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Sum-product estimates for self similar subsets of $\mathbb{Z}$ and Fractal zeta functions

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Abstract. This paper solves the general Erdös-Szemeredi conjecture for some classes of increasing families of finite subsets of self-similar subsets of the integers. It does so by applying zeta function methods for discrete self similar sets.

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1 Introduction

The Erdös-Szemeredi conjecture [ES] predicts a basic feature about the minimal number of distinct $h-$fold sums or products that can be created from an arbitrary finite set of integers. Despite much interest and some partial results, most notably by Chang [C], few examples appear to exist for which the conjecture is actually proved. Indeed, the only reasonably general class of sets known to us are arithmetic progressions where the verification of the conjecture is elementary.

The purpose of this article is to exhibit a large class of families of finite subsets of $\mathbb{Z}$, one that is distinctly different from arithmetic progressions, for which the conjecture can be proved by means of "fractal" zeta function theory. These sets are all subsets of "self-similar" sets (see Definition 3).

In our recent work [EL-2], we have shown how various Falconer type problems for "compatible" self similar sets of $\mathbb{Z}^n$ can be solved using a multivariate Tauberian

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theorem when applied to a fractal zeta function, provided one also has proved essential functional properties of the zeta function. This method extends very simply to solve the Erdös-Szemeredi conjecture for certain increasing families of finite subsets of any self similar subset of $\mathbb{Z}$ since any such set is automatically also compatible.

The main result is as follows.

Theorem 1. *Let $\mathcal{F} \subset \mathbb{Z}$ be a self similar set (see definition 3 below). For $x > 0$ set*

$$\mathcal{F}(x) = \mathcal{F} \cap [-x, x].$$

Then for any integer $h \geq 2$,

$$max\{|\mathcal{F}(x)^h|, |h\mathcal{F}(x)|\} \gg_\varepsilon |\mathcal{F}(x)|^{h-\varepsilon},$$

where $\mathcal{F}(x)^h :=$ $h-$fold products, $h\mathcal{F}(x) :=$ $h-$fold sums of $\mathcal{F}(x)$, and $|\cdot|$ denotes cardinality.

The careful reader will have noted that each of the finite sets *must* equal some $\mathcal{F}(x)$, as defined in the Theorem. Our method does *not* establish the predicted lower bound for increasing families of *arbitrary* finite subsets of $\mathcal{F}$, though it does seem that Theorem 1 is a reasonable first step.

A natural extension of Theorem 1, is the following. Let $P \in \mathbb{Z}[X_1, \ldots, X_h]$ be a polynomial of degree $d > 0$. Define for any $x > 0$ the $P-$fold sumset $P\mathcal{F}(x)$ by

$$P\mathcal{F}(x) = \{P(m_1, \ldots, m_h) \mid m_j \in \mathcal{F}(x) \ \forall j = 1, \ldots, h\}.$$

By using in addition to our zeta method, a deep result of T. Browning and D.R. Heath-Brown [BHB] on the density of rational points in algebraic varieties, we obtain the following extension of Theorem 1:

Theorem 2. *Let $\mathcal{F} \subset \mathbb{Z}$ be a self similar set. Let $P \in \mathbb{Z}[X_1, \ldots, X_h]$, $h \geq 2$ be a form of degree $d \geq 2$ that defines a non singular projective hypersurface $\{P = 0\} \subset \mathbb{P}^{h-1}(\overline{\mathbb{Q}})$. Assume also that*

1. *for all $j = 1, \ldots, h$, the coefficient of X_j^d in P is non zero;*
2. *the upper Minkowski dimension of $\mathcal{F}$ verifies $e(\mathcal{F}) > 1 - \frac{1}{h}$.*

Then, for any $\varepsilon > 0$,

$$|P\mathcal{F}(x)| \gg_\varepsilon |\mathcal{F}(x)|^{\frac{h}{e(\mathcal{F})}\left(e(\mathcal{F})-1+\frac{1}{h}\right)-\varepsilon}.$$

2 Ingredients for the proof

We first recall needed definitions and basic properties of fractal zeta functions used in the proof of Theorem 1. These properties only require $\mathcal{F} \subset \mathbb{R}^n$.

Definition 1. *A similarity on $\mathbb{R}^n$ is an affine linear transformation*

$$L : \mathbf{x} \to cT\mathbf{x} + \mathbf{b},$$

where $c > 0$ and T is an orthogonal transformation.

Definition 2. *Let $\mathcal{F}$ be a countable discrete subset of $\mathbb{R}^n$. Define the upper Minkowski dimension of $\mathcal{F}$ by:*

$$e(\mathcal{F}) = u\,dim_M\,\mathcal{F} := \limsup_{R\to\infty} \frac{\ln\left[\#\mathcal{F}(R)\right]}{\ln R} \in [0,\infty],$$

where $\mathcal{F}(R) := \{\mathbf{x} \in \mathcal{F};\ \|\mathbf{x}\| \leq R\}\ \ \forall R$. We say that $\mathcal{F}$ has finite upper Minkowski dimension whenever the limit is finite.

