Transformations of a series in function fields

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1. INTRODUCTION

Let
\[ \eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) \quad (\text{Im}(z) > 0) \]
be the Dedekind \( \eta \)-function. Dedekind [2] described the transformation of \( \log \eta(z) \) under the substitution \( z' = (az + b)/(cz + d) \), \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \). More precisely, he proved that for \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) with \( a \neq 0, c > 0 \),

\[ \log \eta \left( \frac{az + b}{cz + d} \right) = \log \eta(z) + \frac{1}{2} \log \left( \frac{cz + d}{i} \right) + \frac{\pi i (a + d)}{12c} - \pi i D(a, c), \]

where \( D(a, c) \) is the Dedekind sum defined by

\[ D(a, c) = \frac{1}{4c} \sum_{k=1}^{c-1} \cot \left( \frac{\pi ak}{c} \right) \cot \left( \frac{\pi k}{c} \right) \]

for coprime integers \( a \) and \( c > 0 \). We can use (1.1) to prove the so-called reciprocity law given by

\[ D(a, c) + D(c, a) = -\frac{1}{4} + \frac{1}{12} \left( \frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right) \]

for coprime positive integers \( a, c \). Details of the proofs of (1.1) and (1.3) can be found in the book [6].

An analogy exists between number fields and function fields. For example, \( \mathbb{Z} \), \( \mathbb{Q} \), and \( \mathbb{R} \) are analogous to \( A \), \( K = \mathbb{F}_q(T) \), and \( K_{\infty} = \mathbb{F}_q((1/T)) \), respectively. In [1] and [5], we introduced a function field analog \( s(a, c) \) (see Section 2) of \( D(a, c) \), and established its reciprocity law. In this report, we use the Dedekind sum \( s(a, c) \) in function fields to describe the transformation of a certain series under the substitution \( z' = (az + b)/(cz + d) \), \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) \).

As an application, we prove the reciprocity law for \( s(a, c) \).

2. REVIEW OF THE DEDEKIND SUM

Let \( A = \mathbb{F}_q[T] \) and \( K = \mathbb{F}_q(T) \). Let \( K_{\infty} = \mathbb{F}_q((1/T)) \) be the completion of \( K \) at \( \infty = (1/T) \), and let \( C_{\infty} \) be the completion of an algebraic closure of \( K_{\infty} \).
2.1. The Carlitz exponential function. Let $D_0 = 1$, $D_n = [n][n-1]^{q} \cdots [1]^{q^{n-1}}$ for $n > 0$ and $[n] = T^n - T$. Let $e(z)$ be the Carlitz exponential function defined by

$$e(z) = \sum_{n=0}^{\infty} \frac{z^{q^n}}{D_n},$$

which is entire over $C_{\infty}$. By definition, it holds that $de(z)/dz = e'(z) = 1$. The map $e : C_{\infty} \to C_{\infty}$ is $\mathbb{F}_q$-linear and surjective. The kernel $L := \text{Ker}(e)$ is a free $A$-module of rank one. It is easy to see that $e(z)$ is $L$-periodic: $e(z + l) = e(z)$ for $l \in L$. Let $\overline{\pi}$ denote a generator of $L$. The function $e(z)$ can be written as

$$e(z) = z \prod_{0 \neq l \in L} \left( 1 - \frac{z}{l} \right).$$

From this, we have

$$\frac{1}{e(z)} = \frac{e'(z)}{e(z)} = \sum_{l \in L} \frac{1}{l + z}.$$ 

The reader is referred to Goss [4] for additional details of $e(z)$.

2.2. The Dedekind sum. Let $a, c$ be the coprime elements of $A \setminus \{0\}$. The (inhomogeneous) Dedekind sum $s(a, c)$ is defined as

$$s(a, c) = \frac{1}{c} \sum_{0 \neq \mu \in A/cA} e\left( \frac{\overline{\pi} a \mu}{c} \right)^{-1} e\left( \frac{\overline{\pi} \mu}{c} \right)^{-1}. $$

When $c$ is a unit of $A$, $s(a, c)$ is defined to be zero. This is an analog of the classical Dedekind sum $D(a, c)$ defined in (1.2). For any $\epsilon \in \mathbb{F}_q \setminus \{0\}$,

$$(2.1) \quad s(\epsilon a, c) = \epsilon^{-1} s(a, c).$$

By replacing $\mu$ with $\epsilon \mu$ ($\epsilon \in \mathbb{F}_q \setminus \{0\}$) in the definition of $s(a, c)$, we see that $s(a, c) = 0$ if $q > 3$.

