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Sharp interface limit for the stochastic Allen-Cahn equations

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Abstract

In this paper, we treat our recent results about sharp interface limit for the stochastic Allen-Cahn equations in several settings. Especially, we focus on the generation and motion of interface. Finally, we show the simulation concerned with these models.

1 Introduction

Allen-Cahn equation is a reaction-diffusion equation which has a bistable reaction term;

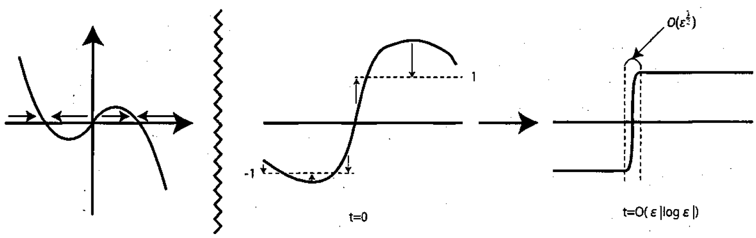
$$\begin{cases} \dot{u}^{\varepsilon}(t,x)=\Delta u^{\varepsilon}(t,x)+\frac{1}{\varepsilon}f(u^{\varepsilon}(t,x)), & t>0,\ x\in\mathbb{R}, \\ u^{\varepsilon}(0,x)=u_{0}^{\varepsilon}(x) \end{cases} \tag{1.1}$$

where $\Delta:=\frac{\partial}{\partial x}$. This equation is parametrized by a small parameter $\varepsilon>0$. The reaction term has the conditions;

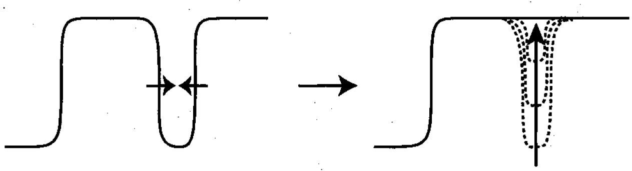
$$\begin{cases} \text{(i) } f \text{ has only three zeros } \pm 1 \text{ and } 0, \\ \text{(ii) } f'(\pm 1)<0,\ f'(0)>0, \\ \text{(iii) } f \text{ is odd } (A(f):=\int_{-1}^{1}f(u)du=0), \\ \text{(iv) } f(u)\leq C(1+|u|^{q}) \text{ with some } C,q>0, \\ \text{(v) } f'(u)\leq c \text{ with some } c>0. \end{cases} \tag{1.2}$$

The conditions (i) and (ii) mean that the reaction term is bistable. The existence and the uniqueness of the solution are assured by the conditions (iv) and (v). We impose the condition (iii) for some technical reasons, however, the condition $A(f)=0$ is rather important. We can take $f(u)=u-u^{3}$ as a typical example.

We can regard the PDE (1.1) as a one-dimensional dynamics by ignoring the diffusion term, because the reaction term $\frac{1}{\varepsilon}f(u)$ is larger than the other term. Hence, the solution tends to ± 1 in an early time, and interfaces appear between the two phases ± 1. We call this process "generation of interface", which occurs in the time of order $O(\varepsilon|\log\varepsilon|)$. After the generation, the interfaces move slowly. The constant $A(f)$ corresponds to a speed of traveling waves. However, the waves become standing waves because $A(f)=0$ from the condition (iii) of (1.2). Thus, the interface motion becomes extremely slow. Indeed, Carr-Pego [2] proved that the proper time scale for the interface motion is of order $O(\exp(-\frac{C}{\varepsilon}))$. In the article of Chen [1], it is called "super slow motion". We note that the width of interface is of order $O(\varepsilon^{\frac{1}{2}})$.



The annihilation of interface is also studied by Chen [1]. When the width of two interfaces once reaches smaller than $O(\varepsilon^{\frac{1}{2}})$, interfaces are annihilated in the speed of order $O(1)$, and form a new phase.



