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Velocity and acceleration at a point on the paths $A \natural_t B$ and $A \sharp_{t,r} B$

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1. Introduction.

This report is based on [13]. Let A and B be strictly positive linear operators on a Hilbert space $\mathcal{H}$. An operator T on $\mathcal{H}$ is said to be positive (denoted by $T \geq 0$) if $(T\xi, \xi) \geq 0$ for all $\xi \in \mathcal{H}$ and T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Fujii and Kamei [1] defined the following relative operator entropy for $A, B > 0$:

$$S(A|B) \equiv A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

Furuta [6] defined generalized relative operator entropy as follows (see also [8]):

$$S_\alpha(A|B) \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \quad (\alpha \in \mathbf{R}).$$

We know immediately $S_0(A|B) = S(A|B)$.

Yanagi, Kuriyama and Furuichi [16] introduced Tsallis relative operator entropy as follows:

$$T_\alpha(A|B) \equiv \frac{A \sharp_\alpha B - A}{\alpha} \quad (\alpha \in (0,1]),$$

where $A \sharp_\alpha B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$ is the weighted geometric operator mean (cf. [15]). Since $\lim_{x\to 0}\frac{a^x - 1}{x} = \log a$ holds for $a > 0$, we have $T_0(A|B) \equiv \lim_{\alpha \to 0} T_\alpha(A|B) = S(A|B)$.

For $A, B > 0$, we define a path $A \natural_t B$ as follows ([2, 3, 5, 7, 12, 14] etc.):

$$A \natural_t B \equiv A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}} \quad (t \in \mathbf{R}),$$

which is passing through $A = A \natural_0 B$ and $B = A \natural_1 B$. If $t \in [0,1]$, the path $A \natural_t B$ coincides with $A \sharp_t B$ (cf. [15]). So we can extend Tsallis relative operator entropy $T_\alpha(A|B)$ for $\alpha \in \mathbf{R}$. We remark that $A \natural_t B = B \natural_{1-t} A$ holds for $t \in \mathbf{R}$ (cf. [7]). We know immediately that

$$S(A|B) = \frac{d}{dt} A \natural_t B\bigg|_{t=0} \text{ and } S_\alpha(A|B) = \frac{d}{dt} A \natural_t B\bigg|_{t=\alpha}.$$

$T_\alpha(A|B)$ can be regarded as the average rate of change of $A \sharp_x B$ from $x = 0$ to $x = \alpha$. We illustrate an image for $S(A|B)$, $S_\alpha(A|B)$ and $T_\alpha(A|B)$ in Figure 1.

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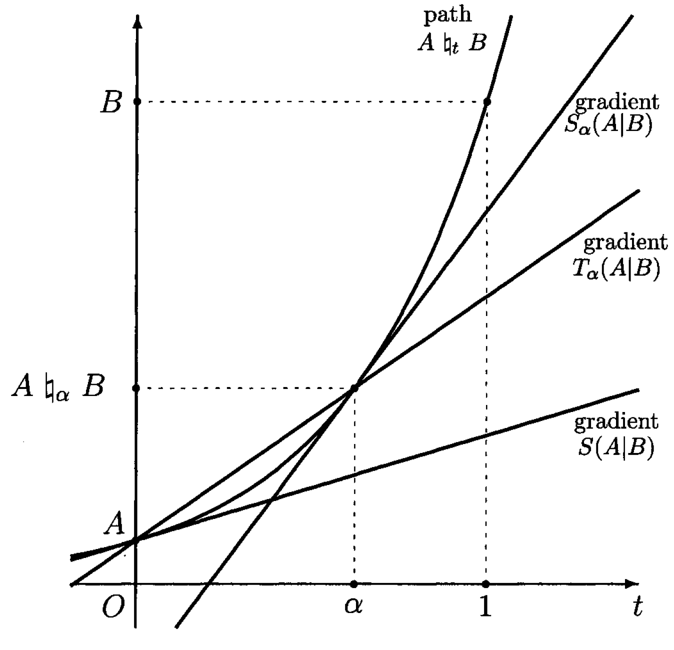

Figure 1. An image of $S(A|B)$, $S_\alpha(A|B)$ and $T_\alpha(A|B)$.