Definition 3. *A discrete set $\mathcal{F} \subset \mathbb{R}^n$ is self similar if its upper Minkowski dimension $e(\mathcal{F})$ is finite and positive, and there exists a finite set of similarities $\{L_j = c_jT_j + \mathbf{b}_j\}_1^r$ such that each scale factor $c_i > 1$ and*[1]

$$\mathcal{F} \equiv \bigcup_{i=1}^{r} L_j(\mathcal{F}) \quad \textit{and} \quad \#\big(L_{j_1}(\mathcal{F}) \cap L_{j_2}(\mathcal{F})\big) < \infty \ \ \forall j_1 \neq j_2.$$

$\mathcal{F}$ is "compatible" if the T_j pairwise commute.

Remark. Interesting examples of such sets are the Pascal triangle mod p, whose zeta function was studied in [E], and Pascal pyramid mod p, studied in [EL-1].

Definition 4. *The fractal zeta function determined by a discrete self similar set $\mathcal{F} \subset \mathbb{R}^n$ is the Dirichlet series*

$$\zeta_{\mathcal{F}}(s) = \sum_{\mathbf{m}\in\mathcal{F}'} \frac{1}{\|\mathbf{m}\|^s} \qquad (\mathcal{F}' := \mathcal{F} \setminus \{\mathbf{0}\}),$$

where $\|m\|$ = Euclidean norm on $\mathbb{R}^n$.

The basic properties of the fractal zeta function $\zeta_{\mathcal{F}}(s)$ were determined in [EL-1]. More precisely, Theorems 1 and 2 of [EL-1] can be summarized as follows:

[1] The notation $F \equiv G$ means that $(F \setminus G) \cup (G \setminus F)$ is a finite set.

Theorem 3. *Let $\mathcal{F}$ be a compatible discrete self similar subset of $\mathbb{R}^n$ as in Definition 3. Then*

1. *$\zeta_{\mathcal{F}}(s)$ converges absolutely and defines a holomorphic function in the halfplane $\{Re\, s > e(\mathcal{F})\}$ and $s = e(\mathcal{F})$ is its abscissa of convergence.*

2. *$\zeta_{\mathcal{F}}(s)$ has a meromorphic continuation with moderate growth to the whole complex plane $\mathbb{C}$.*

3. *The polar locus $\mathcal{P}(\mathcal{F})$ of $\zeta_{\mathcal{F}}(s)$ is a subset of*

$$\bigcup_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}\bigcup_{k\in\mathbb{N}}\left\{s-k \mid \sum_{j=1}^{r}\boldsymbol{\lambda}_j^{\boldsymbol{\alpha}}c_j^{-s}=1\right\},$$

   where for each $j = 1, \ldots, r$ $\boldsymbol{\lambda}_j = (\lambda_{j,1}, \ldots, \lambda_{j,n})$ is a vector of (complex) eigenvalues of the adjoint T_j^* of T_j.

4. *The upper Minkowski dimension $e(\mathcal{F})$ is a simple pole of $\zeta_{\mathcal{F}}(s)$ and satisifes $e(\mathcal{F}) = \sup \mathcal{P}(\mathcal{F})$.*

5. *$e(\mathcal{F})$ is the largest positive solution of the equation $\sum_{j=1}^{r} c_j^{-s} = 1$.*

Remark. If $\mathcal{F}$ is discrete self similar subset of $\mathbb{R}$ (i.e. if $n = 1$), then it is necessarily also compatible.

For any $P(\mathbf{x}) \in \mathbb{R}[X_1, \ldots, X_n]$ the fractal zeta function with weight P is the Dirichlet series

$$\zeta_{\mathcal{F}}(s, P) = \sum_{\mathbf{m}\in\mathcal{F}'} \frac{P(\mathbf{m})}{\|\mathbf{m}\|^s}.$$

Other weight functions can also be used, but these suffice for our purposes here.

If $\mathcal{F}$ is a compatible discrete self similar subset of $\mathbb{R}^n$ as in Definition 3, the following three analytic properties of any weighted zeta function $\zeta_{\mathcal{F}}(s, P)$ are basic for applications. They are proved in Lemma 2 of [EL-2].

Theorem 4. *1. The zeta function $\zeta_{\mathcal{F}}(s, P)$ converges absolutely and defines a holomorphic function in the halfplane $\{Re(s) > e(\mathcal{F}) + deg(P)\}$.*

2. *There exists a meromorphic extension of $\zeta_{\mathcal{F}}(s, P)$ to $\mathbb{C}$ with moderate growth in any open subset of a vertical strip of finite width whose distance from the polar locus is positive.*

3. *The polar locus $\zeta_{\mathcal{F}}(s, P)$ is a subset of*

$$\bigcup_{\boldsymbol{\alpha}\in\mathbb{N}_0^n}\bigcup_{k\in\mathbb{N}}\left\{s+|\boldsymbol{\alpha}|-k \mid \sum_{j=1}^{r}\boldsymbol{\lambda}_j^{\boldsymbol{\alpha}}c_j^{-s}=1\right\}.$$

Remarks. (i) Theorem 4, when combined with a weighted Perron formula [I] and a careful description of the polar locus of $\zeta_{\mathcal{F}}(s)$ near its boundary of analyticity, determines an explicit asymptotic for a weighted average of coefficients of $\zeta_{\mathcal{F}}(s)$.