3. A SERIES RELATED TO $s(a, c)$

Let $\Omega = C_{\infty} \setminus K_{\infty}$ be the Drinfeld upper half-plane, which is an analog of the classical upper half-plane $H := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$. The group $GL_2(A)$ acts on $\Omega$ by fractional linear transformations

$$\left( \begin{array}{ll} a & b \\ c & d \end{array} \right) z = (az + b)/(cz + d).$$

Let

$$\xi(z) = \sum_{0 \neq a \in A} \frac{1}{ae(\overline{\pi}az)},$$

which is convergent for $z \in \Omega$. This can be written as

$$\xi(z) = \begin{cases} 
0 & \text{if } q > 3, \\
-\sum_{a \in A_+} 1/ae(\overline{\pi}az) & \text{if } q = 3, \\
\sum_{a \in A_+} 1/ae(\overline{\pi}az) & \text{if } q = 2,
\end{cases}$$

where $A_+$ is the set of monic elements in $A$.

In the classical case, it is known that for $\gamma \in SL_2(\mathbb{Z})$ and $z \in H$,

$$\log \eta(\gamma z) - \log \eta(z) = -2\pi i \int_{z}^{\gamma z} G_2(\tau) d\tau,$$
where $G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} ne^{2\pi in\tau}$ is the Eisenstein series of weight 2. We have a similar result for $\xi(z)$. Since $de(z)/dz = 1$, $d\xi(z)/dz = \pi \sum_{a \in A^+} e(\pi az)^{-2}$. As is well known, $g(z) := 1 - (T^q - T) \sum_{a \in A^+} e(\pi az)^{-1-q}$ is a weight $q-1$ modular form for $GL_2(A)$ (see [3], (9.2)). Therefore, we see that for $\gamma \in GL_2(A)$ and $z \in \Omega$,

$$\xi(\gamma z) - \xi(z) = \begin{cases} -\pi \int_z^{\gamma z} \frac{g(\tau)^{-1}}{T^3 - T} d\tau & \text{if } q = 3, \\ -\pi \int_z^{\gamma z} \left( \frac{g(\tau)^{-1}}{T^2 - T} \right)^2 d\tau & \text{if } q = 2. \end{cases}$$

4. Transformation formula for $\xi(z)$

We provide the results for the transformations for $\xi(z)$.

**Proposition 1.** (1) For $\varepsilon \in \mathbb{F}_q \setminus \{0\}$, $\xi(\varepsilon z) = \varepsilon^{-1} \xi(z)$.

(2) For $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(A)$, $\xi(\gamma z) = \det \xi(z)$.

**Theorem 2.** We have

$$\xi(-1/z) = \xi(z) - \frac{\alpha(2)}{\overline{\pi}} (z + \frac{1}{z}) - \frac{\alpha(1)^2}{\overline{\pi}},$$

where $\alpha(n) = \sum_{0 \neq a \in A} a^{-n}$.

We require the following lemma to prove Theorem 4.

**Lemma 3.** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $GL_2(A)$ with $c \neq 0$. Then,

$$\gamma z = \frac{a}{c} - \frac{\det \gamma}{c(cz + d)}.$$

**Theorem 4.** For $\gamma = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \in GL_2(A)$,

(4.1) $\xi(\gamma z) = \det \gamma \left[ \xi(z) - \frac{\alpha(2)}{\pi c} \left( cz + d + \frac{1}{cz + d} \right) \right] + \frac{\alpha(1)^2}{\pi c} + \overline{\pi}s(a, c)$.

The following is the main result.

**Theorem 5.** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. If $a \neq 0$ and $c \neq 0$, then

(4.2) $\xi(\gamma z) = \det \gamma \left[ \xi(z) - \frac{\alpha(2)}{\pi c} \left( cz + d + \frac{1}{cz + d} \right) \right] + \frac{\alpha(1)^2}{\pi c} + \overline{\pi}s(a, c)$.