In [10], we introduced noncommutative ratio $\mathcal{R}(v; A, B)$ on the path $A \natural_t B$ as follows:

Definition. *For $A, B > 0$ and $v \in \mathbf{R}$, we define*

$$\mathcal{R}(v; A, B) \equiv (A \natural_v B)A^{-1}.$$

We have the following relation by applying noncommutative ratio $\mathcal{R}(v; A, B)$ to $S_u(A|B)$ [10]:

$$\mathcal{R}(v; A, B)S_u(A|B) \;=\; S_{u+v}(A|B) \;=\; \frac{d}{dt} A \natural_t B\bigg|_{t=u+v}.$$

This relation means that $\mathcal{R}(v; A, B)$ is the ratio of $S_u(A|B)$ and $S_{u+v}(A|B)$.

For $A, B > 0$, $t \in [0, 1]$ and $r \in [-1, 1]$, the operator power mean $A \sharp_{t,r} B$ is defined as follows:

$$A \sharp_{t,r} B \equiv A^{\frac{1}{2}} \left\{(1-t)I + t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r\right\}^{\frac{1}{r}} A^{\frac{1}{2}} \;=\; A \natural_{\frac{1}{r}} \{A \nabla_t (A \natural_r B)\}.$$

We remark that $A \sharp_{t,r} B = B \sharp_{1-t,r} A$ holds for $t \in [0, 1]$ and $r \in [-1, 1]$ (cf. [9, 11]). The operator power mean is a path combining $A = A \sharp_{0,r} B$ and $B = A \sharp_{1,r} B$, and interpolates the arithmetic operator mean, the geometric operator mean and the harmonic operator mean.

arithmetic operator mean

$$A \nabla_t B = (1-t)A + tB$$

$$\big\uparrow_{r=1}$$

$$A \sharp_{t,r} B \underset{r\to 0}{\longrightarrow} \begin{array}{l}\text{geometric operator mean}\\ A \natural_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}\end{array}$$

$$\big\downarrow_{r=-1}$$

harmonic operator mean

$$A \Delta_t B = (A^{-1} \nabla_t B^{-1})^{-1}$$

For $A,\ B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$, expanded relative operator entropy $S_{\alpha,r}(A|B)$ and generalized Tsallis relative operator entropy $T_{\alpha,r}(A|B)$ are defined as follows (cf. [9]):

$$\begin{aligned}
S_{\alpha,r}(A|B) &\equiv \left.\frac{d}{dt} A \sharp_{t,r} B\right|_{t=\alpha} \\
&= A^{\frac{1}{2}}\left[\left\{(1-\alpha)I + \alpha\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-1}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right]A^{\frac{1}{2}} \\
&= (A \sharp_{\alpha,r} B)(A \nabla_\alpha (A \natural_r B))^{-1} S_{0,r}(A|B)\ (r \neq 0), \\
S_{\alpha,0}(A|B) &\equiv \lim_{r\to 0} S_{\alpha,r}(A|B) \ = \ S_\alpha(A|B), \\
T_{\alpha,r}(A|B) &\equiv \frac{A \sharp_{\alpha,r} B - A}{\alpha} \quad (\alpha \neq 0),\ T_{0,r}(A|B) \equiv \lim_{\alpha\to 0} T_{\alpha,r}(A|B) \ = \ T_r(A|B).
\end{aligned}$$

We remark that $S_{0,r}(A|B) = T_r(A|B)$, $S_{1,r}(A|B) = -T_r(B|A)$ and $T_{1,r}(A|B) = B - A$ hold for $r \in [-1,1]$.