Reminding that the width of the interfaces is of order $O(\varepsilon^{\frac{1}{2}})$, the shape of interface becomes sharp as $\varepsilon \to 0$. Our goal is to specify the dynamics of the interface and its proper time scale when we take the limit $\varepsilon \to 0$ in the stochastic case. We call this limit "sharp interface limit".

2 Generation of interface in one-dimensional stochastic case

Now we consider a stochastic Allen-Cahn equation;

$$\begin{cases} \dot{u}^\varepsilon(t,x) = \partial_{xx}u^\varepsilon(t,x) + \frac{1}{\varepsilon}f(u^\varepsilon(t,x)) + \varepsilon^\gamma a(x)\dot{W}(t,x), & t>0,\ x\in\mathbb{R}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x),\ x\in\mathbb{R}, \quad u^\varepsilon(t,\pm\infty) = \pm 1,\ t>0, \end{cases} \tag{2.1}$$

where $a \in C_0^\infty(\mathbb{R})$ and $\dot{W}(t,x)$ is a space-time white noise which formally has a covariance structure;

$$E[\dot{W}(t,x)\dot{W}(s,y)] = \delta(t-s)\delta(x-y).$$

Thus, the solution is defined by a mild solution or a solution in the sense of generalized function. This case was well studied by Funaki [3]. He considered the case that $u_0^\varepsilon \to \chi_{\xi_0}$ in $L^2(\mathbb{R})$ where the function χ_ξ is a step function defined by $\chi_\xi(x) = 1$ if $x \geq \xi$ and $\chi_\xi(x) = -1$ if $x < \xi$. In other word, an interface is generated at the initial time. He proved that $\bar{u}^\varepsilon \Longrightarrow \chi_{\xi_t}$ as $\varepsilon \to 0$ where $\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-\frac{1}{2}}t, x)$, and the process ξ_t obeys an SDE;

$$d\xi_t = \alpha_1 a(\xi_t)dB_t + \alpha_2 a(\xi_t)a'(\xi_t)dt, \tag{2.2}$$

and start at ξ_0 where α_1 and $\alpha_2 \in \mathbb{R}$ depend on f. Namely, the proper time scale is of order $O(\varepsilon^{-2\gamma-\frac{1}{2}})$ and the interface motion is described by the SDE (2.2).

2.1 Settings and main result

We consider a more general initial value, and prove the generation of interface in [6]. The initial value $u_0^\varepsilon \in C^2(\mathbb{R})$ satisfies

$$\begin{cases} \text{(i)}\|u_0^\varepsilon\|_\infty + \|u_0^{\varepsilon\prime}\|_\infty + \|u_0^{\varepsilon\prime\prime}\|_\infty \le C_0, \\ \text{(ii)There exists a unique } \xi_0 \in [-1,1] \text{ independent of } \varepsilon > 0 \text{ such that } u_0^\varepsilon(\xi_0) = 0, \\ \text{(iii)}|u_0^\varepsilon(x)| \ge C_1\varepsilon^{\frac{1}{2}} \ (|x-\xi_0| \ge C_2\varepsilon^{\frac{1}{2}}), \\ \text{(iv)}|u_0^\varepsilon(x) - 1| + |u_0^{\varepsilon\prime}(x)| + |u_0^{\varepsilon\prime\prime}(x)| \le \varepsilon^\kappa C_3 \exp(-\frac{\sqrt{\mu}x}{2}) \ (x \ge 1), \\ \text{(v)}|u_0^\varepsilon(x) + 1| + |u_0^{\varepsilon\prime}(x)| + |u_0^{\varepsilon\prime\prime}(x)| \le \varepsilon^\kappa C_3 \exp(\frac{\sqrt{\mu}x}{2}) \ (x \le -1), \end{cases} \tag{2.3}$$

for some κ, C_0, C_1, C_2, $C_3 > 0$ and $\mu := f'(0)$.

