SOME RESULTS ON THE ISOVARIANT BORSUK-ULAM CONSTANTS

Ikumitsu NAGASAKI
Department of Mathematics
Kyoto Prefectural University of Medicine

ABSTRACT. In the previous article [4], we introduced the isovariant Borsuk-Ulam constant of a compact Lie group and provided an estimate of this constant for the unitary group U(n). In this article, we shall continue the study of the isovariant Borsuk-Ulam constants for simple compact Lie groups and announce some results of [5].

1. REVIEW OF THE ISOVARIANT BORSUK-ULAM CONSTANT

Let G be a compact Lie group. A (continuos) $G$-map $f : X \to Y$ between $G$-spaces is called $G$-isovariant if $f$ preserves the isotropy groups; i.e., $G_{f(x)} = G_x$ for every $x \in X$. The isovariant Borsuk-Ulam theorem was first studied by A. G. Wasserman [9]. In particular, the following result is deduced from Wasserman’s results.

**Theorem 1.1** (Isovariant Borsuk-Ulam theorem). Let $G$ be a solvable compact Lie group. If there exists a $G$-isovariant map $f : S(V) \to S(W)$ between linear $G$-spheres, then

$$\dim V - \dim V^G \leq \dim W - \dim W^G$$

holds.

We call $G$ a Borsuk-Ulam group (BUG for short) if the isovariant Borsuk-Ulam theorem holds for $G$. Therefore solvable $G$ is a Borsuk-Ulam group. A fundamental problem is: Which groups are Borsuk-Ulam groups? This is not completely solved; however, several examples are known, see [6, 7, 9]. Wasserman also conjectures that all finite groups are Borsuk-Ulam groups. On the other hand, a connected compact Lie group being a Borsuk-Ulam group other than a torus is not known.

In [4], we introduced the isovariant Borsuk-Ulam constant $c_G$ as follows.

---

2010 Mathematics Subject Classification. Primary 55M20; Secondary 57S15, 57S25.

Key words and phrases. isovariant Borsuk-Ulam theorem; Borsuk-Ulam group; isovariant Borsuk-Ulam constant; isovariant map; representation theory.
**Definition.** The isovariant Borsuk-Ulam constant $c_G$ of $G$ is defined to be the supremum of $c \in \mathbb{R}$ such that:

If there is a $G$-isovariant map $f : S(V) \to S(W)$, then

$$c(\dim V - \dim V^G) \leq \dim W - \dim W^G$$

holds. (If $G = 1$, then set $c_G = 1$ as convention.)

Clearly $c_G = 1$ if and only if $G$ is a Borsuk-Ulam group.

In equivariant case, the (equivariant) Borsuk-Ulam constant $a_G$ is introduced and studied by Bartsch [2]. In particular, if $G$ is not a $p$-toral group, then $a_G = 0$. Contrary to this, in section 3, we present the positivity of $c_G$ for any compact Lie group $G$.

We here recall some properties of $c_G$ that are generalization of Wasserman's results. The detail is described in [5].

**Proposition 1.2.**

1. If $1 \to K \to G \to Q \to 1$ is an exact sequence of compact Lie groups, then

$$\min\{c_K, c_Q\} \leq c_G \leq c_Q.$$  

In particular, if $c_K = 1$, then $c_G = c_Q$.

2. If $1 = H_0 \triangleleft H_1 \triangleleft H_2 \triangleleft \cdots \triangleleft H_r = G$, then

$$\min_{1 \leq i \leq r}\{c_{H_i/H_{i-1}}\} \leq c_G.$$  

Using this proposition, we have

**Corollary 1.3.** $c_{G_1 \times \cdots \times G_r} = \min_i\{c_{G_i}\}$.

**Corollary 1.4.** Let $G$ be a connected compact Lie group. Then $c_G = \min_i\{G_i\}$, where $G_i$ are simple factors of $G$.

2. **Main results — Estimates of $c_G$**

Let $G$ be a simple compact Lie group. Let $T$ denote the maximal torus $T$ of $G$. We set

$$d_G = \sup \left\{ \frac{\dim U^T}{\dim U} \left| U : \text{nontrivial irreducible } G\text{-representation} \right. \right\},$$

called the zero weight ratio of $G$. The following is a key result for estimation of $c_G$.

**Proposition 2.1** ([5]). $c_G \geq K_G := 1 - d_G$.

By representation theory, $d_G$ can be determined, see [5] for the proof.