In [5], $S(A|B)$ and $S_{0,r}(A|B)$ are given as the viewpoints of the velocity on the paths $A \natural_t B$ and $A \sharp_{t,r} B$ at $t = 0$ respectively since $S(A|B) = \left.\frac{d}{dt} A \natural_t B\right|_{t=0}$ and $S_{0,r}(A|B) = \left.\frac{d}{dt} A \sharp_{t,r} B\right|_{t=0}$. According to this viewpoint, it is natural to call $S_\alpha(A|B)$ and $S_{\alpha,r}(A|B)$ the velocities on the paths $A \natural_t B$ and $A \sharp_{t,r} B$ at $t = \alpha$ respectively.

In this report, we introduce the accelerations $\mathcal{A}_\alpha(A|B)$ and $\mathcal{A}_{\alpha,r}(A|B)$ on the paths $A \natural_t B$ and $A \sharp_{t,r} B$ at $t = \alpha$, and we can show that the properties of the acceleration are inherited from those of velocity. In section 2, we discuss some properties of velocity $S_\alpha(A|B)$ and the acceleration $\mathcal{A}_\alpha(A|B)$. In section 3, we show properties of the velocity $S_{\alpha,r}(A|B)$ and the acceleration $\mathcal{A}_{\alpha,r}(A|B)$ on the path $A \sharp_{t,r} B$ which are similar to those shown in the section 2.

2. Velocity and acceleration on the path $A \natural_t B$.

Since $S_\alpha(A|B)$ can be considered as the velocity on the path $A \natural_t B$ at $t = \alpha$. In this section, we introduce the acceleration on the path $A \natural_t B$ and we show that the acceleration inherits the properties from $S_\alpha(A|B)$.

Since the relative operator entropy $S_\alpha(A|B)$ is regarded as the velocity on the path $A \natural_t B$ at $t = \alpha$, it is natural to call the derivative of $S_t(A|B)$ the acceleration on $A \natural_t B$.

Definition 2.1. *For $A, B > 0$ and $\alpha \in \mathbf{R}$, we define the acceleration on the path $A \natural_t B$ at $t = \alpha$ as follows:*

$$\mathcal{A}_\alpha(A|B) \equiv \frac{d}{dt} S_t(A|B) \Big|_{t=\alpha}.$$

The acceleration $\mathcal{A}_\alpha(A|B)$ is represented explicitly as follows:

Theorem 2.2. *Let $A, B > 0$ and $\alpha \in \mathbf{R}$. Then*

$$\mathcal{A}_\alpha(A|B) = S_\alpha(A|B)A^{-1}S(A|B) = S_\alpha(A|B)(A \natural_\alpha B)^{-1}S_\alpha(A|B).$$

In particular,

$$\mathcal{A}_0(A|B) = S(A|B)A^{-1}S(A|B).$$

Proof. For $a > 0$, we have

$$\frac{d}{dt} a^t \log a = a^t (\log a)^2.$$

Then

$$\begin{aligned}
\mathcal{A}_\alpha(A|B) &= \frac{d}{dt} S_t(A|B) \Big|_{t=\alpha} \\
&= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \left(\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)^2 A^{\frac{1}{2}} \\
&= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}A^{-1}A^{\frac{1}{2}} \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\
&= S_\alpha(A|B)A^{-1}S(A|B),
\end{aligned}$$

which shows the first equality. On the other hand, we have

$$\begin{aligned}
S_\alpha(A|B)A^{-1}S(A|B) &= S_\alpha(A|B)(A \natural_\alpha B)^{-1}(A \natural_\alpha B)A^{-1}S(A|B) \\
&= S_\alpha(A|B)(A \natural_\alpha B)^{-1}S_\alpha(A|B).
\end{aligned}$$

□

Remark. The equation $\ddot{\gamma}(t) - \dot{\gamma}(t)\big(\gamma(t)\big)^{-1}\dot{\gamma}(t) = 0$ for a smooth function $\gamma(t)$ is called geodesic equation. If $\gamma(t) = A \natural_t B$, then $\dot{\gamma}(t) = S_t(A|B)$ and $\ddot{\gamma}(t) = \mathcal{A}_t(A|B)$. Hence, the function $\gamma(t) = A \natural_t B$ satisfies the geodesic equation by Theorem 3.2. It shows that the path $A \natural_t B$ is the geodesic in $B(\mathcal{H})^+$ which is a manifold consisting of all positive invertible operators on a Hilbert space $\mathcal{H}$ (see [4]).

