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On mean-field approximation of particle systems with annihilation and spikes

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On a filtered probability space let us consider the following interactions of $N(\geq 2)$ Brownian particles each of which diffuses on the nonnegative half line $\mathbb{R}_{+}$ and is attracted towards the average position of all the particles. When a particle i attains the boundary 0, it is annihilated (default) and a new particle (also called i) spikes immediately in the middle of particles. More precisely, let us denote by $X_{t} := (X_{t}^{1}, \ldots, X_{t}^{N})$ the positions of these particles, where $X_{t}^{i}(\geq 0)$ is the position of particle i at time $t\geq 0$ for $i=1, \ldots, N$. With the average $\overline{X}_{t} := (X_{t}^{1}+\cdots+X_{t}^{N})/N$ the dynamics of the system is determined by

$$\begin{aligned} X_{t}^{i} &= X_{0}^{i}+\int_{0}^{t}b(X_{s}^{i}, \overline{X}_{s})\mathrm{d}s+W_{t}^{i}+\int_{0}^{t}\overline{X}_{s-}\Big(\mathrm{d}M_{s}^{i}-\frac{1}{N}\sum_{j\neq i}\mathrm{d}M_{s}^{j}\Big); \quad t\geq 0, \\ M_{t}^{i} &:= \sum_{k=1}^{\infty}1_{\{\tau_{k}^{i}\leq t\}}, \quad \tau_{k}^{i} := \inf\Big\{s>\tau_{k-1}^{i} : X_{s-}^{i}-\frac{\overline{X}_{s-}}{N}\sum_{j\neq i}(M_{s}^{j}-M_{s-}^{j})\leq 0\Big\}, \end{aligned} \tag{1}$$

for $i=1, \ldots, N, k\in \mathbb{N}$, where $W_{t} := (W_{t}^{1}, \ldots, W_{t}^{N}), t\geq 0$ is an N-dimensional Brownian motion, M_{t}^{i} is the cumulative number of defaults by time $t\geq 0$, τ_{k}^{i} is the k-th default time with $\tau_{0}^{i}=0$ of particle i. Here we assume that $b:\mathbb{R}_{+}^{2}\rightarrow \mathbb{R}$ is (globally) Lipschitz continuous, i.e., there exists a constant $\kappa>0$ such that

$$|b(x_{1}, m_{1})-b(x_{2}, m_{2})|\leq\kappa(|x_{1}-x_{2}|+|m_{1}-m_{2}|) \tag{2}$$

for all $x_{1}, x_{2}, m_{1}, m_{2}\in \mathbb{R}_{+}$, and we also impose the condition

$$\sum_{i=1}^{N}b(x^{i}, \overline{x}) \equiv 0 \tag{3}$$

for every $x := (x^{1}, \ldots, x^{N})\in \mathbb{R}_{+}^{N}$ and $\overline{x} := (x^{1}+\cdots+x^{N})/N$ on the drift function $b(\cdot, \cdot)$.

Given a standard Brownian motion $W.$ we shall consider a system $X. := (X_{\cdot}^{1}, \ldots, X_{\cdot}^{N})$, $M. := (M_{\cdot}^{1}, \ldots, M_{\cdot}^{N}))$ described by (1) with (2)-(3) on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} := (\mathcal{F}_{t}, t\geq 0)$. In particular, we are concerned with (1) that there might be multiple defaults at the same time with positive probability, i.e.,

$$\mathbb{P}\big(\exists(i, j)\ \exists t\in[0, \infty) \text{ such that } X_{t}^{i}=X_{t}^{j}=0\big)>0.$$

We shall construct a solution to (1) with a specific boundary behavior of defaults until the time $\overline{\tau}_{0} := \inf\{s>0 : \max_{1\leq i\leq N}X_{s}^{i}=0\}$. Let us define the following map $\Phi(x) := (\Phi^{1}(x), \ldots, \Phi^{N}(x)) : [0, \infty)^{N}\mapsto[0, \infty)^{N}$ and set-valued function $\Gamma : \mathbb{R}_{+}^{N}\rightarrow\{1, \ldots, N\}$ defined by $\Gamma_{0}(x) := \{i\in\{1, \ldots, N\} : x^{i}=0\}$,

