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A Support Problem for Superprocesses in Terms of Random Measure

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§1. Introduction

The purpose of this expository article is to investigate the support problem for a special class of superprocesses in terms of random measure. In the theory of measure-valued stochastic processes, compact support problems have been discussed for many years. For instance, in the case of typical super-Brownian motion $X = \{X_t; t \geq 0\}$, Iscoe (1988) proved that if the initial measure $X_0(dx)$ has a compact support, then for every $t > 0$, X_t possesses a compact support. Let $\mathcal{B}_+ \equiv \mathcal{B}_+(\mathbb{R}^n)$ be the totality of nonnegative Borel measurable functions on $\mathbb{R}^n$, and let $L \equiv L(dx)$ be a locally finite random measure on $\mathbb{R}^n$. For $\mathcal{B}_+ \ni f$, we define $\langle f, L\rangle := \displaystyle\int f(x)L(dx)$. Furthermore, $M_F(\mathbb{R}^n)$ denotes the totality of finite Borel measures on $\mathbb{R}^n$ equipped with weak convergence topology. We define a differential operator P by

$$P := \frac{1}{2}\sum_{k=1}^{n} a_k(x)\frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{n} b_k(x)\frac{\partial}{\partial x_k} + c(x)(\cdot) \tag{1}$$

where we assume that $a_k, b_k, c \in C_b^\infty(\mathbb{R}^n)$ satisfy $\exists\delta > 0 : a_j > \delta > 0$. As a matter of fact, our target process $X = (\{X_t, t \geq 0\}, P_\mu)$ in terms of measure L is an $M_F(\mathbb{R}^n)$-valued Markov process, and its Laplace transition functional is given by

$$\mathbb{E}_\mu[e^{-\langle\varphi, X_t\rangle}] = e^{-\langle u(t),\mu\rangle}. \tag{2}$$

Here the function $u(t) \equiv u(t,x)$ satisfies

$$\begin{cases} \partial_t u = Pu - \dot{L}(dx)u^2 \\ u(t,x)|_{t=0+} = \varphi(x) \end{cases} \tag{3}$$

where the symbol $\dot{L}(dx)$ means $\frac{L(dx)}{dx}$. For brevity's sake, in what follows we shall proceed the argument simply for $d = 1$. Our discussion on construction of superprocesses can be extended up to multi-dimensional case. However, the argument on the compact support problem for superprocesses is restricted to one-dimensional case.

§2. Main result

For $\mu \in M_F(\mathbb{R})$, the support of μ, say, $\mathrm{supp}(\mu)$ is defined by

$$\mathrm{supp}(\mu) := \{A \in \mathcal{B}(\mathbb{R}) : \mu(A^c) = 0\}. \tag{4}$$

While, the global support of superprocess $X(\cdot)$, say, $\mathrm{Gsupp}(X)$ is defined by

$$\mathrm{Gsupp}(X) := \bigcup_{t\geq 0} \mathrm{supp}(X_t(dx)). \tag{5}$$

It is a key point that we relate the support $\mathrm{Gsupp}(X)$ of superprocess X_t in terms of locally finite measure $L = L(dx)$ on $\mathbb{R}$ to a nonlinear singular elliptic boundary problem.

Let $d = 1, a(x) > 0$. We consider the associated boundary problem: for a differential operator $P = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$,

$$\begin{cases} Pv = v^2(x)\dfrac{L(dx)}{dx}, & a_1 < x < a_2 \\ v(a_1) = \beta_1, \quad v(a_2) = \beta_2. \end{cases} \tag{6}$$

When we denote the solution of (6) by $v(x;\beta_1,\beta_2)$, since $\exists\{\beta_1^{(n)}\}_n \nearrow \infty$, $\exists\{\beta_2^{(n)}\}_n \nearrow \infty$, the problem (6) possesses a unique solution $v(x;\beta_1^{(n)},\beta_2^{(n)})$. Note that the solution $v(x)$ is a continuous convex function defined on the interval $I = [a_1, a_2]$. Moreover, for $\forall a_1 \leqslant x_0 \leqslant x \leqslant a_2$, $v(x)$ satisfies

$$\begin{aligned} v(x) &= v(x_0) + \Phi_0(x_0)(x - x_0) + \int_{x_0}^{x} \Phi_1(y)v(y)dy \\ &\quad + \int_{x_0}^{x} dy \int_{x_0}^{y} \Phi_2(z)v(z)dz + \int_{x_0}^{x} dy \int_{x_0}^{y} \frac{2v^2(z)}{a(z)}L(dz), \end{aligned} \tag{7}$$

where

$$\Phi_0(x) = v'(x+) + \frac{2b(x)}{a(x)}, \qquad \Phi_1(x) = \frac{2b(x)}{a(x)},$$
$$\Phi_2(x) = \frac{2b(x)a'(x) - 2b'(x)a(x) + 2a(x)c(x)}{a(x)^2}.$$