5. Outline of the proof of Theorem 5

5.1. Case $\gamma \in SL_2(A)$. Let

$$R_1 = \sum_{0 \neq f \in A \atop f \equiv 0 \pmod{c}} \frac{1}{fe(\overline{\pi} f \gamma z)} - \sum_{0 \neq f \in A \atop f \equiv 0 \pmod{c}} \frac{1}{fe(\overline{\pi} f z)},$$

$$R_2 = \sum_{0 \neq f \in A \atop f \equiv 0 \pmod{c}} \frac{1}{fe(\overline{\pi} f \gamma z)} - \sum_{0 \neq f \in A \atop f \equiv 0 \pmod{c}} \frac{1}{fe(\overline{\pi} f z)}.$$
Then, we have $\xi(\gamma z) - \xi(z) = R_1 + R_2$, for which we compute $R_1$ and $R_2$, separately.

We first compute $R_1$. By Lemma 3, $1/e(\pi f \gamma z) = 1/e(-\pi f/c(cz + d))$. Hence,

$$R_1 = \frac{1}{\pi} \sum_{f \not\equiv 0 \pmod{c}} \sum_{g \in A} \frac{c(cz + d)}{f(gc(cz + d) - f)} + \frac{1}{\pi} \sum_{f \not\equiv 0 \pmod{c}} \sum_{g \in A} \frac{1}{f(g - fz + h)}.$$

Setting $f' = f/c$, $R_1$ becomes

$$R_1 = \frac{1}{\pi c} \sum_{0 \not\equiv f' \in A} \sum_{g \in A} \frac{cz + d}{f'(g(cz + d) - f')} + \frac{1}{\pi c} \sum_{0 \not\equiv f' \in A} \sum_{0 \not\equiv g \in A} \frac{1}{f'(g - f'cz + h)}.$$

We set $h = -fd$ and then divide $R_1$ into the part $g = 0$ and the part $g \neq 0$. Then,

$$R_1 = -\frac{\alpha(2)}{\pi c} (cz + d + \frac{1}{cz + d}) + \frac{1}{\pi c} \sum_{0 \not\equiv f' \in A} \sum_{0 \not\equiv g \in A} \frac{cz + d}{f'(g(cz + d) - f')} + \frac{1}{\pi c} \sum_{0 \not\equiv f' \in A} \sum_{0 \not\equiv g \in A} \frac{1}{f'(g - f'(cz + d))}.$$

In the last of the two double summation terms above, we interchange $f'$ and $g$. Then, $R_1$ becomes

$$R_1 = -\frac{\alpha(2)}{\pi c} (cz + d + \frac{1}{cz + d}) + \frac{\alpha(1)^2}{\pi c}.$$

We next compute the two sums in $R_2$. As for the first sum, we have

$$R_3 := \sum_{f \not\equiv 0 \pmod{c}} \frac{1}{f e(\pi f \gamma z)} = \sum_{0 \not\equiv f \not\equiv 0 \pmod{c}} \sum_{f \equiv \mu \pmod{c}} \frac{1}{f e(\pi f \gamma z)}.$$

When $f \equiv \mu \pmod{c}$, $f$ can be written as $f = \mu + ch$ for a certain $h \in A$. Hence, by Lemma 3,

$$\frac{\pi}{e(\pi f \gamma z)} = c \sum_{g \in A} \frac{1}{cg + a \mu - f/cz + d}.$$

Setting $r = cg + a \mu$, $\pi/e(\pi f \gamma z)$ can be written as

$$c(cz + d) \sum_{r \equiv a \mu \pmod{c}} \frac{1}{r(cz + d) - f}.$$

Thus, we have

$$R_3 = \frac{c}{\pi} \sum_{0 \not\equiv \mu \in A} \sum_{f \equiv \mu \pmod{c}} \sum_{g \equiv a \mu \pmod{c}} \frac{cz + d}{f(g(cz + d) - f)}.$$