Theorem 2.1. *If u_0^ε satisfies (2.3) and $\bar{u}^\varepsilon(t,x) := u^\varepsilon(\varepsilon^{-2\gamma-\frac{1}{2}}t, x)$, then there exist a.s. positive random variable $C(\omega)$ and stochastic processes ξ_t^ε such that*

$$P(\|\bar{u}^\varepsilon(t,\cdot) - \chi_{\xi_t^\varepsilon}(\cdot)\|_{L^2(\mathbb{R})} \le \delta \ for\ all\ t \in [C(\omega)\varepsilon^{2\gamma+\frac{3}{2}}|\log\varepsilon|, T]) \to 1 \quad (\varepsilon \to 0).$$

Moreover, if ξ_0 is a unique zero of u_0^ε as in (2.3), the distribution of the process ξ_t^ε on $C([0,T],\mathbb{R})$ weakly converges to that of ξ_t and ξ_t obeys the SDE (2.2) starting at ξ_0.

This result means that the generation of interface occurs until the time of order $O(\varepsilon|\log\varepsilon|)$, and the dynamics of interface is described by the SDE (2.2).

2.2 Outline of the proof

We explain an outline of the proof. The Allen-Cahn equation is also described by an L^2-gradient flow of Ginzburg-Landau free energy $\mathcal{H}^\varepsilon(u) := \int_\mathbb{R} \left\{\frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon}F(u)\right\} dx$ where $F' = -f$. The minimizers of $\mathcal{H}^\varepsilon$ with the boundary condition $u(\pm\infty) = \pm 1$ is $M^\varepsilon := \{m(\varepsilon^{-\frac{1}{2}}(x-\xi)) | \xi \in \mathbb{R}\}$ where m satisfies an ODE;

$$\begin{cases} \Delta m + f(m) = 0, \ m(0) = 0, \ m(\pm\infty) = \pm 1, \\ m \text{ is monotonous increasing.} \end{cases}$$

We consider a scale change of the solution $v(t,x) = u^\varepsilon(t, \varepsilon^{\frac{1}{2}}x)$ and $M := M^1$. We can decompose $v \in L^2(\mathbb{R}) + m$ into $s(v) + m(\cdot - \eta(v))$ such that $\|s(v)\|_{L^2} = dist_{L^2}(v, M)$ where $\eta \in \mathbb{R}$, and this is a unique decomposition if $dist_{L^2}(v,M) \le \exists\beta$. We call the coordinate $(s(v), \eta(v))$ Fermi coordinate. Then, we can consider that the interface is generated when $v(t,x)$ goes into a tubular neighborhood of M. Thus, we prove the decay of $\|s(v)\|_{L^2}$ by using an energy estimate. After entering the neighborhood of M, the scaled solution never goes out of this neighborhood with high probability, and we can connect it to the result of Funaki [3].

3 Generation of interface in multi-dimensional case

In this section, we consider a multi-dimensional stochastic Allen-Cahn equation with Neumann boundary condition;

$$\begin{cases} \dot{u}^\varepsilon(t,x) = \Delta u^\varepsilon(t,x) + \dfrac{1}{\varepsilon} f(u^\varepsilon(t,x)) + \dot{W}_t^\varepsilon(x), \quad t > 0, \ x \in D, \\ u^\varepsilon(0,x) = u_0(x), \quad x \in D, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, \quad t > 0, \ x \in \partial D, \end{cases} \tag{3.1}$$

where $D \subset \mathbb{R}^d$ is a domain with a smooth boundary and ν is a unit normal vector on ∂D. The external noise is defined by $\dot{W}_t^\varepsilon(x) := \varepsilon^\gamma \dot{W}_t^{Q_d}(x)$, where $W_t^{Q_d}(x)$ is a Q-Brownian motion which has a covariance structure $E[W_t^{Q_d}(x)W_s^{Q_d}(y)] = (t \wedge s)Q_d(x,y)$. The function $Q_d : D \times D \to \mathbb{R}$ is a positive, symmetric and compactly supported smooth function.

3.1 Main result

Now we state the generation of interface for multi-dimensional Allen-Cahn equation.

