Barnes type integrals, and satisfy the estimates

$$R^-_{1,\alpha,\beta,m,N_1}(s,a,\lambda;z) = O\{(|\operatorname{Im} s|+1)^{\max(0,\lfloor 2-\operatorname{Re} s\rfloor)}|z|^{-\operatorname{Re}\alpha-N_1}\} \tag{2.11}$$

and

$$R^-_{2,\alpha,\beta,m,N_2}(s,a,\lambda;z) = O\{(|\operatorname{Im}(s+z)|+1)^{\max(0,\lfloor 2-\operatorname{Re}(s+z)\rfloor)}|z|^{-\operatorname{Re}\beta-N_2}\} \tag{2.12}$$

both as $z \to \infty$ through $\delta \leq |\arg z| \leq \pi - \delta$ with any small $\delta > 0$. Here the constant implied in the O-symbol in (2.11) depends at most on α, β, a, $\operatorname{Re} s$, m, N_j $(j = 1, 2)$ and δ, while that in (2.12) at most on α, β, a, $\operatorname{Re} s$, $\operatorname{Re} z$, m, N_j $(j = 1, 2)$ and δ.

The case $(s, z) = (\sigma, it)$ with $\sigma, t \in \mathbb{R}$ of Theorem 4 above asserts the following result.

Corollary 4.1. *Let α, β, a, λ be as in Theorem 2 and σ any real number. Then for any integers $N_j \geq 0$ $(j = 1, 2)$ the asymptotic expansion*

$$\begin{aligned}\mathcal{RL}_{t;\tau}^{\alpha,\beta}(\phi^*)^{(m)}(\sigma+i\tau,a,\lambda) \\ &= \Gamma\binom{\alpha+\beta}{\beta} e^{-\pi i\alpha\operatorname{sgn} t}\left\{\sum_{n=0}^{N_1-1}\frac{(-1)^{n+m}(\alpha)_n(1-\beta)_n}{n!}\right. \\ &\quad\left.\times\phi^*_{\alpha+n-m}(\sigma,a,\lambda)(e^{-\pi i\operatorname{sgn} t/2}|t|)^{-\alpha-n} + O(|t|^{-\operatorname{Re}\alpha-N_1})\right\} \\ &\quad + \Gamma\binom{\alpha+\beta}{\alpha} e^{\pi i\beta\operatorname{sgn} t}\left\{\sum_{n=0}^{N_2-1}\frac{(-1)^{n+m}(\beta)_n(1-\alpha)_n}{n!}\right. \\ &\quad\left.\times\phi^*_{\beta+n-m}(\sigma+it,a,\lambda)(e^{\pi i\operatorname{sgn} t/2}|t|)^{-\beta-n} + O(|t|^{\max(0,\lfloor 2-\sigma\rfloor)-\operatorname{Re}\beta-N_2})\right\}\end{aligned} \tag{2.13}$$

holds as $t \to \pm\infty$, where the constants implied in the O-symbols depend at most on α, β, σ, m and N_j $(j = 1, 2)$.

3. Statement of results: the second parameter

We next state our results on the Laplace-Mellin transform (1.8) of $\phi(s,z,\lambda)$ with respect to the second parameter z.

Theorem 5. *Let α be any complex numbers with $\operatorname{Re}\alpha>0$, a and λ any real parameters with $a>0$. Then for any integer $N\geq 0$, in the region $\operatorname{Re}s>1-N$ except at $s=1$ the formula*

$$\mathcal{LM}^{\alpha}_{z;\tau}\phi(s,a+\tau,\lambda)=\sum_{n=0}^{N-1}\frac{(-1)^n(s)_n(\alpha)_n}{n!}\phi(s+n,a,\lambda)z^n + R^{+}_{\alpha,N}(s,a,\lambda;z) \tag{3.1}$$

holds for $|\arg z|<\pi$. Here $R^{+}_{\alpha,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{+}_{\alpha,N}(s,a,\lambda;z)=O(|z|^N) \tag{3.2}$$

as $z\to 0$ through $|\arg z|\leq\pi-\delta$ with any small $\delta>0$, where the constant implied in the O-symbol depends at most on α, a, s, N and δ.