We show some properties of the velocity and the acceleration on the path $A \natural_t B$. The next lemma is fundamental in our discussion.

Lemma 2.3. ([11]) *For $A, B > 0$ and $x, y, \alpha \in \mathbf{R}$,*

$$(A \natural_y B) \natural_\alpha (A \natural_x B) = A \natural_{(1-\alpha)y+\alpha x} B$$

holds.

Let $A \natural_x B$ and $A \natural_y B$ ($x, y \in \mathbf{R}$) be arbitrary points on the path $A \natural_t B$. Concerning the velocity $S_\alpha(A \natural_y B|A \natural_x B)$ and the acceleration $\mathcal{A}_\alpha(A \natural_y B|A \natural_x B)$ at $t = \alpha$, we have the following relations.

Theorem 2.4. *Let $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$. Then*

$$\begin{array}{lrcl} (1) & S_\alpha(A \natural_y B | A \natural_x B) & = & (x-y) S_{(1-\alpha)y+\alpha x}(A|B). \\ (2) & \mathcal{A}_\alpha(A \natural_y B | A \natural_x B) & = & (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x}(A|B). \end{array}$$

Proof. (1) This relation was already proved in [11].

(2) By (1) in Theorem 2.4, Theorem 2.2 and Lemma 2.3, we have

$$\begin{aligned} &\mathcal{A}_\alpha(A \natural_y B | A \natural_x B) \\ &\quad = S_\alpha(A \natural_y B | A \natural_x B)((A \natural_y B) \natural_\alpha (A \natural_x B))^{-1} S_\alpha(A \natural_y B | A \natural_x B) \\ &\quad = (x-y)^2 S_{(1-\alpha)y+\alpha x}(A|B)(A \natural_{(1-\alpha)y+\alpha x} B)^{-1} S_{(1-\alpha)y+\alpha x}(A|B) \\ &\quad = (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x}(A|B). \end{aligned}$$

□

The next Corollary 2.5 is an immediate consequence of Theorem 2.4.

Corollary 2.5. *For $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$, the following hold:*

(1) $S_\alpha(B|A) = -S_{1-\alpha}(A|B)$ *and* $\mathcal{A}_\alpha(B|A) = \mathcal{A}_{1-\alpha}(A|B)$.

(2) $S_\alpha(A|A \natural_x B) = x S_{\alpha x}(A|B)$ *and* $\mathcal{A}_\alpha(A|A \natural_x B) = x^2 \mathcal{A}_{\alpha x}(A|B)$.

(3) $S_\alpha(A \natural_y B | A \natural_{y+1} B) = S_{\alpha+y}(A|B)$ *and* $\mathcal{A}_\alpha(A \natural_y B | A \natural_{y+1} B) = \mathcal{A}_{\alpha+y}(A|B)$.

Conversely, the statements (1) and (2) or statements (2) and (3) in Corollary 2.5 imply Theorem 2.4, so we have

Theorem 2.6. *For $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$, the following statements hold and they are equivalent.*

$$\begin{array}{ll} (1) & S_\alpha(A \natural_y B | A \natural_x B) = (x-y) S_{(1-\alpha)y+\alpha x}(A|B). \\ (2) & S_\alpha(B|A) = -S_{1-\alpha}(A|B) \text{ and } S_\alpha(A|A \natural_x B) = x S_{\alpha x}(A|B). \\ (3) & S_\alpha(A|A \natural_x B) = x S_{\alpha x}(A|B) \text{ and } S_\alpha(A \natural_y B | A \natural_{y+1} B) = S_{\alpha+y}(A|B). \end{array}$$

Proof. We have only to show that (2) implies (1) and (3) implies (1).

