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TAUBERIAN AND COTAUBERIAN MULTIPLIERS OF THE GROUP ALGEBRAS $L_1(G)$

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ABSTRACT. We describe some recent results on the multipliers of the group algebras $L_1(G)$ which are tauberian or cotauberian, where G is a locally compact abelian group. We show the connections between those results, and we state some open questions on the topic.

1. INTRODUCTION

Tauberian operators were introduced by Kalton and Wilansky in [13] in order to study a problem of summability from an abstract point of view, and cotauberian operators were introduced in [18] as those operators whose conjugate is tauberian. These two classes of operators have found many applications in Banach space theory (see [10, Chapter 5]).

The tauberian operators from $L_1(\mu)$ into a Banach space were studied in [9], where the question whether all tauberian operators $T: L_1(\mu) \to L_1(\mu)$ are upper semi-Fredholm (have closed range and finite dimensional kernel) was raised. A negative answer to this question was given in [12]: there exists a tauberian operator $T: L_1(0,1) \to L_1(0,1)$ with non-closed range. The corresponding question for the multipliers of the Banach algebra $L_1(G)$ which are tauberian or cotauberian, where G is a locally compact abelian group, was studied in [4] and [5]. Observe that the multipliers of $L_1(G)$ coincide with the convolution operators T_μ associated to the Borel measures μ on G [14, Chapter 0]. It was proved in [4] that the tauberian operators T_μ are invertible when the group G is non-compact, and that they are Fredholm when G is compact and they have closed range or the singular continuous part (with respect to the Haar measure on G) of the associated measure μ is zero. Moreover, it was proved in [5] that the cotauberian operators T_μ are invertible when G is non-compact, and that they are Fredholm when G is compact. These results provide new characterizations of the Fredholm multipliers of the group algebras $L_1(G)$ described in [1, Theorem 5.97]. Here we present the results of [4, 5], describe the relations between them, and point out some problems that remain open.

Throughout the paper X and Y are (complex) Banach spaces, we consider (continuous linear) operators $T: X \to Y$, and we denote by $R(T)$ and $N(T)$ the range and the kernel of T. An operator $T: X \to Y$ is called *tauberian* if its second conjugate $T^{**}: X^{**} \to Y^{**}$ satisfies $T^{**-1}(Y) = X$, and the operator T is called *cotauberian* when its conjugate $T^*: Y^* \to X^*$ is tauberian. Moreover an operator $K: X \to Y$ is called *weakly compact* if $K^{**}(X^{**}) \subset Y$. Note that given $T: X \to Y$ tauberian (cotauberian) and $K: X \to Y$ weakly compact, the sum $T + K$ is tauberian (cotauberian).

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An operator $T : X \to Y$ is *upper semi-Fredholm* if it has closed range and finite dimensional kernel, and T is *Fredholm* if it has closed finite codimensional range and finite dimensional kernel. Upper semi-Fredholm operators are tauberian [10, Theorem 2.1.5], and from our point of view can be considered as "trivial" tauberian operators.

For basic results on Fredholm theory, tauberian operators, multipliers of Banach algebras and Fourier analysis we refer to [1], [10] [14] and [17].

2. Preliminary results

We denote by G a locally compact abelian group (a LCA group, for short), m is the Haar measure on G, $L_1(G)$ is the space of m-integrable complex functions on G endowed with the L_1-norm $\|\cdot\|_1$, and $M(G)$ denotes the space of complex Borel measures on G endowed with the variation norm. The space $L_1(G)$ can be identified with the subspace of those $\mu \in M(G)$ that are absolutely continuous with respect to m by associating to $f \in L_1(G)$ the measure $m_f \in M(G)$ defined by $m_f(A) = \int_A f(x)dm(x)$. The space $L_1(G)$ with the convolution $(f \star g)(x) = \int_G f(x-y)g(y)dm(y)$ is a commutative Banach algebra.