Apostol [1] introduced the generalized Bernoulli polynomials $B_n(x,y)$ $(n=0,1,\ldots)$ defined for any complex x and y by the Taylor series expansion

$$\frac{ze^{xz}}{ye^z-1}=\sum_{n=0}^{\infty}\frac{B_n(x,y)}{n!}z^n$$

centered at $z=0$. The function $B_k(x,y)$, coincides with the usual Bernoulli polynomial $B_k(x)$ if $y=1$, is a polynomial in x of degree k with coefficients in $\mathbb{Q}(y)$.

Theorem 6. *Let α, s, a, λ be as in Theorem 5. Then for any integer $N\geq 0$, in the region $\operatorname{Re}s>-N$ except at $s\in\{\alpha+l\mid l\in\mathbb{Z}\}$, upon setting $N'=N-\lfloor\operatorname{Re}(\alpha-s)\rfloor$, the formula*

$$\begin{aligned}\mathcal{LM}^{\alpha}_{z;\tau}&\phi(s,a+\tau,\lambda)\\ &=\sum_{n=-1}^{N-1}\frac{(-1)^{n+1}(s)_n}{(n+1)!}\Gamma\binom{\alpha-s-n}{\alpha}B_{n+1}(a,e(\lambda))z^{-s-n}\\ &\quad+\sum_{n=0}^{N'-1}\frac{(-1)^n(\alpha)_n}{n!}\Gamma\binom{s-\alpha-n}{s}\phi(s-\alpha-n,a,\lambda)z^{-\alpha-n}\\ &\quad+R^{-}_{\alpha,N}(s,a,\lambda;z)\end{aligned} \tag{3.3}$$

holds for $|\arg z|<\pi$. Here $R^{-}_{\alpha,N}(s,a,\lambda;z)$ is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate

$$R^{-}_{\alpha,N}(s,a,\lambda;z)=O(|z|^{-\operatorname{Re}s-N}) \tag{3.4}$$

in $|\arg z|\leq\pi-\delta$ for any small $\delta>0$, where the constant implied in the O-symbol depends at most on α, a, s, N and δ.

Let $\psi(s)$ denote Euler's psi function given by (4.1) below, and $\gamma_0(a,e(\lambda))$ the 0th generalized Euler-Stieltjes constant defined by the Laurent series expansion

$$\phi(s,a,\lambda)=\frac{B_0(a,e(\lambda))}{s-1}+\gamma_0(a,e(\lambda))+O(s-1) \tag{3.5}$$

centered at $s = 1$. We further define for any integers l and n with $n \geq -1$ the coefficients $C_{\alpha,l,n}(s, e(\lambda))$ by

$$(3.6) \qquad C_{\alpha,l,n}(a, e(\lambda)) = \begin{cases} B_0(a, e(\lambda))\{\psi(1) + \psi(l) - \psi(\alpha + l - 1)\} & \\ \quad +\gamma_0(a, e(\lambda)) & \text{if } n = -1, \\ B_{n+1}(a, e(\lambda))\{\psi(n+1) + \psi(n+l+1) & \\ \quad -\psi(\alpha + l + n)\} - (n+1)\phi'(-n, a, \lambda) & \text{if } n \geq 0, \end{cases}$$

where the prime on ϕ signifies hereafter that $\partial/\partial s$. It is in fact possible to transfer from the expansion in Theorem 6 to those for the excluded cases by taking the limits $s \to \alpha + l$ for any $l \in \mathbb{Z}$.

Corollary 6.1. *For any integer $N \geq 0$ the following asymptotic expansions hold as $z \to \infty$ through the sector $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$:*

i) *when $s = \alpha + l$ $(l = 1, 2, \ldots)$,*

$$(3.7) \qquad \begin{aligned} &\mathcal{LM}^{\alpha}_{z;\tau}\phi(\alpha + l, a + \tau, \lambda) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{l-2} \frac{(-1)^n(\alpha + l)_{n-l}(l - n - 1)!}{n!} \phi(l - n, a, \lambda) z^{-\alpha-n} \\ &+ \frac{(-1)^{l-1}}{\Gamma(\alpha)} \sum_{n=-1}^{N-1} \frac{(\alpha + l)_n}{(n+1)!(n+l)!} z^{-\alpha-l-n} \\ &\times \{B_{n+1}(a, e(\lambda)) \log z + C_{\alpha,l,n}(a, e(\lambda))\} + O(|z|^{-\operatorname{Re}\alpha - l - N}); \end{aligned}$$

