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2成分 Camassa-Holm 方程式の多重ソリトン解とその簡約

Multisoliton solutions of the two-component Camassa-Holm equation and their reductions

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Abstract: The two-component Camassa-Holm (CH2) equation models the propagation of nonlinear surface gravity waves on shallow water. It has several remarkable features. Among them, it is a completely integrable system. By employing a direct method in soliton theory, we develop a systematic procedure for constructing multisoliton solutions of the CH2 equation, and explore their properties. Then, we show that the two integrable reductions are possible for the CH2 equation by means of appropriate scaling limits, leading to the CH and two component Hunter-Saxton equations. The reduced form of multisoliton solutions is presented for both equations.

1. Introduction

We consider the following two-component generalization of the Camassa-Holm (CH) equation (CH2 equation hereafter)

$$m_t + um_x + 2mu_x + \rho\rho_x = 0, \quad \rho_t + (\rho u)_x = 0. \tag{1.1}$$

Here, $u = u(x,t), \rho = \rho(x,t)$ and $m = m(x,t) \equiv u - u_{xx} + \kappa^2$ are real-valued functions of time t and a spatial variable x, and the subscripts x and t appended to u and ρ denote partial differentiation. The parameter κ in the expression of m is assumed to be a non-negative real number. In the physical context, the CH2 system arises as a model equation for shallow-water waves. Actually, it was derived from the Green-Naghdi equations by using an asymptotic analysis, where u is the leading order approximation of the horizontal velocity whereas ρ is related to the depth of the fluid at the leading order [1]. The same system was also derived from the basic Euler system for an incompressible fluid with a constant vorticity [2].

One remarkable feature of the CH2 equation is that it is a completely integrable system. Indeed, it has the Lax representation given by [1, 2]

$$\Psi_{xx} = \left(-\lambda^2\rho^2 + \lambda m + \frac{1}{4}\right)\Psi, \quad \Psi_t = \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{u_x}{2}\Psi. \tag{1.2}$$

Various reductions are possible for the CH2 equation while preserving its integrability. Specifically, if one puts $\rho = 0$, then the system reduces to the CH equation [3]

$$u_t + 2\kappa^2 u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1.3}$$

Another reduction is the two-component Hunter-Saxton (HS2) equation which can be derived by the short-wave limit of the CH2 equation [1]. It has the same form as Eq. (1.1) with the variable m replaced by $-u_{xx} + \kappa^2$.

In this paper, we develop a systematic procedure for constructing the multisoliton solutions of the CH2 equation, and explore their properties. The reduction procedure is performed for the soliton solutions of the CH2 equation to obtain the corresponding solutions of the CH and HS2 equations. Here, we describe only the main results, and the details will be reported elsewhere.

2. Exact method of solution

There exist several exact methods of solution for solving nonlinear evolution equations. Among them, we employ the direct method which has been initiated by Hirota [4, 5]. This method is particularly useful in obtaining soliton solutions. The method works effectively if one reduces the CH2 equation to a more tractable form by a reciprocal transformation. Following the standard procedure, the parametric representation of the N soliton solution will be constructed, where N is an arbitrary positive integer.

2.1. Reciprocal transformation

First of all, we introduce the reciprocal transformation $(x,t) \to (y,\tau)$ according to

$$dy = \rho\, dx - \rho u\, dt, \quad d\tau = dt. \tag{2.1a}$$

Then, the x and t derivatives transform as

$$\frac{\partial}{\partial x} = \rho\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \rho u\frac{\partial}{\partial y}. \tag{2.1b}$$

Applying the transformation (2.1) to Eq. (1.1), we obtain the system of PDEs for u and ρ

$$\left(\frac{m}{\rho^2}\right)_\tau + \rho_y = 0, \quad \rho_\tau + \rho^2 u_y = 0. \tag{2.2a, b}$$

It then follows from (2.1b) that the variable $x = x(y,\tau)$ obeys a system of linear PDEs

$$x_y = \frac{1}{\rho}, \quad x_\tau = u. \tag{2.3a, b}$$

The system of equations (2.3) is integrable since its compatibility condition $x_{\tau y} = x_{y\tau}$ is assured by virtue of (2.2b).