When $f \equiv \mu \pmod{c}$, using $k := a \mu - \mu$, $f + k \equiv a \mu \pmod{c}$. As for the second summation in $R_2$, noting that $a, c$ are coprime, we obtain

$$R_4 := \sum_{f \not\equiv 0 \pmod{c}} \frac{1}{f e(\pi f z)} = \sum_{0 \not\equiv f \not\equiv 0 \pmod{c}} \sum_{f \equiv \mu \pmod{c}} \frac{1}{(f + k)e(\pi (f + k) z)}.$$
Now, we compute $\overline{\pi}/e(\overline{\pi}(f + k)z)$ as follows.

$$
\frac{\overline{\pi}}{e(\overline{\pi}(f + k)z)} = -\sum_{g \in A} \frac{1}{g - (f + k)z + h},
$$

where $h \in A$ is a fixed element. Because $f \equiv \mu \pmod{c}$,

$$
\frac{a\mu - k - d(f + k)}{c} = -\frac{bc\mu - d(f - \mu)}{c} \in A.
$$

Letting $h = (a\mu - k - d(f + k))/c$, $\overline{\pi}/e(\overline{\pi}(f + k)z)$ becomes

$$
-c \sum_{r \equiv \mu \pmod{c}} \frac{1}{r - k - (f + k)(cz + d)}.
$$

Hence, we have

$$
R_4 = -\frac{c}{\overline{\pi}} \sum_{0 \neq \mu \in A/cA} \sum_{f \equiv \mu \pmod{cA}} \sum_{g \equiv a\mu \pmod{cA}} \frac{1}{(f + k)(g - k - (f + k)(cz + d))}.
$$

Noting that $f \equiv \mu \pmod{c}$ if and only if $f + k \equiv a\mu \pmod{c}$, $R_4$ becomes

$$
-c \sum_{0 \neq \mu \in A/cA} \sum_{f \equiv \mu \pmod{cA}} \sum_{g \equiv a\mu \pmod{cA}} \frac{1}{g(f - g(cz + d))}.
$$

Using $R_3$ and $R_4$, we obtain

$$
R_2 = R_3 - R_4 = \frac{c}{\overline{\pi}} \sum_{0 \neq \mu \in A/cA} \sum_{f \equiv \mu \pmod{cA}} \sum_{g \equiv a\mu \pmod{cA}} \frac{1}{fg}.
$$

As

$$
\sum_{f \equiv \mu \pmod{cA}} \frac{1}{f} = \sum_{s \equiv A} \frac{1}{\mu + cs} = \frac{\overline{\pi}}{ce(\overline{\pi}\mu/c)},
$$

$$
\sum_{g \equiv a\mu \pmod{cA}} \frac{1}{g} = \sum_{t \equiv A} \frac{1}{a\mu + ct} = \frac{\overline{\pi}}{ce(\overline{\pi}a\mu/c)},
$$

we see that $R_2 = \overline{\pi}s(a, c)$. Therefore, we conclude that $R_1 + R_2$ is the right-hand side of (4.2).

5.2. Case $\gamma \in GL_2(A)$. The cases $q > 3$ and $q = 2$ are trivial. It suffices to show the case of $q = 3$ and $\det \gamma = -1$. Noting that $\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$ belongs to $SL_2(A)$, using the result obtained in Subsection 5.1, we have

$$
\xi(\gamma z) = \xi(\begin{pmatrix} a & -b \\ c & -d \end{pmatrix}(-z))
$$

$$
= \xi(-z) - \frac{\alpha(2)}{\overline{\pi}c} \left( c(-z) - d + \frac{1}{c(-z) - d} \right) + \frac{\alpha(1)^2}{\overline{\pi}c} + \overline{\pi}s(a, c),
$$

which is the right-hand side of (4.2).
6. APPLICATION

In this section, we prove the following theorem.

**Theorem 6 (Reciprocity law [1, 5]).** Let \( a, c \in A \setminus \{0\} \) be coprime.

1. If \( q = 3 \), then
   \[
   s(a, c) + s(c, a) = \frac{1}{T^3 - T} \left( \frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right).
   \]

2. If \( q = 2 \), then
   \[
   s(a, c) + s(c, a) = \frac{1}{T^4 + T^2} \left( \frac{a}{c} + \frac{c}{a} + \frac{1}{a} + \frac{1}{c} + \frac{1}{ac} + 1 \right).
   \]

We have already proved this theorem ([1, 5]), by computing the residues of a rational function. Using Theorem 5, we now provide another proof.