$$\Gamma_{k+1}(x) := \Big\{i\in\{1, \ldots, N\}\backslash\bigcup_{\ell=1}^{k}\Gamma_{\ell}(x) : x^{i}-\frac{\overline{x}}{N}\cdot\Big|\bigcup_{\ell=1}^{k}\Gamma_{\ell}(x)\Big|\leq 0\Big\}; \quad k=0,1,2, \ldots, N-3$$

$$\Gamma(x) := \bigcup_{k=0}^{N-2} \Gamma_k(x), \quad \Phi^i(x) := x^i + \overline{x}\Big(\Big(1+\frac{1}{N}\Big)\cdot \mathbf{1}_{\{i\in\Gamma(x)\}} - \frac{1}{N}\cdot|\Gamma(x)|\Big) \tag{4}$$

for $x = (x^1, \ldots, x^N) \in \mathbb{R}_+^N$, $i = 1, \ldots, N$ with $\overline{x} := (x^1 + \cdots + x_N)/N \ge 0$. Note that $\Phi([0,\infty)^N \setminus \{\mathbf{0}\}) \subseteq [0,\infty)^N \setminus \{\mathbf{0}\}$ and $\Phi(\mathbf{0}) = \mathbf{0} = (0, \ldots, 0)$.

Lemma 1 ([3]). *Given a standard Brownian motion $W.$ and the initial configuration $X_0 \in (0,\infty)^N$ one can construct the process $(X., M.)$ which is the unique, strong solution to (1) with (2), (3) on $[0, \overline{\tau}_0]$, such that if there is a default, i.e., $|\Gamma(X_{t-})| \ge 1$ at time t, then the post-default behavior is determined by the process with $X_t^i = \Phi^i(X_{t-})$ for $i = 1, \ldots, N$.*

Now let us discuss the system (1) with (2)-(3) as a mean-field approximation for nonlinear equation of McKEAN-VLASOV type. For the sake of concreteness, let us assume $b(x,m) = -a(x-m)$, $x, m \in [0,\infty)$ for some $a > 0$. By the theory of propagation of chaos (e.g., TANAKA (1984), SHIGA & TANAKA (1985) and SZNITMAN (1991)) as $N \to \infty$, the dynamics of the finite-dimensional marginal distribution of limiting representative process is expressed by

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathbb{E}[\mathcal{X}_t])\mathrm{d}s + W_t + \int_0^t \mathbb{E}[\mathcal{X}_{s-}]\mathrm{d}(\mathcal{M}_s - \mathbb{E}[\mathcal{M}_s])\,; \quad t \ge 0, \tag{5}$$

where $W.$ is the standard Brownian motion, $\mathcal{M}_t := \sum_{k=1}^{\infty} \mathbf{1}_{\{\tau^k \le t\}}$, $\tau^k := \inf\{s > \tau^{k-1} : \mathcal{X}_{t-} \le 0\}$, $k \ge 1$, $\tau^0 = 0$. Then taking expectations of both sides of (5), we obtain $\mathbb{E}[\mathcal{X}_t] = \mathbb{E}[\mathcal{X}_0]$, $t \ge 0$. When $\mathcal{X}_0 = x_0$ a.s. for some $x_0 > 0$, substituting this back into (5), we obtain

$$\mathcal{X}_t = \mathcal{X}_0 - a\int_0^t (\mathcal{X}_s - \mathcal{X}_0)\mathrm{d}s + W_t + \mathcal{X}_0(\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t])\,; \quad t \ge 0.$$

Transforming the state space from $[0,\infty)$ to $(-\infty, 1]$ by $\widehat{\mathcal{X}}_t := (x_0 - \mathcal{X}_t)/x_0$, we see

$$\widehat{\mathcal{X}}_t = -\int_0^t a\widehat{\mathcal{X}}_s\mathrm{d}s + \widehat{W}_t - \widehat{\mathcal{M}}_t + \mathbb{E}[\widehat{\mathcal{M}}_t]\,; \quad t \ge 0, \tag{6}$$

where we denote $\widehat{W}. = W./x_0$, $\widehat{\mathcal{M}}. = \mathcal{M}.$.