ii) *when $s = \alpha - m$ $(m = 0, 1, 2, \ldots)$,*

$$(3.8) \qquad \begin{aligned} &\mathcal{LM}^{\alpha}_{z;\tau}\phi(\alpha - m, a + \tau, \lambda) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{n=-1}^{m-1} \frac{(-1)^{n+1}(\alpha - m)_n(m - n - 1)!}{(n+1)!} B_{n+1}(a, e(\lambda)) z^{-\alpha+m-n} \\ &+ \frac{(-1)^{m+1}}{\Gamma(\alpha)} \sum_{n=m}^{N-1} \frac{(\alpha - m)_n}{(n+1)!(n-m)!} z^{-\alpha+m-n} \\ &\times \{B_{n+1}(a, e(\lambda)) \log z + C_{\alpha,-m,n}(a, e(\lambda))\} + O(|z|^{-\operatorname{Re}\alpha + m - N}), \end{aligned}$$

Here the constants implied in the O-symbols depend at most on α, a, l, m, N and δ.

We proceed to state our results on the Riemann-Liouville transform (1.9) of $\phi(s, z, \lambda)$ with respect to the second parameter z.

Theorem 7. *Let α, β and s be complex numbers with $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$, and a, λ real parameters with $a > 0$. Then for any integer $N \geq 0$, in the region $\operatorname{Re} s > 1 - N$ except at $s = 1$, the formula*

$$(3.9) \qquad \begin{aligned} \mathcal{RL}^{\alpha,\beta}_{z;\tau}\phi(s, a + \tau, \lambda) = &\sum_{n=0}^{N-1} \frac{(-1)^n(s)_n(\alpha)_n}{(\alpha + \beta)_n n!} \phi(s + n, a, \lambda) z^n \\ &+ R^{+}_{\alpha,\beta,N}(s, a, \lambda; z) \end{aligned}$$

holds for $|\arg z| < \pi$. *Here* $R^{+}_{\alpha,\beta,N}(s,a,\lambda;z)$ *is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate*

$$R^{+}_{\alpha,\beta,N}(s,a,\lambda;z) = O(|z|^{N}) \tag{3.10}$$

as $z \to 0$ *through* $|\arg z| \le \pi - \delta$ *with any small* $\delta > 0$, *where the constant implied in the* O*-symbol depends at most on* α, a, s, N *and* δ.

The limiting case $N \to +\infty$ in Theorem 7 asserts the following result.

Corollary 7.1. *Let* α, β, a, λ *be as in Theorem 7. Then we have in the disk* $|z| < a$ *the power series expansion, except at* $s = 1$,

$$\mathcal{RL}^{\alpha}_{z;\tau}\phi(s,a+\tau,\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n (s)_n (\alpha)_n}{(\alpha+\beta)_n n!}\phi(s+n,a,\lambda)z^n. \tag{3.11}$$

Theorem 8. *Let* α, β, s, a, λ *be as in Theorem 7. Then for any integer* $N \ge 0$, *in the region* $\operatorname{Re} s > -N$ *except at* $s \in \{\alpha + l \mid l \in \mathbb{Z}\}$, *upon setting* $N' = N - \lfloor \operatorname{Re}(\alpha - s) \rfloor$, *the formula*

$$\begin{aligned} \mathcal{RL}^{\alpha,\beta}_{z;\tau}&\phi(s,a+\tau,\lambda) \\ &= \sum_{n=-1}^{N-1} \frac{(-1)^{n+1}(s)_n}{(n+1)!}\Gamma\binom{\alpha-s-n,\alpha+\beta}{\alpha,\alpha+\beta-s-n} B_{n+1}(a,e(\lambda))z^{-s-n} \\ &\quad + \sum_{n=0}^{N'-1} \frac{(-1)^n(\alpha)_n}{n!}\Gamma\binom{s-\alpha-n,\alpha+\beta}{s,\beta-n}\phi(s-\alpha-n,a,\lambda)z^{-\alpha-n} \\ &\quad + R^{-}_{\alpha,\beta,N}(s,a,\lambda;z) \end{aligned} \tag{3.12}$$

holds for $|\arg z| < \pi$. *Here* $R^{-}_{\alpha,\beta,N}(s,a,\lambda;z)$ *is the remainder term expressed by a certain Mellin-Barnes type integral, and satisfies the estimate*

$$R^{-}_{\alpha,\beta,N}(s,a,\lambda;z) = O(|z|^{-\operatorname{Re} s - N}) \tag{3.13}$$

as $z \to \infty$ *through* $|\arg z| \le \pi - \delta$ *with any small* $\delta > 0$, *where the constant implied in the* O*-symbol depends at most on* α, a, N, s *and* δ.