Then we can obtain an explicit expression of the approximate solution. For $\psi \in C^+(\mathbb{R}), \mathrm{supp}(\psi) \subset (-K,K), \theta > 0$, when we denote by $v_K(t,x;\theta\psi)$ the solution of

$$u(t,x) = 0, \qquad x \in (-K,K)^c$$

$$\begin{aligned} u(t,x) &= \theta \int_0^t \int_{-K}^{K} p_K(t-s,x,y)\psi(y)dyds \\ &\quad - \int_0^t \int_{-K}^{K} p_K(t-s,x,y)u^2(s,y)L(dy)ds, \quad x \in (-K,K), \end{aligned} \tag{8}$$

then a simple fact $v_K \geq 0$ yields concurrently to $v_K \nearrow$ in t かつ $v_K \nearrow$ in ψ, and furthermore it follows immediately that

$$v_K(t,x;\theta\psi) \leqslant \sup_{t,x} \int_0^t \int_{-K}^K p_K(t-s,x,y)\theta\psi(y)dyds < \infty.$$

On the other hand, $v_K(\theta, t, x; a_1, a_2)$ denote the solution of (8) with the test function replaced by $\psi = 1_{[a_1,a_2]^c}$.

For simplicity, we assume henceforth that $\mathrm{supp}(X_0) \subset [a_1, a_2] \subset (-K, K)$, $b(x) = 0, c(x) > 0$. We shall represent the positive support probability of superprocess X_t by the solution of (6). The argument of Iscoe (1988) for occupation time processes $\int_0^t X_s^K ds$ or $\int_0^t X_s ds$ implies that

$$\begin{aligned} &E^L_{X_0}[\exp\left\{-\theta \int_0^\infty X_s^K([a_1,a_2]^c)ds\right\}] \\ &\qquad = \exp\left\{-\int_{-\infty}^\infty v_K(\theta,x;a_1,a_2)X_0(dx)\right\} \end{aligned} \tag{9}$$

holds. And besides we have

$$v_K(\theta,x;a_1,a_2) = \lim_{t\to\infty}(\lim_{n\to\infty} v_K(\theta\psi_n;t,x)),$$

and we can deduce that $v(x) \equiv v_K(\theta,x;a_1,a_2)$ satisfies that its second derivative v'' is a signed measure, and also that for $x \in (-K, K)$,

$$\begin{aligned} \frac{dv}{dx}(x\pm) = &\int_{x_0}^{x\pm} \frac{2c(y)v(y)}{a(y)}dy + \int_{x_0}^{x\pm} \frac{2v^2(y)}{a(y)}L(dy) \\ &- 2\theta \int_{x_0}^{x\pm} 1_{[a_1,a_2]^c}(y)dy + (\mathrm{Constant}). \end{aligned}$$

Thus the representation of probability for the support can be derived.

$$\begin{aligned} &P^L_{x_0}(\mathrm{supp}(X_t) \cap [a_1,a_2]^c = \emptyset, \quad \forall t \geq 0) \\ &= \lim_{K\to\infty} P^L_{X_0}(\mathrm{supp}(X_t^K) \cap [a_1,a_2]^c = \emptyset, \forall t \geq 0) \\ &\Longleftarrow \text{by virtue of the right continuity of the path } X_t^K(\omega) \\ &= \lim_{K\to\infty} P^L_{X_0}\left(\int_0^\infty X_s^K([a_1,a_2]^c)ds = 0\right) \\ &\Longleftarrow \text{by the expression of the occupation time process (9)} \\ &= \lim_{K\to\infty}\lim_{\theta\to\infty} \exp\left\{-\int_{-\infty}^\infty v_K(\theta,x;a_1,a_2)X_0(dx)\right\} \\ &= \lim_{n\to\infty} \exp\left\{-\int_{a_1}^{a_2} v(x;\beta_1^{(n)},\beta_2^{(n)})X_0(dx)\right\} \end{aligned} \tag{10}$$

By virtue of the above-mentioned facts we can get the following principal result, the theorem for compact support.