(2) $\Rightarrow$ (1) It is trivial if $y = 0$. If $y \neq 0$, we have

$$A \natural_x B = A \natural_{\frac{x}{y}} (A \natural_y B) = (A \natural_y B) \natural_{\frac{y-x}{y}} A.$$

Then

$$\begin{aligned} S_\alpha(A \natural_y B | A \natural_x B) &= S_\alpha(A \natural_y B | (A \natural_y B) \natural_{\frac{y-x}{y}} A) = \frac{y-x}{y} S_{\frac{\alpha(y-x)}{y}}(A \natural_y B | A) \\ &= -\frac{y-x}{y} S_{1-\frac{\alpha(y-x)}{y}}(A | A \natural_y B) = (x-y) S_{(1-\alpha)y+\alpha x}(A|B). \end{aligned}$$

(3) $\Rightarrow$ (1) From Lemma 2.3, we get

$$\begin{aligned} S_\alpha(A \natural_y B | A \natural_x B) &= S_\alpha\big(A \natural_y B | (A \natural_y B) \natural_{x-y} (A \natural_{y+1} B)\big) \\ &= (x-y) S_{\alpha(x-y)}(A \natural_y B | A \natural_{y+1} B) \\ &= (x-y) S_{\alpha(x-y)+y}(A|B) = (x-y) S_{(1-\alpha)y+\alpha x}(A|B). \end{aligned}$$

□

Theorem 2.7. *For $A, B > 0$ and $\alpha, x, y \in \mathbf{R}$, the following statements hold and they are equivalent.*

(1) $\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) = (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x}(A|B)$.

(2) $\mathcal{A}_\alpha(B|A) = \mathcal{A}_{1-\alpha}(A|B)$ *and* $\mathcal{A}_\alpha(A|A \natural_x B) = x^2 \mathcal{A}_{\alpha x}(A|B)$.

(3) $\mathcal{A}_\alpha(A|A \natural_x B) = x^2 \mathcal{A}_{\alpha x}(A|B)$ *and* $\mathcal{A}_\alpha(A \natural_y B|A \natural_{y+1} B) = \mathcal{A}_{\alpha+y}(A|B)$.

Proof. This relation can be obtained by similar way to Theorem 2.6. □

Next, we show the properties of the velocity and the acceleration related to the noncommutative ratio $\mathcal{R}(v; A, B)$.

Theorem 2.8. *For $A, B > 0$ and $u, v \in \mathbf{R}$, the following hold:*

(1) $\mathcal{R}(v; A, B)S_u(A|B) = S_{u+v}(A|B) = S_u(A \natural_v B|A \natural_{v+1} B)$.

(2) $\mathcal{R}(v; A, B)\mathcal{A}_u(A|B) = \mathcal{A}_{u+v}(A|B) = \mathcal{A}_u(A \natural_v B|A \natural_{v+1} B)$.

Proof. (1) This result was shown in [10].

(2) By (1) in Theorem 2.8, Theorem 2.2 and (3) in Corollary 2.5, we have

$$\begin{aligned} \mathcal{R}(v; A, B)\mathcal{A}_u(A|B) &= \mathcal{R}(v; A, B)S_u(A|B)A^{-1}S(A|B) \\ = S_{u+v}(A|B)A^{-1}S(A|B) &= \mathcal{A}_{u+v}(A|B) = \mathcal{A}_u(A \natural_v B|A \natural_{v+1} B). \end{aligned}$$

□

Corollary 2.9. *For $A, B > 0$ and $v \in \mathbf{R}$, the following hold:*

(1) $\mathcal{R}(v; A, B)S(A|B) = S_v(A|B)$.

(2) $\mathcal{R}(v; A, B)\mathcal{A}_0(A|B) = \mathcal{A}_v(A|B)$.

Furthermore, we obtain an extension of Theorem 2.8.

Theorem 2.10. *Let $A, B > 0$ and $\alpha, v, x, y \in \mathbf{R}$. Then*

(1) $\mathcal{R}(v; A, B)S_\alpha(A \natural_y B|A \natural_x B) = S_\alpha(A \natural_{y+v} B|A \natural_{x+v} B)$.