We next define for any integers l and n with $n \ge -1$ the coefficients $C_{\alpha,\beta,l,n}(a,e(\lambda))$ by

$$\begin{aligned} &C_{\alpha,\beta,l,n}(a,e(\lambda)) \\ &= \begin{cases} B_0(a,e(\lambda))\{\psi(1)+\psi(l)-\psi(\alpha+l-1) \\ \quad -\psi(\beta-l+1)\} + \gamma_0(a,e(\lambda)) & \text{if } n=-1, \\ B_{n+1}(a,e(\lambda))\{\psi(n+1)+\psi(n+l+1)-\psi(\alpha+l+n) \\ \quad -\psi(\beta-l-n)\} - (n+1)\phi'(-n,a,\lambda) & \text{if } n \ge 0. \end{cases} \end{aligned} \tag{3.14}$$

It is in fact possible to transfer from the expansion in Theorem 8 to those for the excluded cases by taking the limits $s \to \alpha + l$ for any $l \in \mathbb{Z}$.

Corollary 8.1. *For any integer* $N \ge 0$ *the following asymptotic expansions hold as* $z \to \infty$ *through the sector* $|\arg z| \le \pi - \delta$ *with any small* $\delta > 0$:

i) *when* $s=\alpha+l$ $(l=1,2,\ldots)$,

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}\phi(\alpha+l,a+\tau,\lambda) \tag{3.15}$$

$$=\Gamma\binom{\alpha+\beta}{\alpha}\sum_{n=0}^{l-2}\frac{(-1)^n(\alpha+l)_{n-l}(l-n-1)!}{n!\Gamma(\beta-n)}\phi(l-n,a,\lambda)z^{-\alpha-n}$$

$$+(-1)^{l-1}\Gamma\binom{\alpha+\beta}{\alpha}\sum_{n=-1}^{N-1}\frac{(\alpha+l)_n}{(n+1)!(n+l)!\Gamma(\beta-l-n)}z^{-\alpha-l-n}$$

$$\times\left\{B_{n+1}(a,e(\lambda))\log z+C_{\alpha,\beta,l,n}(a,e(\lambda))\right\}+O(|z|^{-\operatorname{Re}\alpha-l-N});$$

ii) *when* $s=\alpha-m$ $(m=0,1,2,\ldots)$,

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}\phi(\alpha-m,a+\tau,\lambda) \tag{3.16}$$

$$=\Gamma\binom{\alpha+\beta}{\alpha}\sum_{n=-1}^{m-1}\frac{(-1)^{n+1}(\alpha-m)_n(m-n-1)!}{(n+1)!\Gamma(\beta+m-n)}B_{n+1}(a,e(\lambda))z^{-\alpha+m-n}$$

$$+(-1)^{m+1}\Gamma\binom{\alpha+\beta}{\alpha}\sum_{n=m}^{N-1}\frac{(\alpha-m)_n}{(n+1)!(n-m)!\Gamma(\beta+m-n)}z^{-\alpha+m-n}$$

$$\times\left\{B_{n+1}(a,e(\lambda))\log z+C_{\alpha,\beta,-m,n}(a,e(\lambda))\right\}+O(|z|^{-\operatorname{Re}\alpha+m-N}).$$

Here the constants implied in the O-symbols depend at most on α, β, a, l, m, N *and* δ.

4. Applications to Euler's gamma and psi functions

It is known that

$$\lim_{s\to1}\left\{\zeta(s,z)-\frac{1}{s-1}\right\}=-\psi(z)=-\frac{\Gamma'}{\Gamma}(z)\qquad(|\arg z|<\pi) \tag{4.1}$$

(cf. [4, p.26,1.10(9)]). Then both the limiting cases $s\to1$ of Theorems 5 and 7 (when $\lambda\in\mathbb{Z}$) assert the following results.