For $Re\, s > e(\mathcal{F})$ write the distinct values of $\mathbf{m} \in \mathcal{F}' \to \|\mathbf{m}\|$ as $\lambda_1 < \lambda_2 < \cdots$, and the series as

$$\sum_{\ell} \frac{c_\ell}{\lambda_\ell^s}.$$

A typical application, though hardly the only one, shows the existence of a periodic function $\varphi \neq 0$ and $\rho \geq 0$ such that

$$x \to \sum_{\lambda_\ell \leq x} c_\ell (1 - \frac{x}{\lambda_\ell}) = x^{e(\mathcal{F})} (\log x)^\rho \left(\varphi(\log x) + O(\frac{1}{log\, x})\right) \quad \text{as } x \to \infty. \quad (1)$$

(ii) Because Falconer type problems typically involve translation invariant metric invariants as weights, the polynomial P is often of the form $P(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_h) \in \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_h]$ where each $\mathbf{x}_i \in \mathbb{R}^n$ $(n \geq 1)$. As a result, a single variable Tauberian result is far from sufficient to solve a Falconer type problem. Instead, it is necessary to prove a *multivariable* Tauberian theorem that can give, with precision comparable to (1), an asymptotic for

$$x \to \sum_{\{(\mathbf{m}_1, \ldots, \mathbf{m}_h) : \|\mathbf{m}_u\| \leq x \forall u\}} P(\mathbf{m}_1, \ldots, \mathbf{m}_h) \prod_u (1 - \frac{x}{\|\mathbf{m}_u\|}).$$

This requires use of a multivariable fractal zeta function. Probably the simplest example is the zeta function associated to the Erdös distance problem (see [EL-2]) where the weight function equals $P(\mathbf{m}_1, \mathbf{m}_2) = \|\mathbf{m}_1 - \mathbf{m}_2\|^2$:

$$\zeta_{dist}(s_1, s_2) = \sum_{(\mathbf{m}_1, \mathbf{m}_2) \in (\mathcal{F}')^2} \frac{P(\mathbf{m}_1, \mathbf{m}_2)}{\|\mathbf{m}_1\|^{s_1} \|\mathbf{m}_2\|^{s_2}}.$$

The needed Tauberian result is Theorem 6 [EL-2]. This implies (see §3.1 [ibid.]) the existence of a positive weight $\omega = \omega(\mathcal{F}, p) (= 2(e(\mathcal{F}) + 1))$ and a periodic function $\varphi \neq 0$ such that

$$\sum_{\{(\mathbf{m}_1, \mathbf{m}_2) \in \mathcal{F}^2,\ \|\mathbf{m}_u\| \leq x\ \forall u\}} P(\mathbf{m}_1, \mathbf{m}_2) \prod_{u=1}^{2} (1 - \frac{x}{\|\mathbf{m}_u\|}) = x^\omega \cdot \varphi(\log x) + o(x^\omega) \quad x \to \infty.$$

The weight ω is the maximal weight of any vertex on the real part of the boundary of the domain of analyticity of $\zeta_{dist}(s_1, s_2)$. There are two vertices at $(e(\mathcal{F}) + 2, e(\mathcal{F})), (e(\mathcal{F}), e(\mathcal{F}) + 2)$ so that $\omega = 2(e(\mathcal{F}) + 1)$.

3 Main ingredient

Let $\mathcal{F} \subset \mathbb{R}^n$ be a compatible self similar set as above and P a polynomial of the form $P(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_h) \in \mathbb{R}[\mathbf{x}_1, \ldots, \mathbf{x}_h]$ where each $\mathbf{x}_i \in \mathbb{R}^n$ ($n \geq 1$). We define a multivariable fractal zeta function as follows:

$$\zeta_{\mathcal{F}}(\mathbf{s}, P) = \sum_{\mathbf{m}_1, \ldots, \mathbf{m}_h \in \mathcal{F}'} \frac{P(\mathbf{m}_1, \ldots, \mathbf{m}_h)}{\|\mathbf{m}_1\|^{s_1} \ldots \|\mathbf{m}_h\|^{s_h}} \quad \mathbf{s} = (s_1, \ldots, s_h) \in \mathbb{C}^h.$$

Write

$$P(\mathbf{X}_1, \ldots, \mathbf{X}_h) = \sum_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^h) \in S(P)} a(\boldsymbol{\alpha}) \; \mathbf{X}_1^{\boldsymbol{\alpha}^1} \ldots \mathbf{X}_h^{\boldsymbol{\alpha}^h},$$

where $S(P)$ is a finite subset of $(\mathbb{N}_0^n)^h$.

Theorem 4 implies that $\zeta_{\mathcal{F}}(\mathbf{s}, P)$ converges absolutely and defines a holomorphic function in the domain

$$\mathcal{D}(\mathcal{F}, P) := \bigcap_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^h) \in S(P)} \{\mathbf{s} \in \mathbb{C}^h \mid Re(s_j) > e(\mathcal{F}) + |\boldsymbol{\alpha}^j| \;\; \forall j = 1, \ldots, h\} \tag{2}$$

where it satisfies the identity

$$\zeta_{\mathcal{F}}(\mathbf{s}, P) = \sum_{\boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^h) \in S(P)} a(\boldsymbol{\alpha}) \prod_{j=1}^{h} \zeta_{\mathcal{F}}(s_j, \mathbf{X}_j^{\boldsymbol{\alpha}^j}). \tag{3}$$

Theorem 4 implies also that $\zeta_{\mathcal{F}}(\mathbf{s}, P)$ has a meromorphic continuation with moderate growth to the whole space $\mathbb{C}^h$.