Now, the quantity $m = u - u_{xx} + \kappa^2$ in (1.1) can be rewritten in the new coordinate system as

$$m = u + \rho(\ln\rho)_{\tau y} + \kappa^2, \tag{2.4}$$

where we have used (2.2b) to replace u_y by $-\rho_\tau/\rho^2$.

Let us introduce the new dependent variable $Y = Y(y,\tau)$ by the relation

$$\frac{m}{\rho^2} - \frac{\kappa^2}{\rho_0^2} = Y_y. \tag{2.5}$$

Subsituting (2.5) into (2.2a) and then integrating the resultant expression by y under the boundary conditions $Y_\tau \to 0$ and $\rho \to \rho_0\,(> 0)$ as $|y| \to \infty$, we obtain

$$\rho = \rho_0 - Y_\tau. \tag{2.6}$$

The following proposition is the starting point in the present analysis.

Proposition 2.1. *The variables x and Y satisfy the system of PDEs*

$$x_y(\rho_0 - Y_\tau) = 1, \tag{2.7}$$

$$(\rho_0 - Y_\tau)\left(\frac{\kappa^2}{\rho_0^2} + Y_y\right) = x_\tau x_y - [(\rho_0 - Y_\tau)x_{\tau y}]_y + \kappa^2 x_y. \tag{2.8}$$

2.2. Bilinearization

In applying the direct method to the given nonlinear equations, the first step is to transform the equations into the bilinear equations, which we shall now demonstrate. To this end, we introduce the dependent variable transformations

$$x = \frac{y}{\rho_0} + \ln\frac{\tilde{f}}{f} + d, \quad Y = \mathrm{i}\ln\frac{\tilde{g}}{g}, \tag{2.9}$$

where $f, \tilde{f}, g$ and $\tilde{g}$ are tau-functions and d is an arbitrary constant. Then, we establish the following proposition.

Proposition 2.2. *Consider the following system of bilinear equations for $f, \tilde{f}, g$ and $\tilde{g}$:*

$$D_y\tilde{f}\cdot f + \frac{1}{\rho_0}(\tilde{f}f - \tilde{g}g) = 0, \tag{2.10}$$

$$\mathrm{i}D_\tau\tilde{g}\cdot g + \rho_0(\tilde{f}f - \tilde{g}g) = 0, \tag{2.11}$$

$$D_\tau D_y\tilde{f}\cdot f + \frac{1}{\rho_0}D_\tau\tilde{f}\cdot f + \kappa^2 D_y\tilde{f}\cdot f = 0, \tag{2.12}$$

$$D_\tau D_y\tilde{g}\cdot g - \mathrm{i}\frac{\kappa^2}{\rho_0^2}D_\tau\tilde{g}\cdot g + \mathrm{i}\rho_0 D_y\tilde{g}\cdot g = 0, \tag{2.13}$$

where the bilinear operators are defined by

$$D_y^m D_\tau^n f\cdot g = (\partial_y - \partial_{y'})^m(\partial_\tau - \partial_{\tau'})^n f(y,\tau)g(y',\tau')|_{y'=y,\,\tau'=\tau}, \qquad (m,n = 0,1,2,...). \tag{2.14}$$

Then, the solutions of this system of equations solve the equations (2.7) and (2.8).

2.3. Parametric representations of the solutions

Theorem 2.1. *The two-component CH equation (1.1) admits the parametric representations of the solutions*

$$u(y,\tau) = \left(\ln\frac{\tilde{f}}{f}\right)_\tau, \quad \rho(y,\tau) = \rho_0 - \mathrm{i}\left(\ln\frac{\tilde{g}}{g}\right)_\tau, \tag{2.15a}$$

$$x(y,\tau) = \frac{y}{\rho_0} + \ln\frac{\tilde{f}}{f} + d. \tag{2.15b}$$

Remark 2.1. The parametric representations of $1/\rho$ and m/ρ^2 in terms of the tau-functions are also available from (2.3*a*), (2.5) and (2.9). Explicitly, they read

$$\frac{1}{\rho} = \frac{1}{\rho_0} + \left(\ln\frac{\tilde{f}}{f}\right)_y, \quad \frac{m}{\rho^2} = \frac{\kappa^2}{\rho_0^2} + \mathrm{i}\left(\ln\frac{\tilde{g}}{g}\right)_y. \tag{2.16}$$