Theorem 3.1. *Assume that u_0 satisfies $\|u_0\|_\infty + \|u_0'\|_\infty + \|u_0''\|_\infty \le C_0$. If there exist constants $C_1 > 0$, κ and α satisfying $\kappa > \alpha > \frac{1}{2}$, $\kappa > 1$ and $\frac{\alpha}{\mu} + \frac{\kappa}{p} < C_1 < \frac{1}{\mu}$, then there exist positive constants $\tilde{\gamma}_d > 0$ and, for all $\gamma \ge \tilde{\gamma}_d$, we have that*

$$\text{(i)} \lim_{\varepsilon \to 0} P(-1 - \varepsilon^\kappa \le u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \le 1 + \varepsilon^\kappa \text{ for all } x \in D) = 1,$$

$$\text{(ii)} \lim_{\varepsilon \to 0} P(u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \ge 1 - \varepsilon^\kappa \text{ for } x \in D \text{ s.t. } u_0(x) \ge \varepsilon^\beta) = 1,$$

$$\text{(iii)} \lim_{\varepsilon \to 0} P(u^\varepsilon(x, C_1\varepsilon|\log\varepsilon|) \le -1 + \varepsilon^\kappa \text{ for } x \in D \text{ s.t. } u_0(x) \le -\varepsilon^\beta) = 1,$$

where $\beta := 1 - C_1\mu$.

Also in the multi-dimensional case, the interface is generated until the time of order $O(\varepsilon|\log\varepsilon|)$.

3.2 Outline of the proof

The proof is based on the comparison argument. Before the generation of interface, the reaction term is greater than the other term. We ignore the diffusion term, and consider the one-dimensional dynamics which is represented by an SDE;

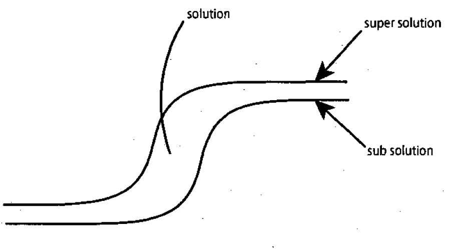
$$\begin{cases} \dot{Y}^\varepsilon(\tau, \xi, x) = f(Y^\varepsilon(\tau, \xi, x)) + \varepsilon^{\gamma + \frac{1}{2}} \dot{W}_\tau^{Q_d}(x), & \tau > 0, \\ Y^\varepsilon(0, \xi, x) = \xi \in [-2C_0, 2C_0]. \end{cases}$$

where $\tau := \varepsilon t$. We set $w_\varepsilon^\pm(t,x) = Y^\varepsilon(\frac{t}{\varepsilon}, u_0^\pm(x) \pm \varepsilon C_2(e^{\frac{\mu t}{\varepsilon}} - 1), x)$ and prove that $w_\varepsilon^\pm(t,x)$ are the super and sub solutions of the SPDE (3.1). In this process of proof, we extend the comparison principle of PDEs to that of SPDEs.

Lemma 3.2. *For every $0 < C_1 < \frac{1}{\mu}$, we have that*

$$P\left(w_\varepsilon^-(t,x) \le u^\varepsilon(t,x) \le w_\varepsilon^+(t,x) \text{ for every } t \in [0, C_1\varepsilon|\log\varepsilon|],\ x \in D\right) \to 1,$$

as $\varepsilon \to 0$.



The main result is implied by a behavior of Y^ε, because the solution u^ε exists between the super and sub solutions $w_\varepsilon^\pm$ which is constructed by using Y^ε.