Define :

1. $S(\mathcal{F}, P) := \{(e(\mathcal{F}) + |\boldsymbol{\alpha}^1|, \ldots, e(\mathcal{F}) + |\boldsymbol{\alpha}^h|) : \; \boldsymbol{\alpha} = (\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^h) \in S(P)\}$;
2. $S_0(\mathcal{F}, P) := \{\omega \in S(\mathcal{F}, P); \; |\omega| = h\, e(\mathcal{F}) + degP\}$;
3. $\Gamma(\mathcal{F}, P) =$ convex hull of $S_0(\mathcal{F}, P) =$ *"polar polyhedron"* of $\zeta_{\mathcal{F}}(\mathbf{s}, P)$;
4. $\mathcal{V}(\mathcal{F}, P) = \{\mathbf{v} \in S_0(\mathcal{F}, P) : \mathbf{v}$ is a vertex of $\Gamma(\mathcal{F}, P)\} :=$ *vertex set* of the polar polyhedron of $\zeta_{\mathcal{F}}(\mathbf{s}, P)$.

Identity (3), Lemma 2 and the multivariate tauberian theorem (Theorem 8) we established in [EL-2] imply the following *key* result:

Theorem 5. *Assume $P \geq 0$ on $\mathcal{F}^h$ and that for any vertex $\mathbf{v} \in \mathcal{V}(\mathcal{F}, P)$,*

$$Res_{s_h=v_h} \dots Res_{s_1=v_1} \zeta_{\mathcal{F}}(\mathbf{s}, P) := \sum_{\alpha=(\alpha^1,\dots,\alpha^h)\in S(P)} a(\alpha) \prod_{j=1}^{h} Res_{s_j=v_j} \zeta_{\mathcal{F}}(s_j, \mathbf{X}_j^{\alpha^j}) \neq 0. \tag{4}$$

Then, for any $\varepsilon > 0$,

$$\sum_{\{(\mathbf{m}_1,\dots,\mathbf{m}_h)\in\mathcal{F}^h,\ \|\mathbf{m}_u\|\leq x\ \forall u\}} P(\mathbf{m}_1, \dots, \mathbf{m}_h) \gg_{\varepsilon} x^{h\,e(\mathcal{F})+degP-\varepsilon} \ \textit{as } x \to \infty.$$

Remark. Formula (3) implies that $\mathcal{D}(\mathcal{F}, P)$, defined in (2), is a "good approximation" for the domain of analyticity of $\zeta_{\mathcal{F}}(\mathbf{s}, P)$, whose boundary in $\mathbb{R}^n$ is known to be a polyhedron. The weight $h\,e(\mathcal{F}) + degP$ is the maximal weight of any vertex of this polyhedron and $\mathcal{V}(\mathcal{F}, P)$ is its vertex set. Formula (3) implies that any vertex in $\mathcal{V}(\mathcal{F}, P)$ is a possible pole of $\zeta_{\mathcal{F}}(\mathbf{s}, P)$. The non-vanishing of iterated residues in formula (4) implies that any point in $\mathcal{V}(\mathcal{F}, P)$ is indeed a pole.

4 Proof of Theorem 1

Set $P_0(X_1, \dots, X_h) = (X_1 \dots X_h)^2$. The fractal zeta function we need for the h–fold product sets $\mathcal{F}(x)^h$ is a Dirichlet series in $\mathbf{s} = (s_1, \dots, s_h)$ which is absolutely convergent in $\bigcap_\ell \{Re s_\ell > e(\mathcal{F}) + 2\}$:

$$\zeta_{prod}^{(h)}(\mathbf{s}) := \zeta_{\mathcal{F}}(\mathbf{s}, P_0) = \sum_{\mathbf{m}\in(\mathcal{F}')^h} \frac{(m_1 \cdots m_h)^2}{|m_1|^{s_1} \cdots |m_h|^{s_h}} = \prod_{\ell=1}^{h} \zeta_{\mathcal{F}}(s_\ell - 2). \tag{5}$$

With notations of §3 above, the vertex set of the polyhedron of $\zeta_{\mathcal{F}}(\mathbf{s}, P_0)$ is $\mathcal{V}(\mathcal{F}, P) = \{\mathbf{v}_0\}$ where $\mathbf{v}_0 = (e(\mathcal{F}) + 2, \dots, e(\mathcal{F}) + 2)$. Moreover,

$$\begin{aligned} Res_{s_h=e(\mathcal{F})+2} \dots Res_{s_1=e(\mathcal{F})+2} \zeta_{\mathcal{F}}(\mathbf{s}, P_0) &= \prod_{j=1}^{h} Res_{s_j=e(\mathcal{F})+2} \zeta_{\mathcal{F}}(s_j - 2) \\ &= \prod_{j=1}^{h} Res_{s_j=e(\mathcal{F})} \zeta_{\mathcal{F}}(s_j) \neq 0. \end{aligned}$$

It follows from Theorem 5 that we have the nontrivial asymptotic bound:

$$\forall \varepsilon > 0, \quad \mathcal{A}_{prod}^{(h)}(x) := \sum_{\{(m_1,\dots,m_h)\in\mathcal{F}^h,\ |m_u|\leq x\ \forall u\}} (m_1 \dots m_h)^2 \gg_{\varepsilon} x^{h\ e(\mathcal{F})+2h-\varepsilon} \text{ as } x \to \infty. \tag{6}$$