2.4. N-soliton solution

Theorem 2.2. *The tau-functions $f, \tilde{f}, g$ and $\tilde{g}$ constituting the N-soliton solution of the system of bilinear equations (2.10)-(2.13) are given by the expressions*

$$f = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j(\xi_j + \phi_j) + \sum_{1\le j<l\le N}\mu_j\mu_l\gamma_{jl}\right], \tag{2.17a}$$

$$\tilde{f} = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j(\xi_j - \phi_j) + \sum_{1\le j<l\le N}\mu_j\mu_l\gamma_{jl}\right], \tag{2.17b}$$

$$g = \sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j(\xi_j + \mathrm{i}\psi_j) + \sum_{1\le j<l\le N}\mu_j\mu_l\gamma_{jl}\right], \tag{2.18a}$$

$$\tilde{g} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\xi_j - \mathrm{i}\psi_j\right) + \sum_{1\le j<l\le N} \mu_j\mu_l\gamma_{jl}\right], \tag{2.18b}$$

where

$$\xi_j = k_j\left(y - c_j\tau - y_{j0}\right), \quad (j = 1, 2, ..., N), \tag{2.19a}$$

$$\mathrm{e}^{-\phi_j} = \sqrt{\frac{(1-\rho_0 k_j)c_j - \rho_0\kappa^2}{(1+\rho_0 k_j)c_j - \rho_0\kappa^2}}, \quad \mathrm{e}^{-\mathrm{i}\psi_j} = \sqrt{\frac{\left(\frac{\kappa^2}{\rho_0} - \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}{\left(\frac{\kappa^2}{\rho_0} + \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}}, \quad (j = 1, 2, ..., N), \tag{2.19b}$$

$$\mathrm{e}^{\gamma_{jl}} = \frac{\kappa^2(c_j - c_l)^2 - \rho_0(k_j - k_l)c_jc_l(c_jk_j - c_lk_l)}{\kappa^2(c_j - c_l)^2 - \rho_0(k_j + k_l)c_jc_l(c_jk_j + c_lk_l)}, \quad (j, l = 1, 2, ..., N; j \neq l), \tag{2.19c}$$

and c_j is the velocity of jth soliton in the (y, τ) coordinate system which is given by the solution of the quadratic equation

$$(1 - \rho_0^2k_j^2)c_j^2 - 2\rho_0\kappa^2c_j - \rho_0^4 = 0, \quad (j = 1, 2, ..., N). \tag{2.20}$$

Here, k_j and y_{j0} are arbitrary complex parameters satisfying the conditions $k_j \neq k_l$ for $j \neq l$. The notation $\sum_{\mu=0,1}$ implies the summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, ..., \mu_N = 0, 1$.

The parametric representation of the N-soliton solution given by (2.15) with the tau-functions (2.17) and (2.18) is characterized by the $2N$ complex parameters k_j and y_{j0} $(j = 1, 2...., N)$. The parameters k_j determine the amplitude and the velocity of the solitons, whereas the parameters y_{j0} determine the position (or phase) of the solitons. If we impose the conditions $\tilde{f} = f^*$ and $\tilde{g} = g^*$ where the asterisk denotes complex conjugate, then the solutions become real functions of x and t. Note, however that they would yield multi-valued functions unless certain conditions are imposed on the parameters $k_j (j = 1, 2, .., N)$. The same situation has been encountered in investigating the structure of the soliton solutions of the CH and modified CH equations [6-8]. We will address this point in the nest section where the detailed analysis of the soliton solutions will be done.

Before proceeding, we investigate the characteristics of the velocity of the soliton. The quadratic equation (2.20) has two roots

$$c_j = \frac{\rho_0}{1 - (\rho_0 k_j)^2}\left(\kappa^2 + d_j\right) = \frac{\rho_0^3}{d_j - \kappa^2}, \quad (j = 1, 2, ..., N), \tag{2.21a}$$

where

$$d_j = \epsilon_j\sqrt{\kappa^4 + \rho_0^2 - \rho_0^4k_j^2}, \qquad (\epsilon_j = \pm 1, \quad j = 1, 2, ..., N). \tag{2.21b}$$