(2) $\mathcal{R}(v; A, B)\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) = \mathcal{A}_\alpha(A \natural_{y+v} B|A \natural_{x+v} B)$.

Proof. (1) By (1) in Theorem 2.4 and (1) in Theorem 2.8, we have

$$\begin{aligned} \mathcal{R}(v; A, B)S_\alpha(A \natural_y B|A \natural_x B) &= (x-y)\mathcal{R}(v; A, B)S_{(1-\alpha)y+\alpha x}(A|B) \\ &= (x-y)S_{(1-\alpha)y+\alpha x+v}(A|B) \\ &= \{(x+v)-(y+v)\}\, S_{(1-\alpha)(y+v)+\alpha(x+v)}(A|B) \\ &= S_\alpha(A \natural_{y+v} B|A \natural_{x+v} B). \end{aligned}$$

(2) By (2) in Theorem 2.4 and (2) in Theorem 2.8, we have

$$\begin{aligned} \mathcal{R}(v; A, B)\mathcal{A}_\alpha(A \natural_y B|A \natural_x B) &= (x-y)^2\mathcal{R}(v; A, B)\mathcal{A}_{(1-\alpha)y+\alpha x}(A|B) \\ &= (x-y)^2\mathcal{A}_{(1-\alpha)y+\alpha x+v}(A|B) \\ &= \{(x+v)-(y+v)\}^2\, \mathcal{A}_{(1-\alpha)(y+v)+\alpha(x+v)}(A|B) \\ &= \mathcal{A}_\alpha(A \natural_{y+v} B|A \natural_{x+v} B). \end{aligned}$$

□

We remark that the following proposition holds on the noncommutative ratio, and also we can give an alternative proof of Theorem 2.10 by using this proposition.

Proposition 2.11. *For $A, B > 0$ and $v, x, y \in \mathbf{R}$, the following hold:*

$$\mathcal{R}(v(x-y); A, B) \;=\; \mathcal{R}(v; A \natural_y B, A \natural_x B).$$

Proof. By using Lemma 2.3, we have

$$\begin{aligned}
\mathcal{R}(v(x-y); A, B) &= (A \natural_{v(x-y)} B)A^{-1} \\
&= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{v(x-y)+y}A^{\frac{1}{2}}A^{-\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{-y}A^{-\frac{1}{2}} \\
&= (A \natural_{(1-v)y+vy} B)(A \natural_y B)^{-1} \;=\; ((A \natural_y B) \natural_v (A \natural_x B))\,(A \natural_y B)^{-1} \\
&= \mathcal{R}(v; A \natural_y B, A \natural_x B).
\end{aligned}$$

□

Alternative proof of Theorem 2.10. (1) By Proposition 2.11 and (1) in Theorem 2.8, we have

$$\begin{aligned}
\mathcal{R}(v; A, B)S_\alpha(A \natural_y B | A \natural_x B) &= \mathcal{R}\left(\frac{v}{x-y}; A \natural_y B, A \natural_x B\right) S_\alpha(A \natural_y B | A \natural_x B) \\
&= S_\alpha((A \natural_y B) \natural_{\frac{v}{x-y}} (A \natural_x B) | (A \natural_y B) \natural_{\frac{v}{x-y}+1} (A \natural_x B)) \\
&= S_\alpha(A \natural_{(1-\frac{v}{x-y})y+\frac{vx}{x-y}} B | A \natural_{-\frac{vy}{x-y}+(\frac{v}{x-y}+1)x} B) \\
&= S_\alpha(A \natural_{y+v} B | A \natural_{x+v} B).
\end{aligned}$$

(2) This relation can be obtained by the similar way to (1). □

3. Velocity and acceleration on the path $A \sharp_{t,r} B$.

In this section, we introduce the velocity and the acceleration on the path $A \sharp_{t,r} B$ and show their properties.