Defining

$$n_{pr}(x) = |\mathcal{F}(x)^h| \quad \text{and setting} \quad \mathcal{F}(x)^h = \{\rho_1 < \rho_2 < \cdots < \rho_{n_{pr}(x)}\},$$

it is clear that

$$\mathcal{A}^{(h)}_{prod}(x) = \sum_{j=1}^{n_{pr}(x)} \rho_j^2 \; N_j$$

for any $j = 1, \ldots, n_{pr}(x)$, $N_j = \#\{(m_1, \ldots, m_h) \in \mathcal{F}(x)^h \mid m_1 \ldots m_h = \rho_j\}$. Since each $m_j \in \mathbb{Z}$, there is a standard bound (see for example [Te]) that is uniform in x:

$$\forall \varepsilon > 0, \; N_j \leq \#\{(m_1, \ldots, m_h) \in \mathbb{Z}^h : m_1 \ldots m_h = \rho_j\} \ll_\varepsilon |\rho_j|^\varepsilon. \qquad (7)$$

In addition, since $|\rho_j| \leq x^h$, it is clear that for any $\varepsilon > 0$

$$\mathcal{A}^{(h)}_{prod}(x) = \sum_{j=1}^{n_{pr}(x)} \rho_j^2 \; N_j \ll_\varepsilon n_{pr}(x) \cdot x^{2h(1+\varepsilon)}.$$

As a result, for any $\varepsilon, \varepsilon' > 0$, we have for all sufficiently large x :

$$x^{h(e(\mathcal{F})+2)-\varepsilon'} \ll_{\varepsilon'} \mathcal{A}_{prod}(x) \ll_\varepsilon n_{pr}(x) \cdot x^{2h(1+\varepsilon)}$$

which implies that for any $\varepsilon > 0$

$$n_{pr}(x) \gg_\varepsilon x^{he(\mathcal{F})-\varepsilon}.$$

Moreover, by the definition of upper Minkowski dimension, we also have:

$$|\mathcal{F}(x)| \ll_\varepsilon x^{e(\mathcal{F})+\varepsilon}.$$

From which, we deduce the predicted lower bound of the Erdös-Szemeredi conjecture for the subsets $\mathcal{F}(x)$ of $\mathcal{F}$:

$$n_{pr}(x) \gg_\varepsilon |\mathcal{F}(x)|^{h\frac{e(\mathcal{F})}{e(\mathcal{F})+\varepsilon}-\varepsilon} \gg_\varepsilon |\mathcal{F}(x)|^{h-\varepsilon}.$$

This, of course, suffices to finish the proof of Theorem 1. □

Remark. It is, however, also of interest to see what the above method gives as a lower bound for $|h\mathcal{F}(x)|$. The evident fractal zeta function equals:

$$\zeta^{(h)}_{sum}(\mathbf{s}) = \zeta_\mathcal{F}(\mathbf{s}, P_1) = \sum_{(m_1,\ldots,m_h)\in(\mathcal{F}')^h} \frac{(m_1 + \cdots + m_h)^2}{|m_1|^{s_1} \cdots |m_h|^{s_h}},$$

where $P_1(X_1,\dots,X_h)=(X_1+\dots+X_h)^2$. It follows that $\zeta^{(h)}_{sum}(\mathbf{s})$ converges absolutely in the domain $\cap_{i=1}^h\{\Re(s_i)>e(\mathcal{F})+2\}$ where it satisfies the identity

$$\zeta^{(h)}_{sum}(\mathbf{s})=\sum_{j=1}^h\zeta_{\mathcal{F}}(s_j-2)\prod_{\substack{i=1\\i\neq j}}^h\zeta_{\mathcal{F}}(s_i)+2\sum_{1\le i<j\le h}\zeta_{\mathcal{F}}(s_i-1)\zeta_{\mathcal{F}}(s_j-1)\prod_{\substack{k=1\\k\neq i,j}}^h\zeta_{\mathcal{F}}(s_k). \quad (8)$$

We deduce that $\zeta^{(h)}_{sum}(\mathbf{s})$ has a meromorphic continuation with moderate growth to $\mathbb{C}^h$.

Denoting the all 1 vector in $\mathbb{R}^h$ by $\mathbf{1}_h$ and $\{\mathbf{e}_j\}_1^h$ the unit vectors, it is not difficult to verify that the vertex set of the polyhedron of $\zeta^{(h)}_{sum}$ is

$$\mathcal{V}(\mathcal{F},P)=\{e(\mathcal{F})\cdot\mathbf{1}_h+2\mathbf{e}_j\mid j=1,\dots,h\}.$$

Moreover, for any $j=1,\dots,h$ if we denote $\mathbf{v}^j=(v_1^j,\dots,v_h^j)$ the vertex $e(\mathcal{F})\cdot\mathbf{1}_h+2\mathbf{e}_j$, we have

$$Res_{s_h=v_h^j}\dots Res_{s_1=v_1^j}\zeta_{\mathcal{F}}(\mathbf{s},P_1)=Res_{s_j=e(\mathcal{F})+2}\zeta_{\mathcal{F}}(s_j-2)\prod_{\substack{i=1\\i\neq j}}^h Res_{s_i=e(\mathcal{F})}\zeta_{\mathcal{F}}(s_i)$$

$$=\prod_{i=1}^h Res_{s_i=e(\mathcal{F})}\zeta_{\mathcal{F}}(s_i)=\left(Res_{s=e(\mathcal{F})}\zeta_{\mathcal{F}}(s)\right)^h\neq 0.$$