To assure the reality of c_j, one must impose the condition for the parameter $\rho_0 k_j$, where $k_j\, (j = 1, 2, ..., N)$ are assumed to be positive real numbers. Actually, It must lie in the interval

$$0 < \rho_0 k_j < \frac{\sqrt{\kappa^4 + \rho_0^2}}{\rho_0}, \quad (j = 1, 2, ..., N). \tag{2.22}$$

Figure 1 plots the velocities $c_+ \equiv c_j(\epsilon_j = +1)$ and $c_- \equiv c_j(\epsilon_j = -1)$ as a function of $\rho_0 k \equiv \rho_0 k_j$. The velocity c_+ is positive for $0 < \rho_0 k < 1$ and negative for $1 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0$. It exhibits the singularity at $\rho_0 k = 1$. Specifically,

$$\rho_0(\kappa^2 + \sqrt{\kappa^4 + \rho_0^2} < c_+ < \infty, \; (0 < \rho_0 k < 1), \tag{2.23a}$$

$$-\infty < c_+ < -\frac{\rho_0^3}{\kappa^2}, \; \left(1 < \rho_0 k < \sqrt{\kappa^4 + \rho_0^2}/\rho_0\right). \tag{2.23b}$$

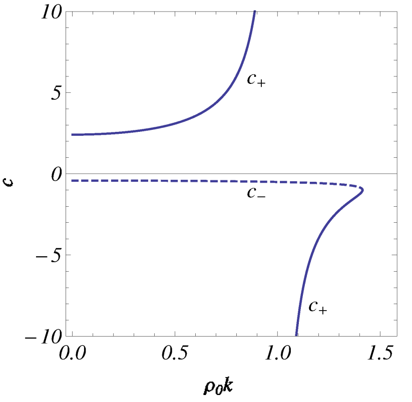

Figure 1. The velocity $c = c_\pm$ of the soliton as a function of $\rho_0 k$ for $\rho_0 = 1$ and $\kappa = 1$: c_+(solid curve), c_-(dashed curve).

On the other hand, the velocity c_- is a continuous function of $\rho_0 k$ and takes negative values in the interval (2.23), as indicated by the inequality

$$-\frac{\rho_0^3}{\kappa^2} < c_- < -\rho_0(\sqrt{\kappa^4+\rho_0^2} - \kappa^2), \ \left(0 < \rho_0 k < \sqrt{\kappa^4+\rho_0^2}/\rho_0\right). \tag{2.24}$$

In particular, $c_- = -\rho_0^3/(2\kappa^3)$ at $\rho_0 k = 1$. It turns out that the soliton with the velocity c_- always propagates to the left whereas the soliton with the velocity c_+ propagates to the right and left depending on the value of $\rho_0 k$. Thus, the two-soliton solution exhibits both the overtaking and head-on collisions.

Using (2.21), the expressions (2.19*b*) become

$$\mathrm{e}^{-\phi_j} = \frac{|(1-\rho_0 k_j)c_j - \rho_0\kappa^2|}{\rho_0\sqrt{\kappa^4+\rho_0^2}} = \frac{\{(1-\rho_0 k_j)c_j - \rho_0\kappa^2\}\mathrm{sgn}\ c_j}{\rho_0\sqrt{\kappa^4+\rho_0^2}}, \quad \mathrm{e}^{-\mathrm{i}\psi_j} = \frac{\kappa^2 c_j + \rho_0^3 - \mathrm{i}\rho_0^2 k_j c_j}{\sqrt{\kappa^4+\rho_0^2}\,|c_j|}, \tag{2.25}$$

where the symbol sgn denotes the sign function. In view of the relation $d_j^2 - d_l^2 = \rho_0^4(-k_j^2 + k_l^2)$ which follows from (2.21*b*), the expression (2.19*c*) becomes

$$\mathrm{e}^{\gamma_{jl}} = \frac{(d_j - d_l)^2 + \rho_0^4(k_j - k_l)^2}{(d_j - d_l)^2 + \rho_0^4(k_j + k_l)^2}. \tag{2.26}$$

3. Properties of soliton solutions

In this section, we first explore the properties of the one-soliton solution in detail and then perform an asymptotic analysis of the general N-soliton solution. Consequently, the formula for the phase shift of each soliton will be derived. The two-soliton case is discussed shortly.