Given $\mu \in M(G)$ and $f \in L_1(G)$ the expression $(\mu \star f)(x) = \int_G f(x-y)d\mu(y)$ defines $\mu \star f \in L_1(G)$ satisfying $\|\mu \star f\|_1 \leq \|\mu\| \cdot \|f\|_1$. Thus for every $\mu \in M(G)$ we obtain a *convolution operator* T_μ on $L_1(G)$ defined by $T_\mu f = \mu \star f$, and satisfying $\|T_\mu\| = \|\mu\|$. Moreover, given $\mu, \nu \in M(G)$, the convolution of measures $\mu \star \nu \in M(G)$ is commutative [17]. Therefore $T_{\mu\star\nu} = T_\mu T_\nu = T_\nu T_\mu$. For each $r \in G$ the *translation operator* T_r on $L_1(G)$ is defined by $(T_r f)(x) = f(x-r)$. Note that T_r is the convolution operator associated to the unit measure δ_r concentrated at $\{r\}$.

The convolution operators acting on $L_1(G)$ can be characterized as those operators $T : L_1(G) \to L_1(G)$ that commute with translations ($T_r T = T T_r$ for each $r \in G$), and coincide with the convolution operators T_μ, $\mu \in M(G)$ [14, Chapter 0].

Let Γ denote the dual group of G. Given $f \in L_1(G)$ and $\mu \in M(G)$, the Fourier transform $\widehat{f} : \Gamma \to \mathbb{C}$ of f and the Fourier-Sieltjes transform $\widehat{\mu} : \Gamma \to \mathbb{C}$ of μ are defined by $\widehat{f}(\gamma) = \int_G f(x)\gamma(-x)dm(x)$ and $\widehat{\mu}(\gamma) = \int_G \gamma(-x)d\mu(x)$.

Let A be a Banach algebra and let B be a subset of a A. The *left annihilator* of B is the set $l(B) := \{x \in A : xB = \{0\}\}$, and the *right annihilator* of B is the set $r(B) := \{x \in A : Bx = \{0\}\}$. Following [14], we say that a Banach algebra A is *without order* if $l(A) = \{0\}$ or $r(A) = \{0\}$, and a mapping $T : A \to A$ is a *multiplier of A* if $x(Ty) = (Tx)y$ for all $x, y \in A$. If G is an LCA group, then $L_1(G)$ is without order.

The second dual space A^{**} of a Banach algebra A is also a Banach algebra endowed with the (first) Arens product [3]. Specifically, given $M, N \in A^{**}$, $f \in A^*$ and $a, b \in A$, we define the product $M \cdot N$ in three steps as follows:

$$\begin{aligned} f \cdot a \in A^* : &\quad \langle f \cdot a, b \rangle := \langle f, ab \rangle \\ N \cdot f \in A^* : &\quad \langle N \cdot f, a \rangle := \langle N, f \cdot a \rangle \\ M \cdot N \in A^{**} : &\quad \langle M \cdot N, f \rangle := \langle M, N \cdot f \rangle. \end{aligned}$$

Thus, for G an LCA group, $L_1(G)^{**}$ is a Banach algebra.

Given $f \in L_\infty(G) \equiv L_1(G)^*$ and $\phi \in L_1(G)$, and denoting $\widetilde{\phi}(x) = \phi(-x)$, we have

$$f \cdot \phi = f \star \widetilde{\phi}. \tag{1}$$

Since G is commutative, the center of $L_1(G)^{**}$ is $L_1(G)$ [15, Corollary 3] i.e.,

$$L_1(G) = \{m \in L_1(G)^{**} : \ m \cdot n = n \cdot m \text{ for each } n \in L_1(G)^{**}\}.$$

When G is compact, $L_1(G)$ is a (closed) ideal of $L_1(G)^{**}$ [19, Proposition 4.2]. Thus $L_1(G)^{**}/L_1(G)$ is a Banach algebra.

For $f \in L_1(\mu)$, we denote $\mathrm{D}(f) = \{t : f(t) \neq 0\}$. We say that a sequence (f_n) in $L_1(\mu)$ is *disjoint* if $\mu(\mathrm{D}(f_k) \cap \mathrm{D}(f_l)) = 0$ for $k \neq l$.

