数理解析研究所講究録
第2015巻 2017年 162-169

Moving frames and Eisenstein invariants

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ABSTRACT. We recall combinatorial reconstitution of the periods of Eisenstein series of congruence subgroups of $SL_2(\mathbb{Z})$, and present some consequence of "moving frames" in a free profinite group.

Plan:
1. Moving frames (review)
2. Eisenstein periods
3. Combinatorics in $\hat{F}_2 = \pi_1^{\text{ét}}($$)$
4. Some applications

1. **Review: Moving frames**

Suppose we are given a sequence of linear transformations on a vector space V:

$$V \xleftarrow{f_3} V \xleftarrow{f_2} V \xleftarrow{f_1} V.$$

Fix a basis $\epsilon_0 = (e_1, \ldots, e_n)$ of V, and let A_i be the representative matrices of f_i $(i = 1, 2, 3)$ respectively in view of the basis ϵ_0. Then, as is well known, the composed transformation $f_3 \circ f_2 \circ f_1$ is represented by the matrix $A_3A_2A_1$.

According to the idea of moving frames, we consider not only the initial basis ϵ_0 but also the moved bases $\epsilon_1 := f_1(\epsilon_0)$ and $\epsilon_2 := f_2f_1(\epsilon_0)$. Then, letting B_i denote the representative matrix of f_i in view of the basis ϵ_{i-1} for $i = 1, 2, 3$, we derive that

$$B_1 = A_1, \quad B_2 = A_1^{-1}A_2A_1, \quad B_3 = A_1^{-1}A_2^{-1}A_3A_2A_1.$$

Consequently we find that the composition $f_3 \circ f_2 \circ f_1$ is represented by the reversely multiplied matrix $B_1B_2B_3$ with respect to ϵ_0.

We have borrowed from Spivak's book [Sp99, Chap. 7] the term "moving frames" as an English translation of E. Cartan's notion "repère mobile". See loc. cit. for more sophisticated applications. A most typical example of that idea may be what is called the Euler angle representation of the space rotations $SO(3) = \{A \in GL_3(\mathbb{R}) \mid {}^tA\,A = 1, \det(A) = 1\}$, which was most impressively encountered to the author in his youth 1983 upon an occasion of reading [YS, Chap.II, §2]: Define special matrices

$$\mathrm{Rot}_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad \mathrm{Rot}_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

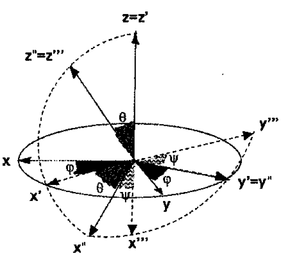
Then, every space rotation in $SO(3)$ can be written as

$$A_{\varphi,\theta,\psi} = \mathrm{Rot}_3(\varphi)\mathrm{Rot}_2(\theta)\mathrm{Rot}_3(\psi) \quad (0 \leq \varphi, \psi \leq 2\pi,\ 0 \leq \theta \leq \pi),$$

uniquely with only exceptions $A_{\varphi,\theta,\psi} = A_{\varphi+\alpha,\theta,\psi-\alpha}$ for $\theta \in \{0, \pi\}$ and $\alpha \in \mathbb{R}$. The above composition of three rotation matrices may be interpreted more naturally if it is read from the left to the right moving xyz-coordinates

$$(x, y, z) \xrightarrow{\varphi} (x', y', z') \xrightarrow{\theta} (x'', y'', z'') \xrightarrow{\psi} (x''', y''', z''')$$

as illustrated in the picture.