Corollary 8.2. *Let* α, a *be as in Theorem 5. Then for any integer* $N\geq1$, *the asymptotic expansion*

$$\mathcal{LM}^{\alpha}_{z;\tau}\psi(a+\tau)=\psi(a)-\sum_{n=1}^{N-1}(-1)^n(\alpha)_n\zeta(1+n,a)z^n+O(|z|^N) \tag{4.2}$$

holds as $z\to0$ *through* $|\arg z|\leq\pi-\delta$ *with any small* $\delta>0$, *where the constant implied in the O-symbol depends at most on* α, N *and* δ.

Corollary 8.3. *Let* α, β *and* a *be as in Theorem 7. Then for any* $N\geq0$, *the asymptotic expansion*

$$\mathcal{RL}^{\alpha,\beta}_{z;\tau}\psi(a+\tau)=\psi(a)-\sum_{n=1}^{N-1}\frac{(-1)^n(\alpha)_n}{(\alpha+\beta)_n}\zeta(1+n,a)z^n+O(|z|^N) \tag{4.3}$$

holds as $z\to0$ *through* $|\arg z|\leq\pi-\delta$ *with any small* $\delta>0$, *where the constant implied in the O-symbol depends at most on* α, β, N *and* δ.

The limiting case $N\to+\infty$ of Corollary 8.3 further asserts the following result.

Corollary 8.4. *Let α, β and a be as in Theorem 8. Then we have in the disk $|z| < a$ the power series expansion*

$$\mathcal{RL}_{z;\tau}^{\alpha,\beta}\psi(a+\tau) = \psi(a) - \sum_{n=1}^{\infty}\frac{(-1)^n(\alpha)_n}{(\alpha+\beta)_n}\zeta(1+n,a)z^n. \tag{4.4}$$

We note that the formulae (4.2)–(4.4) are variants of the classical power series expansion

$$\psi(a+z) = \psi(a) - \sum_{n=1}^{\infty}(-1)^n\zeta(1+n,a)z^n \qquad (|z| < a)$$

(cf. [4, p.45, 1.17(5)] in which the case $a = 1$ is stated).

It is further known that

$$\frac{\partial}{\partial s}\zeta(s,z)\bigg|_{s=0} = \log\left\{\frac{\Gamma(z)}{\sqrt{2\pi}}\right\} \qquad (|\arg z| < \pi) \tag{4.5}$$

(cf. [4, p.26, 1.10(10)]). Then both the limiting cases $s \to 0$ (after differentiation) of Theorems 6 and 8 (when $\lambda \in \mathbb{Z}$) assert the following results.

Corollary 8.5. *Let α and a be as in Theorem 6. Then for any integer $N \geq 1$, except the case $\alpha = m$ $(m = 1, 2, \ldots)$, the asymptotic expansion, upon setting $N' = N - \lfloor \mathrm{Re}\,\alpha \rfloor$,*

$$\begin{aligned} &\mathcal{LM}_{z;\tau}^{\alpha}\log\Gamma(a+\tau) \\ &= \alpha z\{\log z + \psi(\alpha+1) - 1\} + B_1(a)\{\log z + \psi(\alpha)\} + \frac{1}{2}\log 2\pi \\ &+ \sum_{n=1}^{N-1}\frac{(-1)^{n+1}B_{n+1}(a)}{n(n+1)}\Gamma\binom{\alpha-n}{\alpha}z^{-n} \\ &+ \sum_{n=0}^{N'-1}\frac{(-1)^n(\alpha)_n}{n!}\Gamma(-\alpha-n)\zeta(-\alpha-n)z^{-\alpha-n} + O(|z|^{-N}) \end{aligned} \tag{4.6}$$

holds as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on α, N and δ.