The following result was proved in [9] (see also [10, Chapter 4]) when μ is a non-atomic finite measure, but the arguments given there are valid when μ is σ-finite.

Theorem 2.1. [9, Theorems 2 and 6] *Let μ be a σ-finite measure. For an operator $T : L_1(\mu) \to Y$ the following assertions are equivalent:*

(1) *T is tauberian;*
(2) $\liminf_{n\to\infty} \|Tf_n\| > 0$ *for every disjoint normalized sequence (f_n) in $L_1(\mu)$;*
(3) *there exists a number $r > 0$ such that* $\liminf_{n\to\infty} \|Tf_n\| > r$ *for every disjoint normalized sequence (f_n) in $L_1(\mu)$;*
(4) $\liminf_{n\to\infty} \|Tf_n\| > 0$ *for every normalized sequence (f_n) in $L_1(\mu)$ satisfying* $\lim_{n\to\infty} \mu(D(f_n)) = 0$.

The convolution operators with closed range were described by Host and Parreau.

Theorem 2.2. [11, Théorème 1] *Let G be a LCA group and let $\mu \in M(G)$. Then the convolution operator T_μ has closed range if and only if $\mu = \nu \star \xi$, where $\nu, \xi \in M(G)$, ν is invertible and ξ is idempotent.*

Corollary 2.3. *Let G be a LCA group and let $\mu \in M(G)$. Suppose that the convolution operator T_μ has closed range. Then $L_1(G) = R(T_\mu) \oplus N(T_\mu)$.*

Proof. The factorization $\mu = \nu \star \xi$ in Theorem 2.2 and the commutativity of the convolution product of measures give $T_\mu = T_\nu T_\xi = T_\xi T_\nu$. So the result follows from the fact that T_ξ is a projection, since $N(T_\xi) = N(T_\mu)$ and $R(T_\xi) = R(T_\mu)$. □

The point spectrum $\sigma_p(T)$ of an operator $T : X \to X$ is the set of those $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective. When G is compact, the point spectrum of a convolution operator admits the following description.

Proposition 2.4. [16, Example 4.6.2] *Let G be a compact group with dual group Γ, and let $\mu \in M(G)$. Then $\sigma_p(T_\mu) = \widehat{\mu}(\Gamma)$.*

In particular, since $T_r = T_{\delta_r}$ and

$$\widehat{\delta_r}(\gamma) = \int_G \gamma(-x) d\delta_r(x) = \gamma(-r) \tag{2}$$

for each $\gamma \in \Gamma$, we obtain that $\sigma_p(T_r) = \{\gamma(-r) : \gamma \in \Gamma\}$.

Recall that $r \in G$ has *finite order* if there exists $m \in \mathbb{N}$ such that $mr = 0$ (where mr denotes the sum of m copies of r in G). Otherwise we say that r has *infinite order.* It follows from Proposition 2.4 and formula (2) that $\sigma_p(T_r)$ is a finite subset of the unit circle $\mathbb{T}$ when r has finite order.

A measure $\mu \in M(G)$ is said to be *discrete* if it is concentrated in a countable subset of G; i.e., if there exist sequences (x_i) in G and (β_i) in $\mathbb{C}$ so that $\sum_{i=1}^{\infty} |\beta_i| < \infty$ and $\mu = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$.

Proposition 2.5. [16, Theorem 4.11.1] *Let G be a LCA group and let $\mu \in M(G)$ be a discrete measure. Then the spectrum $\sigma(T_\mu)$ of the convolution operator T_μ coincides with the closure of the set $\widehat{\mu}(\Gamma)$.*

3. Tauberian operators

Here we show that, under certain conditions, tauberian convolution operators acting on $L_1(G)$ are Fredholm.

Theorem 3.1. *Let G be a non-compact LCA group. Then every tauberian convolution operator $T_\mu : L_1(G) \to L_1(G)$ is invertible.*

In the proof of Theorem 3.1 we show first that T_μ is bounded below, and then we derive from Corollary 2.3 that T_μ is invertible.

The prof of the result for the case G compact (Theorem 3.5) is done in in several steps.