2. Eisenstein periods

Let $\mathfrak{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ be the complex upper half plane on which $SL_2(\mathbb{Z})$ acts in the usual way. For each $\boldsymbol{x} = \frac{\boldsymbol{u}}{N} = (\frac{u}{N}, \frac{v}{N}) \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$, we have the holomorphic Eisenstein series of weight 2 and 'label' $\boldsymbol{x}$ on $\mathfrak{H}$ defined by

$$E_2^{(\boldsymbol{x})}(\tau) := \sum_{\boldsymbol{a} \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{e^{2\pi i \det(\substack{\boldsymbol{a} \\ \boldsymbol{x}})}}{(2\pi i)^2} \left(\sideset{}{'}\sum_{\substack{(m_1,m_2) \equiv \boldsymbol{a} \\ \bmod N}} \frac{1}{(m_1\tau + m_2)^2} \cdot \frac{1}{|m_1\tau + m_2|^s} \right)_{s \to 0}.$$

The classcial Eisenstein periods of $E_2^{(\boldsymbol{x})}$ for those $\boldsymbol{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ are well known to be encoded in what are called the (generalized) Rademacher functions $\Phi_{\boldsymbol{x}} : SL_2(\mathbb{Z}) \to \mathbb{Q}$, which are good extensions of the period mapping $A \mapsto \int_z^{Az} E_2^{(\boldsymbol{x})}(\tau)d\tau$ for $A \in \Gamma(N)$ with $N\boldsymbol{x} \in \mathbb{Z}^2$. The value of $\Phi_{\boldsymbol{x}}(A) \in \mathbb{Q}$ for every $\boldsymbol{x} \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$ and $A \in SL_2(\mathbb{Z})$ is explicitly calculated in terms of Bernoulli polynomials and Dedekind sums (B.Schoeneberg [Sc74]).

Based on our recent work [N13], we can introduce a (profinite) *combinatorial avatar* "$\mathbb{E}_{\boldsymbol{x}}$" of $\Phi_{\boldsymbol{x}} : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{Q}$. Here, we consider the label $\boldsymbol{x}$ to lie in $\mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$ (adelic row vectors) and replace $\mathrm{SL}_2(\mathbb{Z})$ by a certain profinite group $\pi_1^{\text{ét}}(\mathfrak{M})$ which is:

(1) in the form of a semi-direct product $G_{\mathbb{Q}} \ltimes \hat{B}_3$ of two profinite groups, where $\hat{B}_3$ is a central extension of $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ and $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$;

(2) equipped with a standard representation $\rho : \pi_1^{\text{ét}}(\mathfrak{M}) \to \mathrm{GL}_2(\hat{\mathbb{Z}})$;

as explained soon in more details. Throughout below, we write $A_\sigma \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ for the transposed matrix of $\rho(\sigma)$:

$$A_\sigma = {}^t\rho(\sigma) \qquad (\sigma \in \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3).$$

The main aim of the present article is to illustrate roughly a use of "moving frames" idea to get the following composition law for our invariant $\mathbb{E}_{\boldsymbol{x}}$:

Theorem 2.1 (Composition law [N16b]). *Let $\boldsymbol{x} \in \mathbb{Q}_f^2 := (\mathbb{Q} \otimes \hat{\mathbb{Z}})^2$. Then,*

$$\mathbb{E}_{\boldsymbol{x}}(\sigma_1\sigma_2) = \mathbb{E}_{\boldsymbol{x}A_{\sigma_2}}(\sigma_1) + \det(A_{\sigma_1})\mathbb{E}_{\boldsymbol{x}}(\sigma_2)$$

holds for $\sigma_1, \sigma_2 \in \pi_1^{\acute{e}t}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$. □

Before going further, we quickly introduce a relation between the classical period $\Phi_{\boldsymbol{x}}$ and our avatar $\mathbb{E}_{\boldsymbol{x}}$. Just for now, we recall that the discrete Artin braid group B_3 with three strands fits in a central extension

$$\begin{array}{ccccccc} 1 \to \mathbb{Z} & \to \widetilde{\mathrm{SL}_2(\mathbb{Z})} \cong & B_3 & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{Z}) & \to 1. \\ & & \cup\!\!\!\!\! \text{\small I} & & \cup\!\!\!\!\! \text{\small I} & \\ & & \sigma & \mapsto & \rho(\sigma) & \end{array} \tag{2.2}$$

As seen later in §3, the above ρ extends to a continuous homomorphism

$$\rho : \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3 \longrightarrow \mathrm{GL}_2(\hat{\mathbb{Z}})$$

representing the monodromy actions on the torsion points of an elliptic curve.