4 Stochastic Allen-Cahn equation with Dirichlet boundary conditions

Next, we consider the stochastic Allen-Cahn equation with Dirichlet boundary conditions;

$$\begin{cases} \partial_t u^\varepsilon(t,x) = \partial_{xx} u^\varepsilon + \frac{1}{\varepsilon} f(u^\varepsilon) + \sqrt{2}\varepsilon^\gamma \dot{W}_t(x), & t > 0,\ x \in [-1,1], \\ u^\varepsilon(0,x) = u_0^\varepsilon(x),\ x \in [-1,1], & u^\varepsilon(t,\pm 1) = \pm 1,\ t > 0, \end{cases}$$

where $\dot{W}_t(x)$ is a space-time white noise on $[-1,1]$ and $u_0^\varepsilon \to \chi_{\xi_0}$ as $\varepsilon \to 0$ in $L^2[-1,1]$ for fixed $\xi_0 \in [-1,1]$. In particular, we focus on the motion of interface in this section. From the boundary conditions, the solution is pinned at the boundary $x = \pm 1$. If we formally consider that $a \equiv 1$ in the SDE (2.2), ξ_t moves as a one-dimensional Brownian motion multiplied by a constant. In our case, we can expect that the dynamics of interface should be described as a Brownian motion on $[-1,1]$ which has an reflected wall on $x = \pm 1$ because we impose Dirichlet boundary conditions. However, it is not easy to analyze the behavior of interface near the boundary, because of its singularity.

4.1 Main result

Let P^ε be a probability measure of the solution scaled in time $\bar{u}^\varepsilon(t,x) = u^\varepsilon(\varepsilon^{-2\gamma-\frac{1}{2}}t, x)$ on $C([0,T], L^2[-1,1])$, and let P be that of Markov process $\chi_{\sqrt{2}B(\alpha_1^2 t)}$ on the same space, where $B(t)$ is a reflected Brownian motion on $[-1,1]$ starting from $\xi_0 \in [-1,1]$ and $\alpha_1 := \|\nabla m\|$.

Theorem 4.1. *If $\gamma > \frac{19}{4}$, then P^ε converges to P weakly on $C([0,T], L^2[-1,1])$ as $\varepsilon \to 0$.*

Theorem 4.1 implies that $\bar{u}^\varepsilon \Rightarrow \chi_{\sqrt{2}B(\alpha_1^2 t)}$ as $\varepsilon \to 0$ and the proper time scale is of order $O(\varepsilon^{-2\gamma-\frac{1}{2}})$. As we discussed above, the interface motion at the limit is a reflected Brownian motion.

4.2 Outline of the proof

By considering the solution $u^\varepsilon(t)$ as a $L^2[-1,1]$-valued Markov process, there is a corresponding Dirichlet form $(\mathcal{E}^\varepsilon, \mathcal{D}^\varepsilon)$, and it is defined by

$$\mathcal{E}^\varepsilon(\varphi,\psi) := E^{\mu^\varepsilon}[\langle D\varphi, D\psi \rangle] = E^{\mu^\varepsilon}[\varphi(-\mathcal{L}^\varepsilon)\psi],$$

where $\varphi,\ \psi \in \mathcal{D}^\varepsilon \subset L^2(L^2[-1,1], \mu^\varepsilon)$ and μ^ε is an invariant measure of the SPDE. The operator $\mathcal{L}^\varepsilon$ is a generator of u^ε;

$$\mathcal{L}^\varepsilon F(u) = \langle DF(u), u'' + \frac{1}{\varepsilon} f(u) \rangle + \varepsilon^{2\gamma} Tr(D^2 F)(u),$$

and $\mathcal{L}^\varepsilon$ generates Markov semigroup $\{T_t^\varepsilon\}$ of u^ε. On the other hand, Brownian motion on $[-1,1]$ is associated with Dirichlet form $(\mathcal{E}, \mathcal{D})$;

$$\mathcal{E}(\varphi,\psi) := \frac{1}{2}\int_{-1}^{1} \varphi'(\xi)\psi'(\xi)d\xi, \quad (\varphi,\ \psi \in \mathcal{D}),$$

and the measure $\frac{1}{2}1_{[-1,1]}(\xi)d\xi$ can be regard as a uniform distribution on $[-1,1]$. Weber [8] proved that the invariant measure μ^ε concentrates on $\mathcal{S} := \{\chi_\xi\}_{\xi \in [-1,1]}$ as $\varepsilon \to 0$. Otto et.