Corollary 8.6. *Let α, β and a be as in Theorem 8. Then for any integer $N \geq 1$, except the case $\alpha = m$ $(m = 1, 2, \ldots)$, the asymptotic expansion, upon setting $N' = N - \lfloor \mathrm{Re}\,\alpha \rfloor$,*

$$\begin{aligned} &\mathcal{RL}_{z;\tau}^{\alpha,\beta}\log\Gamma(a+\tau) \\ &= \frac{\alpha z}{\alpha+\beta}\{\log z + \psi(\alpha+1) - \psi(\alpha+\beta+1) - 1\} \\ &+ B_1(a)\{\log z + \psi(\alpha) - \psi(\alpha+\beta)\} + \frac{1}{2}\log 2\pi \\ &+ \sum_{n=1}^{N-1}\frac{(-1)^{n+1}B_{n+1}(a)}{n(n+1)}\Gamma\binom{\alpha-n, \alpha+\beta}{\alpha, \alpha+\beta-n}z^{-n} \\ &+ \sum_{n=0}^{N'-1}\frac{(-1)^n(\alpha)_n}{n!}\Gamma\binom{-\alpha-n, \alpha+\beta}{\beta-n}\zeta(-\alpha-n,a)z^{-\alpha-n} + O(|z|^{-N}) \end{aligned} \tag{4.7}$$

holds as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on α, β, δ and N.

It is possible to transfer from the expansions in Corollaries 8.5 and 8.6 to those for the excluded cases by taking the limits $\alpha \to m$ for any $m = 1, 2, \ldots$.

Corollary 8.7. *Let a be a real parameter with $a > 0$, and $m \geq 1$ an integer. Then for any integer $N \geq 1$ the asymptotic expansion*

$$\begin{aligned}
(4.8)\qquad & \mathcal{LM}^{m}_{z;\tau} \log \Gamma(a+\tau) \\
&= mz\{\log z + \psi(m+1) - 1\} + B_1(a)\{\log z + \psi(m)\} + \frac{1}{2}\log 2\pi \\
&+ \frac{1}{(m-1)!}\sum_{n=1}^{m-1} \frac{(-1)^{n+1}(m-n-1)!}{n(n+1)} B_{n+1}(a) z^{-n} \\
&+ \frac{(-1)^{m-1}}{(m-1)!}\sum_{n=m}^{N-1} \frac{1}{n(n+1)(n-m)!} z^{-n} \\
&\times \left[B_{n+1}(a)\left\{\log z + \psi(n-m+1) + \frac{1}{n}\right\} - (n+1)\zeta'(-n,a)\right] \\
&+ O(|z|^{-N}\log|z|)
\end{aligned}$$

holds as $z \to \infty$ through the sector $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on a, m, N and δ.

Corollary 8.8. *Let $a > 0$ be a real parameter, β any complex number with* $\operatorname{Re}\beta > 0$, *and $m \geq 1$ an integer. Then for any integer $N \geq 1$ the asymptotic expansion*

$$\begin{aligned}
(4.9)\qquad & \mathcal{RL}^{m,\beta}_{z;\tau} \log \Gamma(a+\tau) \\
&= \frac{mz}{\beta+m}\{\log z + \psi(m+1) - \psi(\beta+m+1) - 1\} \\
&+ B_1(a)\{\log z + \psi(m)\} + \frac{1}{2}\log 2\pi \\
&+ \frac{\Gamma(\beta+m)}{(m-1)!}\sum_{n=1}^{m-1} \frac{(-1)^{n+1}(m-n-1)!}{n(n+1)\Gamma(\beta+m-n)} B_{n+1}(a) z^{-n} \\
&+ \frac{(-1)^{m-1}\Gamma(\beta+m)}{(m-1)!}\sum_{n=m}^{N-1} \frac{1}{n(n+1)(n-m)!\Gamma(\beta+m-n)} z^{-n} \\
&\times \Bigg[B_{n+1}(a)\left\{\log z + \psi(n-m+1) + \frac{1}{n} - \psi(\beta+m-n)\right\} \\
&- (n+1)\zeta'(-n,a)\Bigg] + O(|z|^{-N}\log|z|)
\end{aligned}$$

holds as $z \to \infty$ through the sector $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$, where the constant implied in the O-symbol depends at most on a, β, m, N and δ.

We note that the formulae (4.8) and (4.9) are variants of the classical (shifted) Stirling's formula, for any $N \geq 1$,

$$\begin{aligned}
\log \Gamma(a+z) &= z\log z - z + B_1(a)\log z + \frac{1}{2}\log 2\pi \\
&+ \sum_{n=1}^{N-1} \frac{(-1)^{n+1}B_{n+1}(a)}{n(n+1)} z^{-n} + O(|z|^{-N})
\end{aligned}$$

as $z \to \infty$ through $|\arg z| \leq \pi - \delta$ with any small $\delta > 0$ (cf. [4, p.48,1.18(12)]).

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