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A unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter

By

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Abstract

The purpose of this note is to give a unified family of P_{J}-hierarchies (J=I, II, IV, 34) with a large parameter. This note is a short summary of papers [18] and [19].

§ 1. A unified family of P_{J}-hierarchies (J=I, II, IV, 34)

The P_{I}, P_{II}, P_{IV} and P_{34}-hierarchies were studied by Kudryashov ([11],[12]), Gordoa and Pickering ([4]), Shimomura ([13],[14]), Gordoa, Joshi and Pickering ([5]), Clarkson, Joshi and Pickering ([3]). To establish connection formulas for solutions of the higher order Painlevé equations is one of important subjects in algebraic analysis of singular perturbation theory. In the series of papers by Kawai, Koike, Nishikawa and Takei, they introduced a large parameter η to P_{J}-hierarchies $(\mathrm{J}=\mathrm{I}, \mathrm{II}, \mathrm{IV}, 34)$ ([6], [9], [10]) and many important results have been established from a view point of the exact WKB analysis. (See [6], [7], [8], [15], [16], [17] and etc). In what follows, we give a unified family of P_{J}-hierarchies $(\mathrm{J}=\mathrm{I}, \mathrm{II}, \mathrm{IV}, 34)$ with a large parameter η.

Let m be an arbitrary natural number. Let U, V and C denote generating functions of unknown functions u_k, v_k $(k=1, 2, \ldots, m)$ and constants c_k as follows.

$$U(\theta):=\sum_{k=1}^{m+1} u_k\theta^k,\ V(\theta):=\sum_{k=1}^{m+1} v_k\theta^k,\ C(\theta):=\sum_{k=1}^{m} c_k\theta^k.$$

Here θ denotes an independent variable, u_{m+1} and v_{m+1} are arbitrary holomorphic functions of t. Throughout the note, the notation $A\equiv B$ means that $A-B$ is zero

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modulo θ^{m+2}. Let us consider the system of non-linear ordinary differential equations with a large parameter η for these generating functions:

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \times (1-U) + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial H}{\partial U} \\ \frac{\partial H}{\partial V} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{H(U,V)}{1-U} \end{pmatrix}, \tag{1.1}$$

where $H(U, V)$ is a polynomial in U and V with arbitrary complex constants p_i of the following form

$$H(U, V) := (p_1U^2 + p_2V^2)\theta + p_3UV + p_4CU + p_5CV + p_6U + p_7V + p_8C + p_9,$$

and f_1, f_2 are defined by

$$\begin{aligned} f_1 &:= p_7 + (\alpha u_1 + p_5 c_1)\,\theta + (y_1 + (y_1u_1 + y_2)\theta)\theta^m, \\ f_2 &:= -\beta - (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\,\theta + (z_1 + (2z_1u_1 - y_1v_1 + z_2)\theta)\,\theta^m. \end{aligned}$$

Here y_i, z_i are arbitrary holomorphic functions of t and α, β, ε are given by

$$\alpha := p_3 + p_7, \quad \beta := p_6 + p_9 \quad \text{and} \quad \varepsilon := p_4 + p_8,$$

respectively.

If p_i, y_i, z_i are determined as follows, then (1.1) is same as the general member $(P_{\mathrm{J}})_m$ of P_{J}-hierarchy with η (See [19]).

- If $p_2 = -1$, $p_8 = 2$, $p_9 = 1$, $z_2 = 2t$, the others $= 0 \Longrightarrow (P_{\mathrm{I}})_m$.
- If $p_2 = -1$, $p_8 = 2$, $p_9 = 1$, $z_1 = 2\gamma t\,(\gamma \neq 0)$, $z_2 = 4\gamma t c_0$, the others $= 0 \Longrightarrow (P_{34})_m$.
- If $p_2 = 1$, $p_3 = 2$, $p_5 = 2$, the others $= 0 \Longrightarrow (P_{\mathrm{II}})_m$.
- If $p_2 = 1$, $p_3 = 2$, $p_5 = 2$, $y_1 = -2\gamma t$ $(\gamma \neq 0)$, the others $= 0 \Longrightarrow (P_{\mathrm{IV}})_m$.

§ 2. The existence of general formal solutions of (1.1)

We can apply the method given in [2] to the cases I, II:

Case I: $\alpha = p_3 + p_7 \neq 0$, $p_2 \neq 0$.

Case II: $\alpha = p_3 + p_7 = 0$, $\beta = p_6 + p_9 \neq 0$, $p_2 \neq 0$

and we have the following theorem. (For more precise statements, see [19].)

Theorem 2.1. *In the cases I, II, we have formal solutions with* $2m$ *free parameters called instanton-type solutions for* (1.1).

§ 3. Lax pair for (1.1)

Theorem 3.1. *Let us determine* p_1, u_{m+1} *and* v_{m+1} *of* (1.1) *so that they satisfy the following conditions.*

$$\begin{cases} p_1 = 0, \quad p_2 \neq 0, \\ \gamma\alpha\theta^{k-2} = -\alpha u'_{m+1}\theta^{m+1} + (y'_1\theta^m + y'_2\theta^{m+1}), \\ z'_1\theta^m + (z'_1u_1 - y'_1v_1 + z'_2)\theta^{m+1} + (2\beta u'_{m+1} + \alpha v'_{m+1})\theta^{m+1} + \gamma\beta\theta^{k-2} = 0. \end{cases}$$

Here u'_{m+1} *denotes the derivative of* u_{m+1} *with respect to* t, γ *is a non-zero constant, and* k *is determined by the above conditions. Then our system* (1.1) *is equivalent to the compatibility condition of the following equations:*

$$\text{(I)} \quad \left(\gamma\theta^k\frac{\partial}{\partial\theta} - \eta A\right)\psi(\theta, t) = 0, \qquad \text{(II)} \quad \left(\frac{\partial}{\partial t} - \eta B\right)\psi(\theta, t) = 0,$$

where

$$A := \begin{pmatrix} \triangle_1 & (1-U)\theta \\ \triangle_2 & -\triangle_1 \end{pmatrix}, \qquad B := \begin{pmatrix} \Box_1 & 1 \\ \Box_2 & -\Box_1 \end{pmatrix}$$

with

$$\begin{aligned} \triangle_1 &:\equiv -\frac{1}{2}\frac{\partial H}{\partial V} - \frac{p_3}{2}(1-U) + \frac{1}{2}(y_1\theta^m + y_2\theta^{m+1}) - \frac{\alpha}{2}u_{m+1}\theta^{m+1}, \\ \triangle_2 &:\equiv p_2 \times \left(-\frac{\partial H}{\partial U} - \frac{H(U,V)}{1-U} - (z_1\theta^m + (z_1u_1 - y_1v_1 + z_2)\theta^{m+1}) \right. \\ &\qquad \left. - (2\beta u_{m+1} + \alpha v_{m+1})\theta^{m+1}\right), \\ \Box_1 &:= -\frac{1}{2\theta}(\alpha + (\alpha u_1 + p_5c_1)\theta), \\ \Box_2 &:= -\frac{p_2}{\theta}(\beta + (2\beta u_1 + \alpha v_1 + \varepsilon c_1)\theta). \end{aligned}$$

The Lax pair associated with (1.1) plays an important role in analyzing the Stoke geometry of (1.1) (See [18]).

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