If σ lies in the discrete part B_3 of $\hat{B}_3 \subset \pi_1^{\text{ét}}(\mathfrak{M})$, then $\rho(\sigma)$ and A_σ lie in $\mathrm{SL}_2(\mathbb{Z})$.

The following theorem is based on our work [N13].

Theorem 2.3. *One can introduce* $\mathbb{E}_{\boldsymbol{x}}(\sigma) \in \hat{\mathbb{Z}}$ *for* $\sigma \in \pi_1^{\acute{e}t}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3$ *and* $\boldsymbol{x} \in \mathbb{Q}_f^2$ *in a purely combinatorial way (Fox calculus) so that when* $\boldsymbol{x} \in \mathbb{Q}^2$ *and* $\sigma \in B_3$ *with* $A_\sigma \in \mathrm{SL}_2(\mathbb{Z})$,

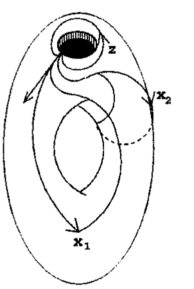
$$\begin{array}{ccccc} \mathbb{E}_{\boldsymbol{x}}(\sigma) & = & -\Phi_{\boldsymbol{x}}(A_\sigma) & + & (\textit{explicit error term}). \\ \Cap & & \Cap & & \Cap \\ \mathbb{Z} & & \mathbb{Q} & & \mathbb{Q} \end{array} \qquad \square$$

Remark 2.4. It is noteworthy to observe that the above error term sweeps out the denominator of $\Phi_{\boldsymbol{x}}(A_\sigma) \in \mathbb{Q}$ to obtain an integer value $\mathbb{E}_{\boldsymbol{x}}(\sigma) \in \mathbb{Z}$. The explicit form of the error term '$K_{\boldsymbol{x}}(A_\sigma) - \frac{1}{12}\rho_\Delta(\sigma)$' is calculated in [N13, Th.7.2.3]. As a consequence, it follows, e.g., that the denominator of $\Phi_{(\frac{u}{N},\frac{v}{N})}(A)$ for $A \in \mathrm{SL}_2(\mathbb{Z})$ is bounded by $12N^2$.

3. Combinatorics in $\hat{F}_2 = \hat{\pi}_{1,1}$

In order to introduce our combinatorial avatar of Eisenstein periods, we shall set up the universal elliptic curves $E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\}$ over the parameter space $\mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\}$. We consider both $E \setminus \{O\}$ and $\mathfrak{M}$ as affine varieties over $\mathbb{Q}$. The natural projection $E \setminus \{O\} \to \mathfrak{M}$ is the Weierstrass family of elliptic curves whose structured chart from a viewpoint of anabelian geometry was discussed in [N13, §5]. In summary, we have a tangential section $\tilde{w} : \mathfrak{M} \dashrightarrow E \setminus \{O\}$ (normalized with $t := -2x/y$) and a tangential fiber $\mathrm{Tate}(q) \hookrightarrow E \setminus \{O\}$. Using the van-Kampen construction of the Tate curve, we also introduced standard loops $\mathbf{x}_1, \mathbf{x}_2, \mathbf{z}$ of $\hat{\pi}_{1,1} := \pi_1^{\text{ét}}(\mathrm{Tate}(q) \otimes \overline{\mathbb{Q}})$ based at $\mathrm{Im}(\tilde{w}) \cap \mathrm{Tate}(q)$ on $E(\mathbb{C}) \setminus \{O\}$ with $[\mathbf{x}_1, \mathbf{x}_2]\mathbf{z} = 1$ ($[\mathbf{x}_1, \mathbf{x}_2] := \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1^{-1}\mathbf{x}_2^{-1}$). Note that $\hat{\pi}_{1,1}$ is isomorphic to a free profinite group $\hat{F}_2$ freely generated by $\mathbf{x}_1, \mathbf{x}_2$.