It follows from Theorem 5 that we have the nontrivial asymptotic bound:

$$\forall\varepsilon>0,\quad A^{(h)}_{sum}(x):=\sum_{\{(m_1,\dots,m_h)\in\mathcal{F}^h,\ |m_u|\le x\ \forall u\}}(m_1+\dots+m_h)^2\gg_\varepsilon x^{h\,e(\mathcal{F})+2-\varepsilon}\text{ as }x\to\infty. \quad (9)$$

Analogously, we define

$$n_{su}(x)=|h\mathcal{F}(x)|\quad\text{and write}\quad h\mathcal{F}(x)=\{\tau_1<\tau_2<\dots<\tau_{n_{su}(x)}\}.$$

It follows that

$$\mathcal{A}_{sum}(x)=\sum_{j=1}^{n_{su}(x)}\tau_j^2M_j,$$

where for all $j=1,\dots,n_{su}(x)$, $M_j:=\#\{(m_1,\dots,m_h)\in\mathcal{F}'(x)^h:\sum_\ell m_\ell=\tau_j\}$. Since each $\tau_j^2\le h^2x^2\ll x^2$, we need a uniform (in τ_j) bound for each M_j. This is given by Lemma 6 [EL-2] which shows that for any $\varepsilon>0$,

$$M_j=\#\{(m_1,\dots,m_h)\in\mathcal{F}(x)^h:\sum_\ell m_\ell=\tau_j\}\ll_\varepsilon x^{h-1+\varepsilon}.$$

Thus, for any $\varepsilon > 0$

$$\mathcal{A}_{sum}(x) = \sum_{j=1}^{n_{su}(x)} \tau_j^2 M_j \ll_\varepsilon n_{su}(x) \cdot x^{h+1+\varepsilon}.$$

Combining this with (9), we deduce

$$x^{he(\mathcal{F})+2-\varepsilon} \ll_\varepsilon n_{su}(x) \cdot x^{h+1+\varepsilon}.$$

So, there is a lower bound for $n_{su}(x)$ that grows in x iff $h(e(\mathcal{F})-1)+1 > 0$, that is, if $e(\mathcal{F}) > 1 - \frac{1}{h}$. However, when this is satisfied, the resulting lower bound

$$n_{su}(x) \gg_\varepsilon |\mathcal{F}(x)|^{\frac{h(e(\mathcal{F})-1)+1}{e(\mathcal{F})} - \varepsilon}$$

is weaker than that obtained for $n_{prod}(x)$ as above.

5 Proof of Theorem 2

5.1 A geometric lemma

Lemma 1. *Let $P \in \mathbb{Z}[X_1, \dots, X_h]$, $h \geq 2$ be a form of degree $d \geq 2$ that defines a non singular projective hypersurface $X = \{P = 0\}$ in $\mathbb{P}^{h-1}(\overline{\mathbb{Q}})$. Then, for any $\varepsilon > 0$, there exists a positive constant $C = C(h, d, \varepsilon) > 0$ which depends only upon h, d and ε such that for any $\ell \in \mathbb{Z}$ and any $B > 0$,*

$$\#\{(m_1, \dots, m_h) \in \mathbb{Z}^h \mid P(m_1, \dots, m_h) = \ell \text{ and } \max_i |m_i| \leq B\} \leq C\ B^{h-1+\varepsilon}.$$

Proof of Lemma 1. **First case:** We assume that $\ell \in \mathbb{Z} \setminus \{0\}$. Define the form $F(x_0, \dots, x_h) = P(X_1, \dots, X_h) - \ell\ X_0^d$ and $Y = \{F(x_0, \dots, x_h) = 0\} \subset \mathbb{P}^h$ the projective hypersurface associated to it.

Assume that Y is singular. It follows that there exists a point $\tilde{\mathbf{x}} = (x_0, x_1, \dots, x_h) \in Y$ such that

$$\forall i = 1, \dots, h \quad \frac{\partial P(x_1, \dots, x_h)}{\partial x_i} = \frac{\partial F(\tilde{\mathbf{x}})}{\partial x_i} = 0 \quad \text{and} \quad -d\ell x_0^{d-1} = \frac{\partial F(\tilde{\mathbf{x}})}{\partial x_0} = 0.$$

We deduce that

$$\forall i = 1, \dots, h \quad \frac{\partial P(x_1, \dots, x_h)}{\partial x_i} = 0 \quad \text{and } x_0 = 0.$$

It follows that $\mathbf{x} = (x_1, \dots, x_h)$ is a singular point of the projectve hypersurface $X = \{P = 0\} \subset \mathbb{P}^{h-1}$. Hence a contradiction.

Thus, the projective hypersurface $Y = \{F(x_0, \dots, x_h) = 0\} \subset \mathbb{P}^h$ is non singular. The corollary of Theorem 1 of [BHB] implies then that for any $\varepsilon > 0$, there exists a positive constant $C = C(h, d, \varepsilon) > 0$ which depends only upon h, d and ε such that for any $B > 0$,

$$\#\{(m_1, \dots, m_h) \in \mathbb{Z}^h \mid F(1, m_1, \dots, m_h) = 0 \text{ and } \max_i |m_i| \leq B\} \leq C\ B^{h-1+\varepsilon}.$$

It follows that for any $B > 0$ and $\ell \in \mathbb{Z}$:

$$\#\{(m_1, \dots, m_h) \in \mathbb{Z}^h \mid P(m_1, \dots, m_h) = \ell \text{ and } \max_i |m_i| \leq B\} \leq C\ B^{h-1+\varepsilon}.$$

This end the proof of lemma 1 in the first case.