**Proof of Theorem 6.** There exist \( b, d \in A \) such that \( ad - bc = 1 \). Then, \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) belongs to \( SL_2(A) \). It holds that \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma z = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z \). We compute the values of the series \( \xi \) on both sides.

Using Theorem 2,
\[
\xi \left( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma z \right) = \xi(\gamma z) - \frac{\alpha(2)}{\pi} \left( \gamma z + \frac{1}{\gamma z} \right) - \frac{\alpha(1)^2}{\pi}.
\]

Combining Lemma 3 with Theorem 5, this can be written as
\[
(6.1) \quad \xi(z) = \frac{\alpha(2)}{\pi c} \left( cz + d + \frac{1}{cz + d} \right) + \frac{\alpha(1)^2}{\pi c} + \pi s(a, c) - \frac{\alpha(2)}{\pi} \left( \frac{a}{c} \right) + \frac{1}{a(cz + d)} - \frac{\alpha(1)^2}{\pi}.
\]

Using (2.1) and Theorem 5, \( \xi \left( \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} z \right) \) becomes
\[
(6.2) \quad \xi(z) = \frac{\alpha(2)}{\pi a} \left( az + b + \frac{1}{az + b} \right) + \frac{\alpha(1)^2}{\pi a} - \pi s(c, a).
\]

Equating (6.1) with (6.2), we obtain
\[
(6.3) \quad s(a, c) + s(c, a) = \frac{\alpha(2)}{\pi^2} \left( \frac{a}{c} + \frac{c}{a} + \frac{1}{ac} \right) + \frac{\alpha(1)^2}{\pi^2} \left( \frac{1}{a} - \frac{1}{c} + 1 \right).
\]

Combining (6.3) with the following lemma enables us to complete the proof.

**Lemma 7.** (1) If \( q = 3 \), then
\[
\alpha(2) = \frac{\pi^2}{T^3 - T}, \quad \alpha(1) = 0.
\]

(2) If \( q = 2 \), then
\[
\alpha(2) = \alpha(1)^2 = \frac{\pi^2}{T^4 + T^2}.
\]
7. AN ANALOG OF THE SAWTOOTH FUNCTION

The sawtooth function \((x)\) is defined by
\[
(x) = \begin{cases} 
\{x\} - 1/2 & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\
0 & \text{if } x \in \mathbb{Z},
\end{cases}
\]
where \(\{x\}\) is the fraction part of \(x\). This function has the following Fourier expansion:
\[
(x) = -\frac{1}{2\pi i} \sum_{0 \neq n \in \mathbb{Z}} \frac{\exp(2\pi inx)}{n}.
\]

Inspired by the definition of \(\xi(z)\), we define an analog of \((x)\) as follows. For a given \(x \in K_\infty\), let \(S_x\) be the set of all \(a \in A\) such that \(e(\pi ax) \neq 0\). Then, we define
\[
F(x) = \begin{cases} 
-\frac{1}{\pi} \sum_{a \in S_x} \frac{1}{ae(\pi ax)} & \text{if } S_x \neq \emptyset, \\
0 & \text{if } S_x = \emptyset.
\end{cases}
\]

This function has the following properties.
- For \(\epsilon \in \mathbb{F}_q \setminus \{0\}\), \(F(\epsilon x) = \epsilon^{-1}F(x)\).
- For \(b \in A\), \(F(b) = 0\).
- For \(b \in A\), \(F(x+b) = F(x)\).

Moreover, the value \(F(x)\) at \(x \in K\) can be described in terms of the Dedekind sum in function fields:

**Theorem 8.** For coprime \(a, c \in A \setminus \{0\}\), \(F(a/c) = s(-a, c)\).

We note that for the function \((x)\), the result corresponding to Theorem 8 does not hold. It would be interesting to investigate \(F(x)\) in the future.

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**REFERENCES**


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