$$\begin{array}{ccc} E \setminus \{O\} := \{y^2 = 4x^3 - g_2x - g_3\} & \hookleftarrow\!\!- - & \mathrm{Tate}(q) \\ \big\downarrow\!\!\big\uparrow \tilde{w} & & \big\downarrow\!\!\big\uparrow \\ \mathfrak{M} := \{(g_2, g_3) \mid \Delta := g_2^3 - 27g_3^2 \neq 0\} & \hookleftarrow\!\!- - & \mathrm{Spec}\, \mathbb{Q}((q)) \end{array}$$



It is natural to employ the images of $\mathrm{Spec}\, \mathbb{Q}((q))$ as base points of those étale fundamental groups of individual spaces in the above diagram. Then, we obtain the basic identification :

$$\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}, \quad \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \ltimes \hat{B}_3.$$

In fact, the moduli space $\mathfrak{M}$ is naturally interpreted as the space of (normalized) cubics, and a topological loop in $\pi_1(\mathfrak{M}(\mathbb{C}))$ is a motion of three points on the plane: we may identify $\pi_1(\mathfrak{M}(\mathbb{C}))$ with the Artin braid group B_3 of three strands, consequently, $\pi_1^{\text{ét}}(\mathfrak{M})$ as the semidirect product of $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the profinite completion $\hat{B}_3$.

The conjugate action in the above splitting $\pi_1^{\text{ét}}(E \setminus \{O\}) = \pi_1^{\text{ét}}(\mathfrak{M}) \ltimes \hat{\pi}_{1,1}$ induces the monodromy action of $\pi_1^{\text{ét}}(\mathfrak{M})$ on $\hat{\pi}_{1,1} = \hat{F}_2$:

$$\begin{array}{ccccc} \pi_1^{\text{ét}}(\mathfrak{M}) = G_{\mathbb{Q}} \rtimes \hat{B}_3 & \xrightarrow{\varphi} & \mathrm{Aut}^*(\hat{F}_2) & \xrightarrow{\bmod \hat{F}_2'} & \mathrm{GL}(\hat{\mathbb{Z}}^2) \\ \Cup & & \Cup & & \Cup \\ \sigma & \longmapsto & \varphi(\sigma) & \longmapsto & \rho(\sigma) \end{array}$$

where $\mathrm{Aut}^*(\hat{F}_2)$ denotes the group of *special automorphisms* defined by

$$\mathrm{Aut}^*(\hat{F}_2) = \{\sigma \in \mathrm{Aut}(\hat{F}_2) \mid \sigma(\langle \mathrm{z} \rangle) = \langle \mathrm{z} \rangle\}.$$

Given $m \geq 1$, $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$ and $(u, v) \in \hat{\mathbb{Z}}^2$, let $\rho(\sigma) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and set

$$\mathcal{S}_{uv}(\sigma) := \sigma(\mathbf{x}_2^{-v}\mathbf{x}_1^{-u})\mathbf{x}_1^{au+bv}\mathbf{x}_2^{cu+dv} \in \hat{F}_2' := [\hat{F}_2, \hat{F}_2].$$

By Ihara's theory (cf. [I99]), with the class of $\mathcal{S}_{uv}(\sigma)$ in the 2nd derived quotient $\hat{F}_2'/\hat{F}_2''$, we may associate a unique element of the complete group algebra $\hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] = \varprojlim_m \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$, where $\bar{\mathbf{x}}_1$, $\bar{\mathbf{x}}_2$ designate the abelianization images of $\mathbf{x}_1, \mathbf{x}_2 \in \hat{F}_2$ respectively. In order to explain this procedure in a more fitting form with the moving frame idea, it is useful to introduce a sequence of maps composed of the Fox derivative ∂_{x_1} with projections

$$\hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\partial_{x_1}} \hat{\mathbb{Z}}[[\hat{F}_2]] \xrightarrow{\text{ab}} \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}^2]] \xrightarrow{\bmod m} \frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$$

and, writing any element of $\frac{\hat{\mathbb{Z}}[\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2]}{(\bar{\mathbf{x}}_1^m - 1, \bar{\mathbf{x}}_2^m - 1)}$ as $\sum_{i,j=0}^{m-1} c_{ij}\bar{\mathbf{x}}_1^i\bar{\mathbf{x}}_2^j$, define

$$\mathbb{E}_m(\sigma; u, v) := \text{constant term } c_{00} \text{ of } \left[\frac{[\partial_{\mathbf{x}_1}(\mathcal{S}_{uv}(\sigma))]^{\text{ab}}}{\bar{\mathbf{x}}_2 - 1}\right]_{\bar{\mathbf{x}}_1^m = \bar{\mathbf{x}}_2^m = 1} \left(\in \hat{\mathbb{Z}}\right).$$

Cf. [N13, (3.2.3)].