**Theorem 2.2.** The zero weight ratios are given in the following table.
SOME RESULTS ON THE ISOVARIANT BORSUK-ULAM CONSTANTS

<table>
<thead>
<tr>
<th>Type of $G$</th>
<th>$A_n ,(n \geq 1)$</th>
<th>$B_n ,(n \geq 2)$</th>
<th>$C_n ,(n \geq 3)$</th>
<th>$D_n ,(n \geq 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_G$</td>
<td>$\frac{1}{n+2}$</td>
<td>$\frac{1}{2n+1}$</td>
<td>$\frac{1}{2n+1}$</td>
<td>$\frac{1}{2n-1}$</td>
</tr>
<tr>
<td>$K_G$</td>
<td>$\frac{n+1}{n+2}$</td>
<td>$\frac{2n}{2n+1}$</td>
<td>$\frac{2n}{2n+1}$</td>
<td>$\frac{2n-2}{2n-1}$</td>
</tr>
</tbody>
</table>

**Table 1. Classical case**

<table>
<thead>
<tr>
<th>Type of $G$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_G$</td>
<td>$\frac{1}{13}$</td>
<td>$\frac{1}{19}$</td>
<td>$\frac{1}{31}$</td>
<td>$\frac{1}{13}$</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$K_G$</td>
<td>$\frac{12}{13}$</td>
<td>$\frac{18}{19}$</td>
<td>$\frac{30}{31}$</td>
<td>$\frac{12}{13}$</td>
<td>$\frac{6}{7}$</td>
</tr>
</tbody>
</table>

**Table 2. Exceptional case**

This implies the following isovariant Borsuk-Ulam type result. Set

$$d(V,W) = \frac{\dim W - \dim W^G}{\dim V - \dim V^G}.$$  

**Corollary 2.3.** If $d(V,W) < K_G$ for $G$ simple, then there is no $G$-isovariant map $f : S(V) \to S(W)$.

3. **Remarks and Applications**

As a consequence of Theorem 2.2, $c_G > 0$ for connected $G$. In [3], we also see that $c_G > 0$ for finite $G$. Therefore we obtain a positivity result on $c_G$ by Proposition 1.2.

**Corollary 3.1.** $c_G > 0$ for any compact Lie group $G$.

This implies that the weak isovariant Borsuk-Ulam theorem holds for any $G$ which was first proved in [3]. We recall the weak isovariant Borsuk-Ulam theorem.

**Definition** (Isovariant Borsuk-Ulam function $\varphi_G : \mathbb{N} \to \mathbb{N}$). $\varphi_G(n)$ is defined as the minimum of $\dim W - \dim W^G$ such that there exists a $G$-isovariant maps $f : S(V) \to S(W)$ with $\dim V - \dim V^G \geq n$.

**Proposition 3.2.** (1) If $n \leq m$, then $\varphi_G(n) \leq \varphi_G(m)$. 
SOME RESULTS ON THE ISOVARIANT BORSUK-ULAM CONSTANTS

(2) $\varphi_G(n+m) \leq \varphi_G(n) + \varphi_G(m)$ (subadditivity).

(3) $\varphi_G(n) \leq n$ for $n \in D_G := \{ \dim V | V^G = 0 \}$.

From the definition of $c_G$, one can see

**Proposition 3.3.**

(1) 
$$c_G = \lim_{n \to \infty} \frac{\varphi_G(n)}{n} = \inf_n \frac{\varphi_G(n)}{n}.$$  

(2) 
$$\varphi(n) \geq c_G n \text{ for } n \in D_G.$$  

**Definition.** We say that the weak isovariant Borsuk-Ulam theorem holds for $G$ if

$$\lim_{n \to \infty} \varphi_G(n) = \infty.$$  

Clearly the positivity of $c_G$ shows

**Corollary 3.4 ([3]).** The weak isovariant Borsuk-Ulam theorem holds for any $G$.

Bartsch [1] showed that when $G$ is finite, the weak Borsuk-Ulam theorem holds for $G$ if and only if $G$ is a finite $p$-group. Our result is an isovariant version of Bartsch’s result.

As an application of the positivity of $c_G$, one can see another isovariant Borsuk-Ulam type theorem using by a similar argument of [1].

**Corollary 3.5.** Let $G$ be a compact Lie group. Then there is no $G$-isovariant map $f : S(V) \to S(W)$ for $W \subsetneq V (V^G = 0)$.

**Remark.** This is an isovariant version of Bartsch’s result that there is no $G$-map $f : S(V) \to S(W)$ for $W \subsetneq V (V^G = 0)$ if and only if $G$ is a $p$-toral, where $G$ is called a $p$-toral if $G$ has an exact sequence $1 \to T \to G \to P \to 1$, $T$: torus, $P$: finite $p$-group.

Also, an isovariant version of an infinite Borsuk-Ulam type theorem holds.

**Corollary 3.6.** Let $G$ be a compact Lie group. Suppose that $\dim V = \infty$ and $\dim V^G < \infty$. If there exists a $G$-isovariant map $f : S(V) \to S(W)$, then $\dim W = \infty$.

**Proof.** Suppose $\dim W < \infty$. The Peter-Weyl theorem [8] shows that there exists a finite-dimensional subrepresentation $V'$ of $V$ with arbitrary higher dimension. Then there exists a $G$-isovariant map $f' := f_{|S(V')} : S(V') \to S(W)$; however, this contradicts $c_G > 0$.  $\square$
SOME RESULTS ON THE ISOVARIANT BORSUK-ULAM CONSTANTS

REFERENCES


DEPARTMENT OF MATHEMATICS, KYOTO PREFECTURAL UNIVERSITY OF MEDICINE, 1-5 SHIMO-GAMO HANGI-CHO, SAKYO-KU, KYOTO 606-0823, JAPAN
E-mail address: nagasaki@koto.kpu-m.ac.jp