Recall that given $r \in G$, the translation operator T_r is an invertible operator on $L_1(G)$ that satisfies $T_r T_s = T_{r+s}$ for every $s \in G$. In the proof of the next result we distinguish the cases in which r has finite or infinite order.

Proposition 3.2. *Let G be a compact group, and let $\lambda \in \sigma_p(T_r)$ for some $r \in G$. Then $T_r - \lambda I$ is not tauberian.*

Next we consider the case of a discrete measure concentrated in a finite number of points of G; namely $\mu = \sum_{l=1}^{k} \alpha_l \delta_{r_l}$. In this case $T_\mu = \sum_{l=1}^{k} \alpha_l T_{r_l}$.

Theorem 3.3. *Let G be a compact group with dual group Γ, and let $\mu = \sum_{l=1}^{k} \alpha_l \delta_{r_l}$ where $r_1, \dots, r_k$ are distinct points in G. Then $T_\mu - \lambda I$ is not tauberian when $\lambda \in \sigma(T_\mu)$.*

To prove this result, we consider first the case $\lambda \in \sigma_p(T_\mu)$, which coincides with $\widehat{\mu}(\Gamma)$ (Proposition 2.4), and then the general case.

Now we consider the case of an arbitrary discrete measure on G; namely $\mu = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$ where (x_i) is a sequence of points of G and (β_i) is a sequence in $\mathbb{C}$ satisfying $\sum_{i=1}^{\infty} |\beta_i| < \infty$. In this case $T_\mu = \sum_{i=1}^{\infty} \beta_i T_{x_i}$.

Proposition 3.4. *Let G be a compact group with dual group Γ, and let $\mu = \sum_{i=1}^{\infty} \beta_i \delta_{x_i}$, where (x_i) is a sequence of distinct points in G and $(\beta_i) \subset \mathbb{C}$ satisfying $\sum_{i=1}^{\infty} |\beta_i| < \infty$. Then $T_\mu - \lambda I$ is not tauberian when $\lambda \in \sigma(T_\mu)$.*

Now we can state our result for the case G compact.

Theorem 3.5. *Let G be a compact group G, let $\mu, \mu_0 \in M(G)$ with μ_0 discrete, and let $f \in L_1(G)$. Then*

1. *If T_μ is tauberian with closed range, then it is Fredholm.*
2. *T_{μ_0} is tauberian if and only if it is invertible.*
3. *$T_{\mu_0 + m_f}$ is tauberian if and only if it is Fredholm.*

4. Cotauberian operators

In this section we show that the cotauberian convolution operators T_μ acting on $L_1(G)$ are always Fredholm, and that T_μ is tauberian if and only if its natural extension to the algebra of measures $M(G)$ is tauberian. We derive some consequences for convolution operators acting on $C_0(G)$ and $L_\infty(G)$, and we answer a question raised in [8] about the measures $\mu \in M(G)$ such that $\nu \in M(G)$ and $\mu \star \nu \in L_1(G)$ imply $\nu \in L_1(G)$.

First we show that the Banach algebras involved in our arguments are without order.

Proposition 4.1. *Let G be a LCA group. Then the algebra $(L_1(G)^{**}, \cdot)$ admits a norm-one right identity; hence it is a Banach algebra without order. Moreover, when the group G is compact, the quotient algebra $L_1(G)^{**}/L_1(G)$ also admits a norm-one right identity and it is a Banach algebra without order.*

The multipliers of algebras without order have a good behavior under duality:

Proposition 4.2. *Let A a Banach algebra without order and let T be a multiplier of A. Then the second conjugate $T^{**} : A^{**} \to A^{**}$ is a multiplier of A^{**}.*

Given a Banach space X, we denote by X^{co} the quotient space X^{**}/X. The second conjugate T^{**} of an operator $T : X \to Y$ induces another operator $T^{co} : X^{co} \to Y^{co}$ which is defined by $T^{co}(m + X) := T^{**}m + Y$ $(m \in X^{**})$, and it is called the *residuum operator* of T. Note that T is tauberian if and only if T^{co} is injective, and T is cotauberian if and only if T^{co} has dense range [10, Proposition 3.1.8 and Corollary 3.1.12].