Proposition 3.1 ([N16b], Theorem A). *It holds that*

$$\mathbb{E}_m(\sigma_1\sigma_2; \boldsymbol{u}) = \mathbb{E}_m(\sigma_1; \boldsymbol{u}A_{\sigma_2}) + (\det\rho(\sigma_1)) \cdot \mathbb{E}_m(\sigma_2; \boldsymbol{u})$$

for $\sigma_1, \sigma_2 \in \mathrm{Aut}^*(\hat{F}_2)$ *and* $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$.

Proof motivation of the above composition law: Given any $\sigma \in \pi_1^{\text{ét}}(\mathfrak{M})$, view the data $\mathbb{E}_m(\sigma) := \left[\mathbb{E}_m(\sigma; u, v)\right]_{(u,v)\in\hat{\mathbb{Z}}^2}$ as a profinite tableau on the plane $\hat{\mathbb{Z}}^2$ with entries $\hat{\mathbb{Z}}$. Let us consider traveling in $\hat{F}_2$ (with portable $\mathbb{E}_m$-board in one hand) along the composition of two automorphisms $\sigma \circ \tau \in \mathrm{Aut}^*(\hat{F}_2)$ and observe effects on the $\mathbb{E}_m$. Noting that the definition of $\mathbb{E}_m$ depends entirely on the choice of free generator system $\underline{\mathbf{x}} = (\mathbf{x}_1, \mathbf{x}_2)$ of $\hat{F}_2$, we are urged to look closely at the diagram

(3.2)
$$\begin{array}{ccccc} & & \overset{\sigma\tau}{\curvearrowleft} & & \\ \hat{F}_2 & \underset{\tau(\underline{\mathbf{x}})}{\overset{\sigma}{\longleftarrow}} & \hat{F}_2 & \underset{\underline{\mathbf{x}}}{\overset{\tau}{\longleftarrow}} & \hat{F}_2 \\ {\scriptstyle\tau}\uparrow & & \downarrow{\scriptstyle\tau^{-1}} & & \\ \hat{F}_2 & \underset{\underline{\mathbf{x}}}{\overset{\sigma'}{\longleftarrow}} & \hat{F}_2 & & \end{array}$$

and especially at the effect of σ with regard to the moved frame $\tau(\underline{\mathbf{x}}) = (\tau(\mathbf{x}_1), \tau(\mathbf{x}_2))$. In fact, one symbolically finds

$$\mathcal{S}_{\boldsymbol{u}}(\sigma\tau) = \mathcal{S}_{\boldsymbol{u}}(\sigma; \text{“rel.}\tau(\underline{\mathbf{x}})\text{”}) \cdot \mathcal{S}_{\sigma'\boldsymbol{u}}(\tau)$$

which approximately leads to

$$\mathbb{E}_m(\sigma\tau, \boldsymbol{u}) \approx (\det\rho(\tau)) \cdot \mathbb{E}_m(\sigma', \boldsymbol{u}) + \mathbb{E}_m(\tau, \boldsymbol{u}A_{\sigma'}).$$

Proposition 3.1 follows then by rewriting: $\sigma_2 = \sigma' = \tau^{-1}\sigma\tau$, $\sigma_1 = \tau$ so that $\sigma_1\sigma_2 = \sigma\tau$. □

Remark 3.3. In [N13], it is shown that the adelic tableau $\mathbb{E}_m(\sigma) \in \hat{\mathbb{Z}}^{\hat{\mathbb{Z}}^2}$ encodes the image of σ by $\pi_1^{\text{ét}}(\mathfrak{M}) \to \mathrm{Aut}^*(F_2/F_2'')$ (the meta-abelian monodromy).