Second case: We assume that $\ell = 0$. Since the projective hypersurface $X = \{P(x_1, \dots, x_h) = 0\} \subset \mathbb{P}^{h-1}$ is non singular. The corollary of Theorem 1 of [BHB] implies then that for any $\varepsilon > 0$, there exists a positive constant $K = K(h, d, \varepsilon) > 0$ which depends only upon h, d and ε such that for any $B > 0$,

$$\begin{aligned} &N(B) \\ &:= \#\{\mathbf{m} = (m_1, \dots, m_h) \in \mathbb{Z}^h \mid P(\mathbf{m}) = 0,\ gcd(m_1, \dots, m_h) = 1 \text{ and } \max_i |m_i| \leq B\} \\ &\leq K\ B^{h-2+\varepsilon}. \end{aligned}$$

It follows that for any $\varepsilon > 0$ and any $B > 1$

$$\begin{aligned} N_0(B) &:= \#\{\mathbf{m} = (m_1, \dots, m_h) \in \mathbb{Z}^h \mid P(m_1, \dots, m_h) = 0 \text{ and } \max_i |m_i| \leq B\} \\ &= \sum_{u \leq B} \#\{\mathbf{m} \in \mathbb{Z}^h \mid P(\mathbf{m}) = 0,\ gcd(m_1, \dots, m_h) = u \text{ and } \max_i |m_i| \leq B\} \\ &= \sum_{u \leq B} \# \left\{\mathbf{m} \in \mathbb{Z}^h \mid P(\mathbf{m}) = 0,\ gcd(m_1, \dots, m_h) = 1 \text{ and } \max_i |m_i| \leq \frac{B}{u}\right\} \\ &= \sum_{u \leq B} N\left(\frac{B}{u}\right) \ll_{h,d,\varepsilon} \sum_{u \leq B} \left(\frac{B}{u}\right)^{h-2+\varepsilon} \ll_{h,d,\varepsilon} B^{h-2+\varepsilon} \left(\sum_{u \leq B} 1\right) \ll_{h,d,\varepsilon} B^{h-1+\varepsilon}. \end{aligned}$$

This end the proof of lemma 1. □

5.2 Proof of Theorem 2

Let $P \in \mathbb{Z}[X_1, \dots, X_h]$ be a polynomial of degree d which satisfies the assumption of Theorem 2. Write $P(X_1, \dots, X_h) = \sum_{\alpha \in S(P)} a_\alpha \mathbf{X}^\alpha$, where the finite set $S(P)$ is the support of P. The fractal zeta function we need for the sets $P\mathcal{F}(x)$ is a Dirichlet series in $\mathbf{s} = (s_1, \ldots, s_h)$:

$$\zeta_{\mathcal{F}}(\mathbf{s}, P^2) = \sum_{\mathbf{m}\in(\mathcal{F}')^h} \frac{P(m_1, \cdots, m_h)^2}{|m_1|^{s_1} \cdots |m_h|^{s_h}}. \tag{10}$$

Theorem 4 implies that $\zeta_{\mathcal{F}}(\mathbf{s}, P)$ converges absolutely in the domain $\{\mathbf{s} \in \mathbb{C}^h \mid \Re(s_j) > e(\mathcal{F}) + 2d \ \forall j\}$ where it satisfies the identity

$$\zeta_{\mathcal{F}}(\mathbf{s}, P^2) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in S(P)} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \prod_{j=1}^{h} \zeta_{\mathcal{F}}(s_j, X_j^{\alpha_j+\beta_j}). \tag{11}$$

Theorem 4 implies also that $\zeta_{\mathcal{F}}(\mathbf{s}, P)$ has a meromorphic continuation to $\mathbb{C}^h$ with moderate growth.

Moreover, with notations of §3, we have:

1. $S(\mathcal{F}, P^2) := \{(e(\mathcal{F}) + \alpha_1 + \beta_1, \ldots, e(\mathcal{F}) + \alpha_h + \beta_h)\,;\ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(P)^2\}$;
2. $S_0(\mathcal{F}, P^2)$
   $:=\{\omega \in S(\mathcal{F}, P^2);\ |\omega| = h\, e(\mathcal{F}) + 2d\}$
   $= \big\{(e(\mathcal{F}) + \alpha_1 + \beta_1, \ldots, e(\mathcal{F}) + \alpha_h + \beta_h)\,;\ (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S(P)^2 \text{ and } |\boldsymbol{\alpha}| = |\boldsymbol{\beta}| = d\big\}$;
3. Denoting the all 1 vector in $\mathbb{R}^h$ by $\mathbf{1}_h$ and $\{\mathbf{e}_j\}_1^h$ the unit vectors, it is not difficult to verify that the polar polyhedron $\Gamma(\mathcal{F}, P^2) = \textit{convex hull of } S_0(\mathcal{F}, P^2)$ is the set
   $$\Gamma(\mathcal{F}, P^2) = \textit{convex hull } \{e(\mathcal{F}) \cdot \mathbf{1}_h + 2d\mathbf{e}_1, \ldots, e(\mathcal{F}) \cdot \mathbf{1}_h + 2d\mathbf{e}_h\}.$$
4. It is also not difficult to verify that the vertex set of the polar polyhedron is
   $$\mathcal{V}(\mathcal{F}, P^2) = \{e(\mathcal{F}) \cdot \mathbf{1}_h + 2d\mathbf{e}_j \mid j = 1, \ldots, h\}.$$