3.1. One-soliton solution

The tau-functions corresponding to the one-soliton solution are given by (2.17) and (2.18) with $N = 1$

$$f = 1 + \mathrm{e}^{\xi+\phi}, \quad \tilde{f} = 1 + \mathrm{e}^{\xi-\phi}, \tag{3.1}$$

$$g = 1 + \mathrm{e}^{\xi+\mathrm{i}\psi}, \quad \tilde{g} = 1 + \mathrm{e}^{\xi-\mathrm{i}\psi}, \tag{3.2}$$

with

$$\xi = k\,(y - c\tau - y_0)\,, \quad c = c_{\pm} = \frac{\rho_0^3}{\pm\sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k^2} - \kappa^2}, \tag{3.3a}$$

$$\mathrm{e}^{-\phi} = \frac{|(1-\rho_0 k)c - \rho_0\kappa^2|}{\rho_0\sqrt{\kappa^4 + \rho_0^2}}, \quad \mathrm{e}^{-\mathrm{i}\psi} = \frac{\kappa^2 c + \rho_0^3 - \mathrm{i}\rho_0^2 kc}{\sqrt{\kappa^4 + \rho_0^2}\,|c|}, \tag{3.3b}$$

where we have put $\xi = \xi_1, k = k_1, c = c_1, \phi = \phi_1, \psi = \psi_1$ and $y_0 = y_{10}$ for simplicity.

The parametric representation of the one-soliton solution is obtained by introducing (3.1) and (3.2) with (3.3) into (2.15). It can be written in the form

$$u = \frac{kc \sinh \phi}{\cosh \xi + \cosh \phi}, \quad \rho = \rho_0 + \frac{kc \sin \psi}{\cosh \xi + \cos \psi}, \tag{3.4a}$$

$$X \equiv x - \tilde{c}t - x_0 = \frac{\xi}{\rho_0 k} + \ln \frac{1 - \tanh \frac{\phi}{2} \tanh \frac{\xi}{2}}{1 + \tanh \frac{\phi}{2} \tanh \frac{\xi}{2}}, \tag{3.4b}$$

with

$$\sinh \phi = \frac{k|c|}{\sqrt{\kappa^4 + \rho_0^2}}, \ \cosh \phi = \sqrt{1 + \frac{k^2 c^2}{\kappa^4 + \rho_0^2}}, \ \sin \psi = \frac{\rho_0^2 kc}{\sqrt{\kappa^4 + \rho_0^2}\,|c|}, \ \cos \psi = \frac{\kappa^2 c + \rho_0^3}{\sqrt{\kappa^4 + \rho_0^2}\,|c|}, \tag{3.4c}$$

where $\tilde{c} = c/\rho_0$ is the velocity of the soliton in the (x, t) coordinate system, $x_0 = y_0/\kappa$ and the constant d in (2.15b) has been chosen such that $\xi = 0$ corresponds to $X = 0$. Let us now describe some important properties of the solution.

(a) Smoothness of the solution

We compute the y derivative of x from (3.4b) to obtain

$$x_y = \frac{1}{\rho_0} - \frac{k \sinh \phi}{\cosh \xi + \cosh \phi}. \tag{3.5}$$

Since $k > 0$ and $\phi > 0$, $x_y \geq x_y|_{\xi=0}$. Using (3.3b) for ϕ gives

$$x_y|_{\xi=0} = \frac{1}{\rho_0}\left(1 - \rho_0 k \tanh \frac{\phi}{2}\right) = \frac{1}{|c|}(\sqrt{\rho_0^2 + \kappa^4} - \kappa^2). \tag{3.6}$$

Thus, if c is finite, then $x_y > 0$, and the map (2.1) becomes one-to-one, assuring that the solution is smooth and nonsingular. Actually, one can show that the derivatives du/dX and $d\rho/dX$ are finite for arbitrary $X \in \mathbb{R}$.