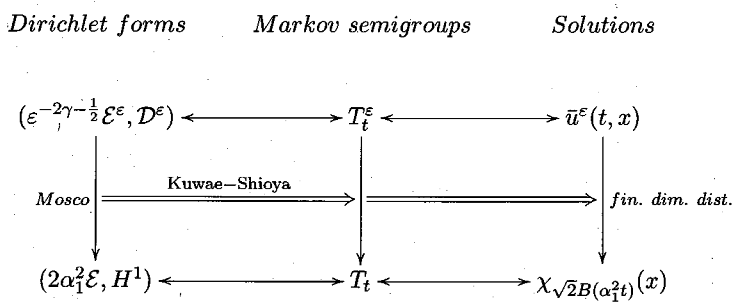
al. [7] also proved that μ^ε converges to μ which is a uniform distribution on $\mathcal{S}$ weakly. From this observation, if we can characterize the convergence $(\mathcal{E}^\varepsilon, \mathcal{D}^\varepsilon) \to (\mathcal{E}, \mathcal{D})$ as $\varepsilon \to 0$, it is natural to prove the result through this convergence, and this is our motivation. We consider Mosco convergence of the quadratic form which is determined by Dirichlet form. Indeed, Mosco convergence and the strong convergence of Markov semigroup is equivalent.

Theorem 4.2 (Kuwae, Shioya [5], Kolesnikov [4]). *Let $(\mathcal{E}^\varepsilon, \mathcal{D}(\mathcal{E}^\varepsilon))$ and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be Dirichlet forms, and let T_t^ε and T_t be semigroups which is associate with these closed forms. Then Mosco convergence $(\mathcal{E}^\varepsilon, \mathcal{D}(\mathcal{E}^\varepsilon)) \to (\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is equivalent to the strong convergence of operator $T_t^\varepsilon \to T_t$ for all $t > 0$.*

The domain $\mathcal{D}$ of the limit is also important. If $\mathcal{D} = H^1(\mathcal{S})$, then Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ corresponds to a reflected Brownian motion. However, if $\mathcal{D} = H_0^1(\mathcal{S})$, then Dirichlet form corresponds to a Brownian motion absorbed in a boundary. Thus, we need to prove that $\mathcal{D} = H^1(\mathcal{S})$. Now we state Mosco convergence of Dirichlet form which corresponds to the solution $\bar{u}^\varepsilon$.

Lemma 4.3. *The Dirichlet form $(\varepsilon^{-2\gamma-\frac{1}{2}}\mathcal{E}^\varepsilon, \mathcal{D}(\mathcal{E}^\varepsilon))$ converges to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ where $\mathcal{E}(\cdot,\cdot) := \frac{1}{2\|\nabla m\|^2}\langle \frac{d}{d\xi}\cdot, \frac{d}{d\xi}\cdot\rangle$ and $\mathcal{D}(\mathcal{E}) := H^1(\mathcal{S})$ in Mosco sense. In particular, $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ associates with Markov process $\{\chi_{\sqrt{2}B(\alpha_1^2 t)}\}$ where $B(t)$ is a reflected Brownian motion on $[-1, 1]$.*

Theorem 4.2 concludes the convergence of Markov semigroup, and this implies the weak convergence of finite dimensional distribution of u^ε on $L^2[-1, 1]$. Combining with the tightness which follows from Funaki [3], we complete the proof of the main result.