Moreover, it follows from (11) that
for any vertex $\mathbf{v}_k = (v_{k,1}, \ldots, v_{k,h}) = e(\mathcal{F}) \cdot \mathbf{1}_h + 2d\mathbf{e}_k \in \mathcal{V}(\mathcal{F}, P)$:

$$Res_{s_h=v_{k,h}} \ldots Res_{s_1=v_{k,1}} \zeta_{\mathcal{F}}(\mathbf{s}, P^2) = \sum_{\boldsymbol{\alpha},\boldsymbol{\beta}\in S(P)} a_{\boldsymbol{\alpha}} a_{\boldsymbol{\beta}} \prod_{j=1}^{h} Res_{s_j=v_{k,j}} \zeta_{\mathcal{F}}(s_j, X_j^{\alpha_j+\beta_j}).$$

Since

1. $\zeta_{\mathcal{F}}(s_k, X_k^{\alpha_k+\beta_k})$ converges absolutely and defines a holomorphic function in the halfplane $\{Re(s_k) > e(\mathcal{F}) + \alpha_k + \beta_k\}$ and;
2. $v_{k,k} = e(\mathcal{F}) + 2d > e(\mathcal{F}) + \alpha_k + \beta_k$ if $\boldsymbol{\alpha} \neq d\mathbf{e}_k$ or $\boldsymbol{\beta} \neq d\mathbf{e}_k$,

we deduce that

$$\begin{aligned}
&Res_{s_h=v_{k,h}} \dots Res_{s_1=v_{k,1}} \zeta_{\mathcal{F}}(\mathbf{s}, P^2) \\
&= (a_{d\mathbf{e}_k})^2 \, Res_{s_k=e(\mathcal{F})+2d} \zeta_{\mathcal{F}}(s_k - 2d) \prod_{j=1,\ j\neq k}^{h} Res_{s_j=e(\mathcal{F})} \zeta_{\mathcal{F}}(s_j) \\
&= (a_{d\mathbf{e}_k})^2 \left(Res_{s=e(\mathcal{F})} \zeta_{\mathcal{F}}(s)\right)^h \neq 0.
\end{aligned}$$

Theorem 5 implies then the nontrivial asymptotic bound:

$$\forall \varepsilon > 0, \quad \mathcal{A}_P(x) := \sum_{\{(m_1,\dots,m_h)\in\mathcal{F},\ |m_u|\leq x\ \forall u\}} P(m_1 \dots m_h)^2 \gg_\varepsilon x^{h\,e(\mathcal{F})+2d-\varepsilon} \text{ as } x \to \infty. \tag{12}$$

Defining

$$n_P(x) := |P\mathcal{F}(x)| \quad \text{and setting} \quad P\mathcal{F}(x) = \{\rho_1 < \rho_2 < \cdots < \rho_{n_P(x)}\},$$

it is clear that

$$\mathcal{A}_P(x) = \sum_{j=1}^{n_P(x)} \rho_j^2 \, N_j,$$

where for any $j = 1, \dots, n_P(x)$,

$$N_j = \#\{(m_1, \dots, m_h) \in \mathcal{F}(x)^h \mid P(m_1, \dots, m_h) = \rho_j\}.$$

Since each $m_j \in \mathbb{Z}$, Lemma 1 above implies the following *uniform* bound in x and ρ_j:

$$\forall \varepsilon > 0,\ N_j \leq \#\{(m_1, \dots, m_h) \in \mathbb{Z}^h : P(m_1, \dots, m_h) = \rho_j \text{ and } |m_j| \leq x\, \forall j\} \ll_\varepsilon x^{h-1+\varepsilon}. \tag{13}$$

In addition, since $|\rho_j| \ll x^d$, it is clear that for any $\varepsilon > 0$

$$\mathcal{A}_P(x) = \sum_{j=1}^{n_P(x)} \rho_j^2 \, N_j \ll_\varepsilon n_P(x) \cdot x^{2d+h-1+\varepsilon}.$$

As a result, for any $\varepsilon, \varepsilon' > 0$, we have for all sufficiently large x :

$$x^{he(\mathcal{F})+2d-\varepsilon'} \ll_{\varepsilon'} \mathcal{A}_P(x) \ll_\varepsilon n_P(x) \cdot x^{2d+h-1+\varepsilon}.$$

which implies that for any $\varepsilon > 0$

$$n_P(x) \gg_\varepsilon x^{h(e(\mathcal{F})-1)+1-\varepsilon}.$$

Moreover, by the definition of upper Minkowski dimension, we also have:

$$|\mathcal{F}(x)| \ll_\varepsilon x^{e(\mathcal{F})+\varepsilon}.$$

From which, we deduce the lower bound:

$$n_P(x) \gg_\varepsilon |\mathcal{F}(x)|^{\frac{h(e(\mathcal{F})-1)+1}{e(\mathcal{F})}-\varepsilon}.$$

This, of course, suffices to finish the proof of Theorem 2. □

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