We know that the path $A \sharp_{t,r} B$ has similar properties to those of $A \natural_t B$. For example,

Lemma 3.1. ([14]) *For $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$,*

$$(A \sharp_{y,r} B) \sharp_{\alpha,r} (A \sharp_{x,r} B) \;=\; A \sharp_{(1-\alpha)y+\alpha x, r} B$$

holds.

Theorem 3.2. ([11]) *For $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$,*

$$S_{\alpha,r}(A|B) \;=\; (A \sharp_{\alpha,r} B)(A \nabla_\alpha (A \natural_r B))^{-1} S_{0,r}(A|B)$$

holds.

Theorem 3.2 shows that $(A \sharp_{\alpha,r} B)(A \nabla_\alpha (A \natural_r B))^{-1}$ is a partial extension of the noncommutative ratio on $A \natural_t B$.

We introduce the acceleration on the path $A \sharp_{t,r} B$ as follows:

Definition 3.3. *For $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$, we define $\mathcal{A}_{\alpha,r}(A|B)$ as*

$$\mathcal{A}_{\alpha,r}(A|B) \equiv \frac{d}{dt}S_{t,r}(A|B)\bigg|_{t=\alpha}.$$

We call it the acceleration on the path $A \sharp_{t,r} B$ at $t = \alpha$.

The acceleration $\mathcal{A}_{\alpha,r}(A|B)$ is represented explicitly as follows:

Theorem 3.4. *Let $A, B > 0$, $\alpha \in [0,1]$ and $r \in [-1,1]$. Then*

$$\begin{aligned}\mathcal{A}_{\alpha,r}(A|B) &= (1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B) \\ &= (1-r)S_{\alpha,r}(A|B)(A \sharp_{\alpha,r} B)^{-1}S_{\alpha,r}(A|B).\end{aligned}$$

Proof. We have shown the case $r = 0$ in Theorem 2.2. Hence, we have only to show the case $r \neq 0$. Since

$$\begin{aligned}&\frac{d}{dt}S_{t,r}(A|B) \\ &= (1-r)A^{\frac{1}{2}}\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-2}\left(\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right)^2 A^{\frac{1}{2}} \\ &= (1-r)A^{\frac{1}{2}}\left[\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{\frac{1}{r}-1}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}\right]A^{\frac{1}{2}} \\ &\quad \times A^{-\frac{1}{2}}\left\{(1-t)I + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right\}^{-1}A^{-\frac{1}{2}}A^{\frac{1}{2}}\frac{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r - I}{r}A^{\frac{1}{2}} \\ &= (1-r)S_{t,r}(A|B)(A \nabla_t (A \natural_r B))^{-1}T_r(A|B) \\ &= (1-r)S_{t,r}(A|B)(A \nabla_t (A \natural_r B))^{-1}S_{0,r}(A|B),\end{aligned}$$

we have

$$\mathcal{A}_{\alpha,r}(A|B) = (1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B).$$

On the other hand, by Theorem 3.2, we have

$$(1-r)S_{\alpha,r}(A|B)(A \nabla_\alpha (A \natural_r B))^{-1}S_{0,r}(A|B) = (1-r)S_{\alpha,r}(A|B)(A \sharp_{\alpha,r} B)^{-1}S_{\alpha,r}(A|B).$$

□

From Theorem 3.4, we have the following properties for $\mathcal{A}_{\alpha,r}(A|B)$.

Corollary 3.5. *For $A, B > 0$, the following hold:*

(1) $\mathcal{A}_{\alpha,0}(A|B) = \mathcal{A}_\alpha(A|B)$ $(\alpha \in [0,1])$.

(2) $\mathcal{A}_{0,r}(A|B) = (1-r)S_{0,r}(A|B)A^{-1}S_{0,r}(A|B)$ $(r \in [-1,1])$.

By Lemma 3.1 and Theorem 3.4, we obtain similar properties discussed in section 2. First, we show similar relations to Theorem 2.4 and Corollary 2.5.

