The mean values of the Dirichlet $L$-functions

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1 Introduction

First, we recall the definition of the Dirichlet $L$-function. Let $\chi$ be a Dirichlet character modulo $k \geq 2$ and

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

be the Dirichlet $L$-function for $\Re s > 1$.

In the present paper, we consider the following mean values of the Dirichlet $L$-functions for a positive integer $r$:

$$S_r(m_1, \ldots, m_{r+1})$$

$$= \left( \frac{2}{\phi(k)} \right)^r \sum_{\chi_1(-1)=(-1)^{m_1}} \cdots \sum_{\chi_r(-1)=(-1)^{m_r}} \left( \prod_{i=1}^{r} L(m_i, \chi_i) \right) L(m_{r+1}, \overline{\chi_1 \cdots \chi_r}),$$

where $m_1, \ldots, m_{r+1} \in \mathbb{N}$ with $m_1 + \cdots + m_{r+1} \equiv 0 \pmod{2}$ and $\chi_1, \ldots, \chi_r$ are the Dirichlet character modulo $k$. The aim of such studies is to express $S_r(m_1, \ldots, m_{r+1})$ in terms of the values of the Riemann zeta-function $\zeta(\ell)$ and Jordan’s totient functions

$$J_\ell(k) = k^\ell \prod_{p \mid k} (1 - p^{-\ell}),$$

where $\ell$ is a positive integer.
In the case $r = 1$, $m_1 = m_2 = 1$ and $k = p$ is a prime, Walum [8] studied $S_1(1, 1)$, and Alkan [1], Qi [7], Louboutin [4] and Zhang [9] independently gave the following explicit formula

$$S_1(1, 1) = \frac{\pi^2}{6k^2} \left( J_2(k) - 3\phi(k) \right),$$

in the case $k \geq 2$. Also Louboutin [5] studied $S_1(m, m)$ for a positive integer $m$ and $k > 2$, and Liu and Zhang [3] studied $S_1(m, n)$ for positive integers $m, n \geq 1$ with the same parity (see [3] Theorem 1.1).

Moreover Alkan [2] considered the case $r = 2$, and he gave the explicit formula

$$S_2(1, 1, 2) = \pi^4 \left( \frac{J_4(k)}{90k^4} - \frac{J_2(k)}{18k^4} \right)$$

for $k > 2$.

## 2 Main theorem

Now we give the main theorem, but first we have to prepare the following some notations which use in the main theorem:

Let $(b_{\alpha})_{\alpha=1}^{7n} = (b_1, b_2, \ldots, b_m)$ be an $m$-th row vector and $(c_{ij})_{1 \leq i, j \leq m}$ be an $m$-th square matrix. Then, for an even positive integer $n$, we put the $\frac{n}{2}$-th column vector

$$A_n = \left( \frac{(-1)^{i}ia_{i,j}}{2^{2i-1}B_{2i}} \right)_{1 \leq i, j \leq \frac{n}{2}}^{-1} \begin{pmatrix} J_2(k) \\ J_4(k) \\ \vdots \\ J_n(k) \end{pmatrix},$$

where $B_m$ is the $m$-th Bernoulli number and

$$a_{i,j} = \begin{cases} (-1)^{i}4^{i-1} & \text{if } j = 1, \\ 2(j - 1)(2j - 1)a_{i-1,j-1} - 4j^2a_{i-1,j} & \text{if } 1 < j < i, \\ -(2i - 1)! & \text{if } j = i, \\ 0 & \text{if } j > i. \end{cases}$$

**Theorem 1.** Let $k > 2$ be a positive integer and $m_1, \ldots, m_{r+1} \in \mathbb{N}$ with $m_1 + \cdots + m_{r+1} \equiv 0 \pmod{2}$.

In the case $r$ is odd and $m_1 = m_2 = \cdots = m_{r+1} = 1$, we have

$$S_r(1, \ldots, 1) = \left( \frac{\pi}{2k} \right)^{r+1} \left\{ \left( \sum_{\alpha=1}^{r+1} \frac{r+1}{2} - \alpha \right) A_{r+1} + (-1)^{r+1} \phi(k) \right\}.$$
and the otherwise,

\[ S_r(m_1, \ldots, m_{r+1}) = \frac{(-1)^{r+1} \pi^{m_1 + \cdots + m_{r+1}}}{2^{r+1} k^{m_1 + \cdots + m_{r+1}} \{ \prod_{i=1}^{r+1} (m_i - 1)! \}} \]

\[ \times \left( \sum_{n=0}^{[\frac{1}{2}]} (-1)^{[\frac{1}{2}]-n} \left( \frac{[\frac{1}{2}]}{n} \right) \sum_{j_1 + \cdots + j_{r+1} = \alpha - n} \cdots \sum_{1 \leq j_r \leq \left\lfloor \frac{m_r}{2} \right\rfloor, 1 \leq j_{r+1} \leq \left\lfloor \frac{m_{r+1}}{2} \right\rfloor} \prod_{v=1}^{r+1} f(m_v, j_v) \right) \]

where for \( m_v \geq 2, \) \( f(m_v, j_v) = a_{\left\lfloor \frac{m_v}{2} \right\rfloor, j_v} \rho(m_v, j_v), \)

\[ \rho(m_v; j_v) = \begin{cases} -2j_v & \text{if } m_v \in 2N + 1, \\ 1 & \text{otherwise.} \end{cases} \]

and we put

\[ t = \# \{ m_v \mid m_v \text{ is odd for } 1 \leq v \leq r + 1 \}. \]

Also, in the case \( m_v = 1, \) the sum and product corresponding \( v \) does not appear.