This transformed process $\widehat{\mathcal{X}}.$ is similar to the nonlinear McKEAN-VLASOV-type stochastic differential equation

$$\widetilde{\mathcal{X}}_t = \widetilde{\mathcal{X}}_0 + \int_0^t \mathrm{b}(\widetilde{\mathcal{X}}_s)\mathrm{d}s + \widetilde{W}_t - \widetilde{\mathcal{M}}_t + \alpha\mathbb{E}[\widetilde{\mathcal{M}}_t]\,; \quad t \ge 0, \tag{7}$$

studied by DELARUE, INGLIS, RUBENTHALER & TANRÉ (2015 a,b). Here $\widetilde{\mathcal{X}}_0 < 1$, $\alpha \in (0,1)$, $\mathrm{b} : (-\infty, 1] \to \mathbb{R}$ is assumed to be Lipschitz continuous with at most linear growth. $\widetilde{W}.$ is the standard Brownian motion, $\widetilde{\mathcal{M}}. = \sum_{k=1}^{\infty} \mathbf{1}_{\{\widetilde{\tau}^k \le \cdot\}}$ with $\widetilde{\tau}^k := \inf\{s > \tau^{k-1} : \widetilde{\mathcal{X}}_{s-} \ge 1\}$, $k \ge 1$, $\widetilde{\tau}^0 = 0$. When we specify $\widetilde{\mathcal{X}}_0 = 0$, $\mathrm{b}(x) = -ax$, $x \in \mathbb{R}_+$, and $\alpha = 1$, the solution $(\widehat{\mathcal{X}}., \widehat{\mathcal{M}}.)$ to (7) reduces to the solution $(\widetilde{\mathcal{X}}., \widetilde{\mathcal{M}}.)$ to (6), however, the previous study of (7) does not guarantee the uniqueness of solution to (7) in the case $\alpha = 1$.

Proposition 1 ([3]). *Assume $\mathbb{E}[\mathcal{X}_0] \ge 1$ and $b(x,m) = -a(x-m)$, $x, m \in [0,\infty)$ for some $a > 0$. There exists a unique strong solution to (5) on $[0,T]$. Moreover, for every $T > 0$, there exists a constant c_T such that every solution to (5) satisfies $(\mathrm{d}/\mathrm{d}t)\mathbb{E}[\mathcal{M}_t] \le c_T$ for $0 \le t \le T$.*

The proof is based on a fixed point argument. For example, when $a = 0$, we may reformulate the solution $(\widehat{\mathcal{X}}_\cdot, \widehat{\mathcal{M}}_\cdot)$ in (6) as

$$\widehat{\mathcal{Z}}_t = \widehat{\mathcal{X}}_t + \widehat{\mathcal{M}}_t = \widehat{W}_t + \mathbb{E}[\widehat{\mathcal{M}}_t], \quad \widehat{M}_t = \lfloor \sup_{0 \le s \le t} (\widehat{\mathcal{Z}}_s)^+ \rfloor ; \quad t \ge 0, \tag{8}$$

where $\lfloor x \rfloor$ is the integer part. Given a candidate solution e_t for $\mathbb{E}[\widehat{\mathcal{M}}_t]$, $t \ge 0$, we shall consider

$$\widehat{\mathcal{Z}}_t^e := \widehat{W}_t + e_t, \quad \widehat{\mathcal{M}}_t^e := \lfloor \sup_{0 \le s \le t} (\widehat{\mathcal{Z}}_s^e)^+ \rfloor ; \quad t \ge 0, \tag{9}$$

where the superscripts e of $\widehat{\mathcal{Z}}_\cdot^e$ and $\widehat{\mathcal{M}}_\cdot^e$ represent the dependence on $e_\cdot$. Then uniqueness of the solution to (6) is reduced to uniqueness of the fixed point $e_\cdot^* = \mathfrak{M}_\cdot(e^*)$ of the map $\mathfrak{M} : C(\mathbb{R}_+, \mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}_+)$ defined by

$$\mathfrak{M}_t(e) := \mathbb{E}\big[\lfloor \sup_{0 \le s \le t} (\widehat{\mathcal{Z}}_s^e)^+ \rfloor\big] = \mathbb{E}[\widehat{\mathcal{M}}_t^e]; \quad t \ge 0. \tag{10}$$