Theorem 1. (Main Result) *Let* $\mu \in M_F(\mathbb{R})$ *and* $\text{supp}(\mu) \subset [a_1, a_2]$. *Suppose that* $d = 1, a(x) > 0, b(x) = 0, c(x) > 0$. *For* $\forall \varepsilon > 0$ ($\varepsilon << 1$: sufficiently small), *there exist proper real numbers* $\exists \underline{x} = \underline{x}(\varepsilon) < a_1$, $\exists \bar{x} = \bar{x}(\varepsilon) > a_2$ *such that* v *is a nonnegative solution of* (7) *on the interval* $(\underline{x}, \bar{x})$, i.e. $v(x) \geq 0$ *for* $x \in (\underline{x}, \bar{x})$. *If* v *satisfies the conditions*

$$\sup_{a_1 \leqslant x \leqslant a_2} v(x) \leqslant \varepsilon, \quad \lim_{x \to \underline{x}} v(x) = \lim_{x \to \bar{x}} v(x) = \infty, \tag{11}$$

then the superprocess $X = \{X_t, t \geq 0\}$ *has the compact support.*

§3. Formulation of superprocess by admissible functional

Let us denote by $X = \{X_t, t \geq 0\}$ the measure-valued branching process corresponding to a locally finite random measure L, and P_μ^L denotes the probability law of the measure-valued process X. Then a measure-valued process (X_t, P_μ^L) in terms of random measure L is given by the following Laplace transition functional.

$$E_\mu^L[e^{-\langle \varphi, X_t \rangle}] = e^{-\langle u(t), \mu \rangle} \qquad \text{with} \quad X_0 = \mu \in M_F(\mathbb{R}). \tag{12}$$

Here the function $u(t, x)$ satisfies the following Cauchy problem.

$$\begin{cases} \partial_t u = Pu - \dfrac{L(dx)}{dx} u^2, \\ u(0, x) = \varphi \in C_b^+(\mathbb{R}). \end{cases} \tag{13}$$

Now, suggested by a formulation by Dawson-Fleischmann (1995), we shall consider the above initial value problem as an integral equation. As a matter of fact, when we write the fundamental solution to the aforementioned Cauchy problem by p, then we have

$$u(t, x) = \int p(t, x, y)\varphi(y)dy - \int_0^t \int p(t - s, x, y)u^2(s, y)L(dy)ds. \tag{14}$$

This means that we consider the mild solution to the above Cauchy problem. We shall assume henceforth:

[Assumption] For any $c > 0$,

$$\int_{-\infty}^{\infty} e^{-cx^2} L(dx) < \infty, \quad \text{a.s.} \tag{15}$$

Recall a method to apply admissible Brownian functional in the studies on superprocesses by E.B. Dynkin (1994). Roughly speaking, it is nothing but a special case that the branching rate term γ in the super-Brownian motion or the Dawson-Watanabe superprocess would be changed into a general additive functional which does not always possess its density. For a finite measure $\tilde{L}$ on $\mathbb{R}$ and a local time $\ell_{t,x}(\omega)$ of Browninan motion B_s, we define the additive functional $K_t^{[\tilde{L}]}(\omega)$ by

$$K_t^{[\tilde{L}]}(\omega) := \int \ell_{t,x}(\omega)\tilde{L}(dx). \tag{16}$$

We shall impose the following admissible conditions.

[Dynkin's Admissibility] For a Brownian motion $(B_t, \Pi_{0,x})$,
(i) $\Pi_{r,x}[K^{[\tilde{L}]}(r,t)] < \infty$, for $\forall r < t, x$
(ii) $\Pi_{r,x}[K^{[\tilde{L}]}(r,t)] \to 0$ uniformly in x $(r, t \to s)$ $\forall s$

Theorem 2. (Dynkin, 1994) *If the transition function $\mathcal{P}(r,\mu;t,C) = P_{r,\mu}(X_t \in C)$ satisfied the following two conditions, then the measure-valued Markov process named (ξ, K, ψ)-superprocess with parameters $X = (X_t, P_{r,\mu})$ can be determined.*

$$\int \mathcal{P}(r,\mu;t,d\nu)e^{-\langle f,\nu\rangle} = \exp\{-\langle v(r),\mu\rangle\}, \tag{17}$$

$$v(r,x) + \Pi_{r,x}\int_r^t \psi(s,v(s))(\xi_s)dK_s = \Pi_{r,x}f(\xi_t). \tag{18}$$