Corollary 4.3. *Let G be a compact LCA group and let $T_\mu : L_1(G) \to L_1(G)$ be a convolution operator. Then the residuum operator T_μ^{co} is a multiplier of the algebra $L_1(G)^{co}$.*

Next we show that cotauberian convolution operators on $L_1(G)$ are tauberian. This result contrasts with the fact that it is easy to find non-trivial cotauberian operators on $L_1(G)$, just take a surjective operator with non-reflexive kernel, but it is much more difficult to obtain a non-trivial tauberian operator (see [12]).

Proposition 4.4. *Let G be a LCA group. Then every cotauberian convolution operator $T_\mu : L_1(G) \to L_1(G)$ is tauberian.*

Corollary 4.5. *Let G be a non-compact LCA group. A convolution operator on $L_1(G)$ is cotauberian if and only if it is invertible.*

Let E be a right identity in $L_1(G)^{**}$ provided by Proposition 4.1. We consider the map $\Gamma_E : M(G) \to L_1(G)^{**}$ defined by

$$\Gamma_E(\mu) := T_\mu^{**}(E), \quad \mu \in M(G).$$

The map Γ_E is an isometric algebra homomorphism of $M(G)$ into $L_1(G)^{**}$ which extends the natural embedding of $L_1(G)$ into $L_1(G)^{**}$ [7, Proposition 2.3].

Since T_μ^{**} is a multiplier of $L_1(G)^{**}$, for each $m \in L_1(G)^{**}$ we have

$$T_\mu^{**}m = (T_\mu^{**}m) \cdot E = m \cdot T_\mu^{**}E = m \cdot \Gamma_E(\mu). \tag{3}$$

Thus T_μ^{**} is a right multiplication operator (by $\Gamma_E(\mu)$). Moreover

$$E \cdot \Gamma_E(\mu) = T_\mu^{**}(E) = \Gamma_E(\mu). \tag{4}$$

Next we give our main result.

Theorem 4.6. *Let G be a LCA group. Then $T_\mu : L_1(G) \to L_1(G)$ is cotauberian if and only if it is Fredholm of index zero.*

To prove Theorem 4.6, we note that T_μ cotauberian implies T_μ tauberian (Proposition 4.4). Then, in the case G non-compact, Theorem 3.1 implies that T_μ is invertible.

In the case G compact, $L_1(G)^{co}$ is a Banach algebra, and we prove that T_μ cotauberian implies that the residuum operator T_μ^{co} acting on $L_1(G)^{co}$ is bijective, and from the inverse of T_μ^{co} we get an inverse of T_μ modulo the compact operators, hence T_μ is Fredholm.

Next we study the relation between a convolution operator $T_\mu : L_1(G) \to L_1(G)$ and its extension $M_\mu : M(G) \to M(G)$ defined by $M_\mu(\nu) = \mu \star \nu$.

Theorem 4.7. *Let G be a LCA group. Then T_μ is tauberian if and only if M_μ is tauberian.*

Proof. Suppose that T_μ is tauberian, and let E be a right identity in $L_1(G)^{**}$. Then the following diagram is commutative:

$$\begin{tikzcd} L_1(G)^{**} \arrow[r, "T_\mu^{**}"] & L_1(G)^{**} \\ M(G) \arrow[u, "\Gamma_E"] \arrow[r, "M_\mu"] & M(G) \arrow[u, "\Gamma_E"'] \end{tikzcd}$$

Now T_μ tauberian implies T_μ^{**} tauberian [10, Theorem 4.4.2]. Therefore $T_\mu^{**}\Gamma_E = \Gamma_E M_\mu$ is tauberian, and hence M_μ is tauberian, in both cases by [10, Proposition 2.1.3].