To solve the equation (10) let us define recursively $e_\cdot^{(0)} \equiv 0$, $e_\cdot^{(n+1)} := \mathfrak{M}_\cdot(e^{(n)})$ for $n \in \mathbb{N}_0$. Then one can verify $e_\cdot^{(n)} \le e_\cdot^{(n+1)}$ for $n \in \mathbb{N}_0$. Let us also define

$$\mathcal{L} := \{ e \in C^1([0, \infty)) : \dot{e}_\cdot \ge 0, e_0 = 0, e_t \le \ell(t) := t / x_0, t \ge 0 \}.$$

Then one can show that if $e^{(n)} \in \mathcal{L}$, then $e^{(n+1)} \in \mathcal{L}$. (For example, if $a = 0$ and $x_0 \ge 1$, then $\widehat{Z}_\cdot = \widehat{W}_\cdot + e_\cdot = (W_\cdot / x_0) + e_\cdot$ for every $e \in \mathcal{L}$, and hence by an application of the renewal theory

$$\mathfrak{M}_t(e) = \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \le s \le t} (\widehat{W}_s + e_s)^+ \ge k) \le \sum_{k=1}^{\infty} \mathbb{P}(\sup_{0 \le s \le t} (W_s + s)^+ \ge k x_0) \le \frac{t}{x_0}$$

for $t \ge 0$.) By utilizing this monotone property of the map $\mathfrak{M}_t$ and the first passage time distribution for diffusions, we verify the contraction property and then find a unique fixed point in the class of continuously differentiable, nonnegative functions bounded by a linear line with slope $1 / x_0$. Note that in some numerical evaluation we observe the slow convergence of PICARD iteration even for the case $x_0 < 1$.

For the stationary distribution of the solution $\mathcal{X}_\cdot$ to (5) we have the following proposition.

Proposition 2 ([3]). *When $a > 0$, the stationary distribution of*

$$\mathcal{X}_t = \mathcal{X}_0 - a \int_0^t (\mathcal{X}_s - x_0) \mathrm{d}s + W_t + x_0 (\mathcal{M}_t - \mathbb{E}[\mathcal{M}_t]); \quad t \ge 0$$

has the density

$$\mathfrak{p}_a(x) := 2 c_0 \Big(\int_0^{x \wedge x_0} e^{a y^2 + 2 x_0 (c_0 + a) y} \mathrm{d}y \Big) e^{-a x^2 - 2 x_0 (c_0 + a) x}; \quad x \ge 0,$$

where $c_0 := \lim_{t \to \infty} \mathrm{d}\mathbb{E}[\mathcal{M}_t] / \mathrm{d}t$ is a unique solution to

$$\frac{c_0}{a} \int_{\sqrt{2/a} c_0 - \sqrt{2a} x_0}^{\sqrt{2/a} c_0 x_0} e^{x^2/2} \Big(\int_x^{\infty} e^{-y^2/2} \mathrm{d}y \Big) \mathrm{d}x = 1.$$

When $a = 0$, we have $c_0 = 1/x_0^2$ and $\mathfrak{p}_0(x) = (1 - e^{-2x/x_0})/x_0$ if $0 < x < x_0$ and $\mathfrak{p}_0(x) = e^{-2x/x_0}(e^2 - 1)/x_0$ if $x > x_0$.

It follows from Proposition 1 that the propagation-of-chaos result holds for the reformulated solution $(\mathcal{Z}_{\cdot}, \mathcal{M}_{\cdot})$ from the original $X_{\cdot}$ in (1). Thus we have the following.

Proposition 3 ([3]). *Let us assume that* X_0^i, $i \in \mathbb{N}$ *are independently, identically distributed with a finite mean. Under the same assumption as in Proposition 1, for every* $k \geq 1$, $\ell \geq 1$, $t_1, \dots t_\ell$, *as* $N \to \infty$ *the vector* $(X_{t_j}^i, M_{t_j}^i)$, $1 \leq i \leq k$, $1 \leq j \leq \ell$ *defined from (1) converges towards the finite dimensional marginals at times* $t_1, \dots, t_\ell$ *of* k *independent copies of* $(\mathcal{X}_{\cdot}, \mathcal{M}_{\cdot})$ *in (5).*

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