§4. Construction of sequence of approximate measure-valued processes

In this section we shall construct a basic process as a limit of increasing sequence of finite measure $M_F(\mathbb{R})$-valued processes realized on the common basic probability space. This provides us with a proto-type in the construction of our target superprocess. For each $K \in \mathbb{N}$, we put

$$E_K := \bigcup_{n=1}^{K} \{n\} \times (-n,n), \tag{19}$$

and we denote by $\tilde{X}_t^K \equiv \tilde{X}_t^K(dx)$ an $M_F(E_K)$-valued process. We shall first of all construct this measure-valued baisc process $\tilde{X}_t^K$ in what follows. For $x \in (-n,n)$, a Markov process w_K on E_K starting at a point (n,x) can be defined as

$$w_K(t) := (\{n\}, w(t)), \qquad \text{for} \quad 1 \leqslant t \leqslant \tau_n$$
$$w_K(\tau_n) := (\{n+1\}, w(\tau_n)), \qquad \tau_n = \inf\{t > 0 : w(t) = \pm n\}$$

where w is a P-diffusion starting at a point x. Notice that the stochastic process w_K dies out finally at time τ_K. Next we consider a randam measure L_K. In fact, we define

$$L_K(\{n\} \times (a,b)) := L((-n,n) \cap (a,b)), \qquad \text{for} \quad n \leqslant K.$$

On this account, we can define the admissible additive functional $\mathcal{K}_t^{[L_K]}(w_K)$ by making use of this random measure L_K, i.e.

$$\mathcal{K}_t^{[L_K]}(w_K) := \int \tilde{\ell}_{t,y}(w_K)L_K(dy) \tag{20}$$

where $\tilde{\ell}_{t,x}$ is a positive random variable given by

$$\tilde{\ell}_{t,x}(w) := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t 1_{(a-\varepsilon,a+\varepsilon)}(w(s))ds. \tag{21}$$

Then an application of the previous Dynkin's existence theorem (Theorem 2) with this admissible additive functional $\mathcal{K}_t^{[L_K]}$ gives us a superprocess, which we denote by $\tilde{X}_t^K = \tilde{X}_t^K(dx)$. That is to say,

$$E_{r,x}^{(L_K)} e^{-\langle\varphi,\tilde{X}_t^K\rangle} = \exp\{-\langle v(r),\mu\rangle\}, \tag{22}$$

$$v(r,x) + \tilde{\Pi}_{r,x}^P \int_r^t v(s,w_K(s))^2 d\mathcal{K}_t^{[L_K]} = \tilde{\Pi}_{r,x}^P \varphi(w_K(t)). \tag{23}$$

Next we shall construct a new approximate sequence of branching measure-valued processes by employing the above-mentioned process, and shall give its characterization. Before constructing the superprocess in question, we consider first the initial measure as its initial value. We choose a finite measure $\mu \in M_F(\mathbb{R})$ as a candidate of the initial measure for our measure-valued process $\tilde{X}_t^K$. For $n \geq 1$, for each subset $B \subset \mathbb{R}$ we define

$$\tilde{X}_0^K(\{n\}\times B) := \mu(B \cap \{[n-1,n) \cup (-n,-n+1]\}). \tag{24}$$

Then, if it is the case of the number $M \in \mathbb{N}$ satisfying $M > K$, the law of the process $\tilde{X}_t^M$ restricted to a set $E_K = \cup_{n=1}^K \{n\}\times(-n,n)$ is equivalent to the law of the process $\tilde{X}_t^K$. In other words,

$$\mathcal{L}(\tilde{X}_t^M \restriction E_K) = \mathcal{L}(\tilde{X}_t^K), \qquad \text{for} \quad \forall M > K.$$

Let us now denote by $P_{X_0}^{L,K}$ the probability law of the measure-valued process $\tilde{X}^K$, and we put $E_\infty := \bigcup_{n=1}^{\infty}\{n\}\times(-n,n)$ and $\tilde{X}^\infty$ denotes an $M(E_\infty)$-valued process. Then note that since the law $\{P_{X_0}^{L,K}\}_K$ of $\tilde{X}^K$ becomes a consistent family, its projective limit induces the law of $M(E_\infty)$-valued process $\tilde{X}^\infty$. Hence, if we define a new $M_F((-K,K))$-valued process X_t^K as

$$X_t^K(B) := \sum_{n=1}^K \tilde{X}_t^\infty(\{n\}\times B), \tag{25}$$

then an increasing sequence of stochastic processes $\{X_t^K(B)\}_K \nearrow$ is obtained.