4. Some applications

Let us briefly pick up a few topics from [N16b].

4.1. **Homogeneity.** The above composition law Proposition 3.1 leads us to the following basic property:

Corollary 4.1 (Homogeneity [N16b] Theorem C). *Let* $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$, $\sigma \in \pi_1^{\acute{e}t}(\mathfrak{M})$. *Then, for each positive integer* $k \in \mathbb{N}$, *it holds that*

$$\mathbb{E}_m(\sigma, \boldsymbol{u}) = \mathbb{E}_{mk}(\sigma, k\boldsymbol{u}).$$

In fact, by virtue of Proposition 3.1, expressing σ as a product of $\sigma_1 \in G_{\mathbb{Q}}$ and $\sigma_2 \in \hat{B}_3$, we may reduce the proof of Corollary to individual cases where $\sigma \in G_{\mathbb{Q}}$ or $\sigma \in \hat{B}_3$. In the latter case, since $B_3 \times \mathbb{Z}^2$ is dense in $\hat{B}_3 \times \hat{\mathbb{Z}}^2$, the result follows from the explicit formula of $\mathbb{E}_{km}(\sigma, k\boldsymbol{u})$ for $\sigma \in B_3$, $\boldsymbol{u} \in \mathbb{Z}^2$ given in Theorem 2.3 (cf. [N13, Th. 7.2.3]). In the former case where $\sigma \in G_{\mathbb{Q}}$, the result follows from an explicit calculation of $\mathbb{E}_m(\sigma, \boldsymbol{u})$ which is based on the Grothendieck-Teichmüller theory on $\pi_1^{\text{ét}}(\text{Tate}(q) \setminus O)$ (see [N16b]).

The above corollary allows us to define the "*adelic Eisenstein function*" $\mathbb{E}_{\boldsymbol{x}}(\sigma)$:

$$\pi_1^{\text{ét}}(\mathfrak{M}) \times \mathbb{Q}_f^2 \ni (\sigma, \boldsymbol{x}) \longmapsto \mathbb{E}_{\boldsymbol{x}}(\sigma) \in \hat{\mathbb{Z}}$$

by assigning $\mathbb{E}_m(\sigma, \boldsymbol{u})$ for any choice of $m \in \mathbb{N}$ and $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$ so that $\boldsymbol{x} = \frac{\boldsymbol{u}}{m} \in \mathbb{Q}_f^2$. Then, Theorem 2.1 is only the reload of Proposition 3.1.

4.2. **Level splitter homomorphism** ([N16b, §7]). Let m, M be positive integers and set $N = \gcd(2, M) \cdot M$. We define the principal congruence subgroup of level N by $\pi_1^{\text{ét}}(\mathfrak{M})[N] := \{\sigma \mid A_\sigma \equiv \mathbf{1} \bmod N\}$. Then, combining results of [N12], [N13] and [N16b], we see that $\mathbb{E}_m(\sigma, \boldsymbol{u}) \bmod M$ has $m \times m$-periodicity in $\boldsymbol{u} \in \hat{\mathbb{Z}}^2$, hence that it induces a homomorphism

$$\mathbb{E}_{m]M} : \pi_1^{\text{ét}}(\mathfrak{M})[mN] \to (\mathbb{Z}/M\mathbb{Z})[(\mathbb{Z}/m\mathbb{Z})^2].$$

Generally, the above level splitter $\mathbb{E}_{m]M}$ affords a non-trivial abelian quotient of $\pi_1^{\text{ét}}(\mathfrak{M})[N]$ and should involve highly arithmetic information about "Eisenstein quotient". We hope to discuss it in more details on some other occasion.

Acknowledgment: The present note is based on the author's talks delivered in Keio University (June 26, 2015), Lille University (June 16, 2016) as well as in the workshop of the volume "Various Aspects of Multiple Zeta Values" held at RIMS, Kyoto Univ. (July 11-14, 2016). The author thanks the organizers for kind invitations.

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