Similarly, denoting by $J : L_1(G) \to M(G)$ the natural isomorphic embedding, we have $JT_\mu = M_\mu J$. Hence, by [10, Proposition 2.1.3], if M_μ is tauberian, so is T_μ. □

Recall that an operator $T : L_1(G) \to L_1(G)$ is tauberian if and only if $m \in L_1(G)^{**}$ and $T^{**}m \in L_1(G)$ imply $m \in L_1(G)$. In particular, if T_μ is tauberian, then $\nu \in M(G)$ and $\mu \star \nu \in L_1(G)$ imply $\nu \in L_1(G)$.

Observation 4.8. *It was asked in* [8] *whether a convolution operator $T_\mu : L_1(G) \to L_1(G)$ is tauberian when the measure μ satisfies the following condition:*

$$\nu \in M(G), \mu \star \nu \in L_1(G) \Rightarrow \nu \in L_1(G). \tag{5}$$

Next we will show that the answer to this question is negative.

Indeed, it was proved in [4] that there exists an atomic measure $\mu_0 \in M(\mathbb{T})$ such that T_{μ_0} is an injective non-tauberian operator, where $\mathbb{T}$ denotes the unit circle. It is enough to choose μ_0 such that its Fourier-Stieltjes transform $\widehat{\mu}_0$ satisfies $0 \in \overline{\widehat{\mu}_0(\mathbb{Z})} \setminus \widehat{\mu}_0(\mathbb{Z})$. The following argument, due to Doss [6], shows that T_{μ_0} satisfies formula (5):

Every $\nu \in M(\mathbb{T})$ can be written as $\nu = \nu_1 + \nu_2$ with $\nu_1 \ll m$ and $\nu_2 \perp m$, where m is the Haar measure on $\mathbb{T}$. Since $\mu_0 \star \nu_1 \in L_1(\mathbb{T})$ and $\mu_0 \star \nu_2$ is supported in a m-null set, $T_{\mu_0}\nu \in L_1(G)$ if and only if $\nu_2 = 0$. □

Note that $\mu \star \widetilde{f} = \widetilde{\widetilde{\mu} \star f}$ for $\mu \in M(G)$ and $f \in L_1(G)$. Also, if a sequence $(f_n) \subset L_1(G)$ is normalized and disjoint, then so is $(\widetilde{f_n})$. Therefore, it follows from [10, Theorem 4.1.3] that T_μ is tauberian if and only if so is $T_{\widetilde{\mu}}$. Hence, by Theorem 4.7, the same happens for M_μ and $M_{\widetilde{\mu}}$, and we get the following result, where $S_\mu : C_0(G) \to C_0(G)$ and its extension $L_\mu : L_\infty(G) \to L_\infty(G)$ are defined by $L_\mu g = \mu \star g$.

Proposition 4.9. *Let G be a non-compact LCA group. Then*

(i) *$L_\mu : L_\infty(G) \to L_\infty(G)$ is tauberian if and only if it is cotauberian, and this is equivalent to L_μ invertible;*

(ii) *$M_\mu : M(G) \to M(G)$ is tauberian if and only if it is invertible;*

(iii) *$S_\mu : C_0(G) \to C_0(G)$ is cotauberian if and only if it is invertible.*

5. Some open questions

The main question that remains open is the following one.

Question 1. Let G be a compact LCA group and let $T_\mu : L_1(G) \to L_1(G)$ be a tauberian operator. Is $T_m u$ Fredholm?

This question admits equivalent formulations:

Question 2. Let G be a compact LCA group and let $T_\mu : L_1(G) \to L_1(G)$ be a tauberian operator. Is T_μ cotauberian?

Observation 4.8 gives a negative answer to a problem raised in [8], but we can reformulate it as follows.

Question 3. Find a condition additional to $\nu \in M(G), \mu \star \nu \in L_1(G) \Rightarrow \nu \in L_1(G)$ implying T_μ tauberian.

We have seen in Theorem 4.7 that $T_\mu : L_1(G) \to L_1(G)$ is tauberian if and only if so is $M_\mu : M(G) \to M(G)$. The second condition is much stronger.

Question 4. Find characterizations of T_μ tauberian in terms of the restrictions

$$M_\mu|_{L_1(|\nu|)} : L_1(|\nu|) \to M(G)$$

for special measures $\nu \in M(G)$ (different from the Haar measure m).

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