5 Simulations

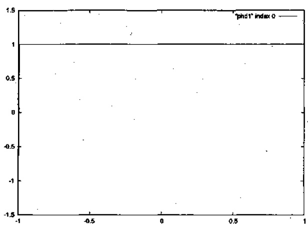
In this section, we simulate the one-dimensional stochastic Allen-Cahn equation;

$$\dot{u}(t,x) = \Delta u(t,x) + af(u(t,x)) + b\dot{W}_t(x), \quad t > 0,\ x \in [-1, 1],$$

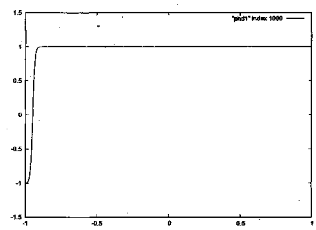
where $a > 0$, $b \in \mathbb{R}$, $f(u) = u - u^3$ and $\dot{W}_t(x)$ is a space-time white noise. We impose Dirichlet boundary conditions $u(\pm 1) = \pm 1$. We use the discretizing method for this simulation.

5.1 Reflection of interface at the boundary

Before the stochastic case, we consider the deterministic case $(b = 0)$. We change the time as $\bar{u}(t) := u(ct)$. The initial value takes value -1 on $x = -1$ and takes 1 on $x \neq -1$. Now we simulate the case that $a = 10^3$, $b = 0$, $c = 10^4$ and $N = 150$.

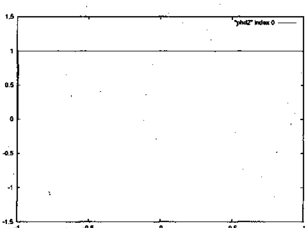


(1) $t = 0$

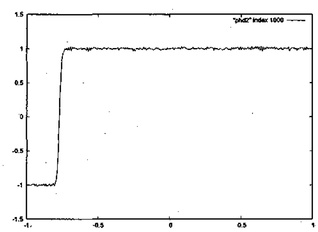


(2) $t = 1000$

We can see that the interface almost stops immediately although we take very long time scale. Actually, this moves, however, the speed of interface is extremely slow. This is the super slow motion. On the other hand, the motion of interface becomes totally different if we take $b = 2$.



(1) $t = 0$

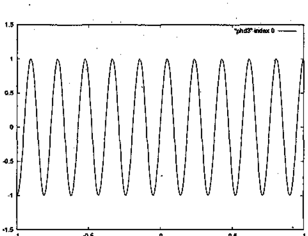


(2) $t = 1000$

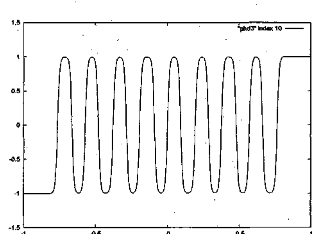
In this case, the solution becomes singular, and the interface perturbs randomly and fast. We can observe a reflected Brownian motion as an interface motion. Moreover, we can expect that we can take the value γ to be smaller than $\frac{19}{4}$ which is lower bound of γ in Theorem 4.1, Section 4.

5.2 Annihilation of interfaces

We also consider the annihilation of interface. First, we simulate the deterministic case. We set the initial value $u_0(x) := \sin \frac{21\pi x}{2}$.



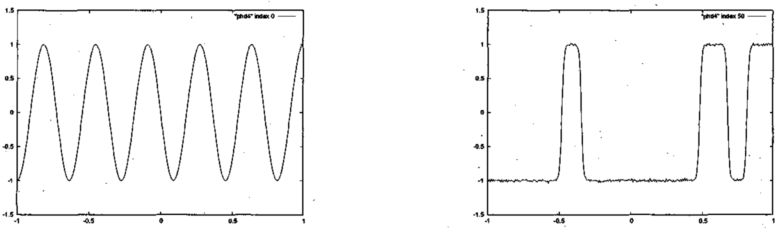
(1) $t = 0$



(2) $t = 10$

The annihilation occurs symmetrically because of the boundary conditions and the definition of the reaction term f. Next we consider the stochastic case. We set the initial value $u_0(x) :=$

$-\sin\frac{11\pi x}{2}$. We change the initial value because the annihilation occurs too fast if we take $u_0(x) := \sin\frac{21\pi x}{2}$.



(1) $t = 0$ (2) $t = 50$

The interfaces move like the independent Brownian motions, and the annihilation randomly occurs.

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