We note that this result is a generalization of results of [6].

3 Sketch of proof

The proof of Theorem 1 is similar to the proof of Theorem 1.1 and 1.2 of [6]. Therefore we give only the sketch of the proof of Theorem 1 in this paper.

The key of the proof is to use the following result of Louboutin ([5] Proposition 3 (1)):

Let \( n \geq 1 \) and \( k > 2 \) be positive integers. Let \( \cot^{(n)} x \) denote the \( n \)-th derivative of \( \cot x. \) If \( \chi \) is a Dirichlet character modulo \( k \) and if \( \chi(-1) = (-1)^n \) then we have

\[ L(n, \chi) = \frac{(-1)^{n-1} \pi^n}{2^{kn(n-1)!}} \sum_{j=1}^{k-1} \chi(j) \cot^{(n-1)}(\pi j/k). \quad (1) \]
Then, by (1) and
\[
\sum_{\chi(-1)=(-1)^{n}} \chi(j_{1})\overline{\chi}(j_{2}) = \begin{cases} 
\frac{\phi(k)}{2} & \text{if } j_{1} \equiv j_{2} \mod k, \ (j_{1}, k) = 1, \\
(-1)^{n}\frac{\phi(k)}{2} & \text{if } j_{1} \equiv -j_{2} \mod k, \ (j_{1}, k) = 1, \\
0 & \text{otherwise},
\end{cases}
\]
we have
\[
S_{r}(m_{1}, \ldots, m_{r+1}) = \frac{(-1)^{r+1}\pi^{m_{1}+\cdots+m_{r+1}}}{2k^{m_{1}+\cdots+m_{r+1}}(\prod_{i=1}^{r+1}(m_{i} - 1)!)}
\times \sum_{l=1}^{k-1} \cot^{(m_{1}-1)}(\pi l/k) \cot^{(m_{2}-1)}(\pi l/k) \cdots \cot^{(m_{r+1}-1)}(\pi l/k).
\]

Now, using the following formula:

For any positive integer \( n \), we have
\[
cot^{(2n-1)} x = \sum_{j=1}^{n} a_{n,j} \sin^{-2j} x,
\]
where \( a_{n,j} \) is defined in Section 2, we can express \( S_{r}(m_{1}, \ldots, m_{r+1}) \) to the terms of \( \sum_{j=1}^{n} a_{n,j} \sin^{-2j}(\pi l/k) \) for \( 1 \leq j \leq (m_{1} + \cdots + m_{r+1})/2 \).

On the other hand, applying (2) to (1), we have
\[
L(2n, \chi_{0}) = \frac{-\pi^{2n}}{2k^{2n}(2n-1)!} \sum_{j=1}^{n} a_{n,j} \sum_{l=1}^{k-1} \sin^{-2j}\left(\frac{\pi l}{k}\right),
\]
where \( \chi_{0} \) is a principal Dirichlet character modulo \( k \). Also, by the Euler product expansion, we have
\[
L(2n, \chi_{0}) = \zeta(2n)k^{-2n}J_{2n}(k).
\]

Then, by (3) and (4), we can express \( \sum_{j=1}^{n} a_{n,j} \sin^{-2j}(\pi l/k) \) to the terms of \( J_{2j} \) for \( 1 \leq j \leq (m_{1}+\cdots+m_{r+1})/2 \). Therefore we can express \( S_{r}(m_{1}, \ldots, m_{r+1}) \) to the terms of \( J_{2j} \) for \( 1 \leq j \leq (m_{1}+\cdots+m_{r+1})/2 \) and we can show Theorem 1.
4 Examples

Lastly we give the following several evaluation formulas for $S_r(m_1, \ldots, m_{r+1})$ in the case $r = 2$:

\[
S_2(1, 1, 4) = \frac{\pi^6}{12k^6} \left( \frac{4}{315} J_6(k) - \frac{4}{45} J_4(k) + \frac{8}{45} J_2(k) \right),
\]

\[
S_2(1, 1, 6) = \frac{\pi^8}{240k^8} \left( \frac{8}{315} J_8(k) - \frac{32}{189} J_6(k) + \frac{8}{45} J_4(k) + \frac{32}{63} J_2(k) \right).
\]

References


[3] H. Liu and W. Zhang, On the mean value of $L(m, \chi)L(n, \overline{\chi})$ at positive integers $m, n \geq 1$, Acta. Arith. 122 no. 1 (2006), 51–56.


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