Theorem 3.6. *Let $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$. Then*

$$\begin{array}{ll} (1) & S_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \ = \ (x-y) S_{(1-\alpha)y+\alpha x,r}(A|B). \\ (2) & \mathcal{A}_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \ = \ (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x,r}(A|B). \end{array}$$

Proof. (1) By using Lemma 3.1, we have

$$\begin{aligned} & S_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \\ & \quad = \frac{d}{dt} (A \natural_{y,r} B) \natural_{t,r} (A \natural_{x,r} B) \Big|_{t=\alpha} \\ & \quad = \lim_{v \to 0} \frac{(A \natural_{y,r} B) \natural_{\alpha+v,r} (A \natural_{x,r} B) - (A \natural_{y,r} B) \natural_{\alpha,r} (A \natural_{x,r} B)}{v} \\ & \quad = \lim_{v \to 0} \frac{A \natural_{(1-(\alpha+v))y+(\alpha+v)x,r} B - A \natural_{(1-\alpha)y+\alpha x,r} B}{v} \\ & \quad = (x-y) \lim_{v \to 0} \frac{A \natural_{(1-\alpha)y+\alpha x+v(x-y),r} B - A \natural_{(1-\alpha)y+\alpha x,r} B}{(x-y)v} \\ & \quad = (x-y) S_{(1-\alpha)y+\alpha x,r}(A|B). \end{aligned}$$

(2) From (1) in Theorem 3.6 and Lemma 3.1, we obtain

$$\begin{aligned} & \mathcal{A}_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \\ & = (1-r) S_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \big((A \natural_{y,r} B) \natural_{\alpha,r} (A \natural_{x,r} B) \big)^{-1} S_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \\ & = (1-r)(x-y)^2 S_{(1-\alpha)y+\alpha x,r}(A|B) (A \natural_{(1-\alpha)y+\alpha x,r} B)^{-1} S_{(1-\alpha)y+\alpha x,r}(A|B) \\ & = (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x,r}(A|B). \end{aligned}$$

□

Corollary 3.7. *For $A, B > 0$, $\alpha, x \in [0,1]$ and $r \in [-1,1]$, the following hold:*

$$\begin{array}{ll} (1) & S_{\alpha,r}(B|A) \ = \ -S_{1-\alpha,r}(A|B) \ \textit{and} \ S_{\alpha,r}(A|A \natural_{x,r} B) \ = \ x S_{\alpha x,r}(A|B). \\ (2) & \mathcal{A}_{\alpha,r}(B|A) \ = \mathcal{A}_{1-\alpha,r}(A|B) \ \textit{and} \ \mathcal{A}_{\alpha,r}(A|A \natural_{x,r} B) \ = \ x^2 \mathcal{A}_{\alpha x,r}(A|B). \end{array}$$

Second, we obtain the relations similar to Theorem 2.6 and Theorem 2.7 as follows:

Theorem 3.8. *For $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$, the following (1) and (2) are equivalent.*

$$\begin{array}{ll} (1) & S_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \ = \ (x-y) S_{(1-\alpha)y+\alpha x,r}(A|B). \\ (2) & S_{\alpha,r}(B|A) \ = \ -S_{1-\alpha,r}(A|B) \ \textit{and} \ S_{\alpha,r}(A|A \natural_{x,r} B) \ = \ x S_{\alpha x,r}(A|B). \end{array}$$

Theorem 3.9. *For $A, B > 0$, $\alpha, x, y \in [0,1]$ and $r \in [-1,1]$, the following (1) and (2) are equivalent.*

$$\begin{array}{ll} (1) & \mathcal{A}_{\alpha,r}(A \natural_{y,r} B | A \natural_{x,r} B) \ = \ (x-y)^2 \mathcal{A}_{(1-\alpha)y+\alpha x,r}(A|B). \\ (2) & \mathcal{A}_{\alpha,r}(B|A) \ = \ \mathcal{A}_{1-\alpha,r}(A|B) \ \textit{and} \ \mathcal{A}_{\alpha,r}(A|A \natural_{x,r} B) \ = \ x^2 \mathcal{A}_{\alpha x,r}(A|B). \end{array}$$

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