Proposition 3. (Characterization) *Let $u_K(t,x)$ be a log-Laplace function of X_t^K. Then X_t^K satisfies the following*

$$E_{X_0^K}[e^{-\langle\varphi,X_t^K\rangle}] = e^{-\langle u_K(t),\mu\rangle}, \quad \text{with} \quad X_0^K(dx) = \mu(dx). \tag{26}$$

Moreover, the function $u_K(t,x)$ satisfies uniquely the follwoing integral equation: for $\varphi \in C_0(\mathbb{R})$,

$$\begin{aligned} u_K(t,x) &= \int_{-K}^{K} p_K(t,x,y)\varphi(y)dy \\ &\quad - \int_0^t \int_{-K}^{K} p_K(t-s,x,y)u_K^2(s,y)L(dy)ds, \end{aligned} \tag{27}$$

$$E[X_t^K(B)] = \int_{-K}^{K} \int_B p_K(t,x,y)\mu(dx)dy, \tag{28}$$

where $p_K(t,x,y)$ is the fundamental solution of the Dirichlet boundary value problem:

$$\partial_t u - Pu = 0, \qquad u|_{\partial(-K,K)} = 0 \tag{29}$$

§5. Existence of superprocess in terms of finite measure

Therefore $M_F(\mathbb{R})$-valued process $X = \{X_t, t \geq 0\}$ with the initial measure $\mu \in M_F(\mathbb{R})$ can be defined by the following limit

$$X_t(dx) := \lim_{K\to\infty} X_t^K(dx). \tag{30}$$

We call this stochastic process X_t a superprocess in terms of randam measure L which represents a random media. Next we shall extend $p_K(t,\cdot,\cdot)$ onto $\mathbb{R}\times\mathbb{R}$. Namely,

$$p_K(t,x,y) = 0 \quad \text{if} \quad x \text{ or } y \notin (-K,K).$$

Then, since $p_K(t,\cdot,\cdot) \nearrow p(t,\cdot,\cdot)$, we may apply the monotone convergence theorem to obtain

$$E[X_t(B)] = \int_{-\infty}^{\infty} \int_B p(t,x,y)\mu(dx)dy \qquad \forall B \in \mathcal{B}(\mathbb{R}). \tag{31}$$

On the other hand, since we have $\{X_t^K(\cdot)\}_K \nearrow$ in K, the sequence of log-Laplace functions $\{u_K(t,\cdot)\}_K$ associated with the sequence of those measure-valued processes is also increasing $\nearrow$ in K. As a consequence, by using the monotone convergence theorem again, the log-Laplace function $u(t,x)$ of the above-mentioned limit process $X_t(dx)$ can also be obtained by

$$u(t,x) = \lim_{K\to\infty} u_K(t,x). \tag{32}$$

Finally, an application of the monotone convergence theorem again leads to the following :

$$\begin{aligned}
u(t,x) &= \lim_{K\to\infty} u_K(t,x) \\
&= \lim_{K\to\infty} \int_{-K}^{K} p_K(t,x,y)\varphi(y)dy \\
&\qquad - \lim_{K\to\infty} \int_0^t \int_{-K}^{K} p_K(t-s,x,y)u_K^2(s,y)L(dy)ds \\
&= \int_{-\infty}^{\infty} p(t,x,y)\varphi(y)dy - \int_0^t \int_{-\infty}^{\infty} p(t-s,x,y)u^2(s,y)L(dy)ds. \qquad (33)
\end{aligned}$$

Remark. It is interesting to note that the above construction requires us only local finiteness of the random measure $L(dx)$.

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References.

[1] Dawson, D. and Fleischmann, K. : Super-Brownian motions in higher dimensions with absolutely continuous measure states. J. Theoret. Probab. **8** (1995), 179–206.

2] Dôku, I. : A certain class of immigration superprocesses and its limit theorem. Adv. Appl. Stat. **6** (2006), 145–205.

[3] Dôku, I. : A limit theorem of superprocesses with non-vanishing deterministic immigration. Sci. Math. Japn. **64** (2006), 563–579.

[4] Dôku, I. : A limit theorem of homogeneous superprocesses with spatially dependent parameters. Far East J. Math. Sci. **38** (2010), 1–38.

[5] Dôku, I. : Tumour immunoreaction and environment-dependent models. Trans. Japn. Soc. Indu. Appl. Math. **26** (2016), 213–252.

[6] Dynkin, E. B. : *An Introduction to Branching Measure-Valued Processes.* Amer. Math. Soc., Providence, RI, 1994.

[7] Iscoe, I. : On the supports of measure-valued critical branching Brownian motion. Ann. Probab. **16** (1988), 200–221.

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