(b) Amplitude-velocity relation

The amplitude-velocity relation of the soliton is an important characteristic of the wave. It can be derived simply from the explicit form of the solution. To this end, let A_ρ be the amplitude of the wave measured from the constant level $\rho = \rho_0$ and A_u be that of the fluid velocity, namely $A_\rho = \rho(X = 0) - \rho_0, A_u = |u(X = 0)|$. It follows from (3.3) and (3.4) that

$$A_\rho = \left(\sqrt{\kappa^4 + \rho_0^2}\,|\tilde{c}| - \kappa^2\tilde{c} - \rho_0^2\right)/\rho_0, \quad A_u = \left||\tilde{c} - \kappa^2| - \sqrt{\kappa^4 + \rho_0^2}\right|, \tag{3.7}$$

where $\tilde{c} = c/\rho_0$. Note from (3.3b) and (3.4a) that

$$u(X = 0) = kc \tanh \frac{\phi}{2} = \left(|\tilde{c} - \kappa^2| - \sqrt{\kappa^4 + \rho_0^2}\right) \operatorname{sgn} \tilde{c}.$$

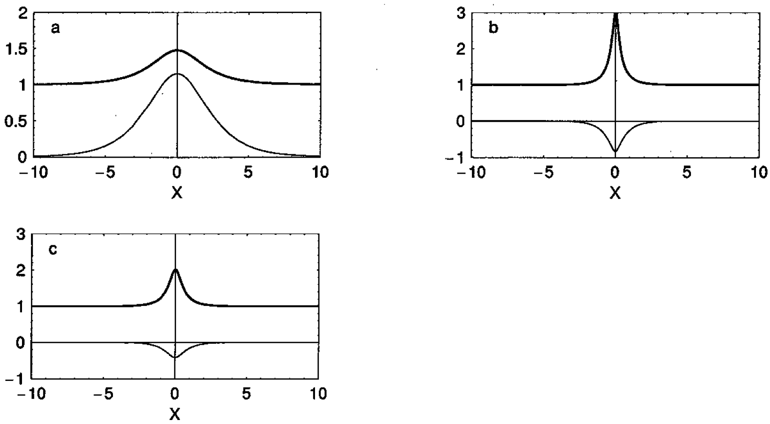

Figure 2. One-soliton solution. u: thin solid curve, ρ: bold solid curve. a: $\kappa = 1, \rho_0 = 1, k = 0.4, \tilde{c} = \tilde{c}_+ = 2.81$, b: $\kappa = 1, \rho_0 = 1, k = 1.0, \tilde{c} = \tilde{c}_+ = -1.25$, c: $\kappa = 1, \rho_0 = 1, k = 1.4, \tilde{c} = \tilde{c}_- = -0.83$.

Invoking the expression of the velocity c from (3.3a), we can see that $A_\rho > 0$ for arbitrary $c = c_\pm$ whereas $u(X = 0) > 0$ for $c > 0$ and $u(X = 0) < 0$ for $c < 0$. These results show that the profile of ρ is always of bright type, but that of u depends on the propagation direction of the soliton. Actually, if c is positive (negative), then u is curved upward (downward).

Figure 2 depicts the typical profile of u and ρ for the right-going soliton (a), and the left-going soliton (b) and (c), respectively.

3.2. N-soliton solution

Here, we investigate the asymptotic behavior of the N-soliton solution for large time. Let $\tilde{c}_n (= c_n/\rho_0\ (n = 1, 2, ..., N)$ be the velocity of the $n$the soliton in the (x, t) coordinate system, and order them in accordance with the relation $\tilde{c}_N < \tilde{c}_{N-1} < ... < \tilde{c}_1$. We take the limit $t \to -\infty$ with the phase variable ξ_n of the nth soliton being fixed. Then, the other phase variables behave like $\xi_1, \xi_2, ..., \xi_{n-1} \to +\infty$, and $\xi_{n+1}, \xi_{n+2}, ..., \xi_N \to -\infty$. Performing an asymptotic analysis for the tau-functions (2.17) and (2.18) and substituting the leading-order approximations for them into (2.15), we obtain the asymptotic form of u, ρ and x

$$u \sim \frac{k_n c_n \sinh \phi_n}{\cosh\left(\xi_n + \delta_n^{(-)}\right) + \cosh \phi_n}, \quad \rho \sim \rho_0 + \frac{k_n c_n \sin \psi_n}{\cosh\left(\xi_n + \delta_n^{(-)}\right) + \cos \psi_n}, \tag{3.8}$$

$$x - \tilde{c}_n t - x_{n0} \sim \frac{\xi_n}{\rho_0 k_n} + \ln \frac{1 - \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(-)}\right)}{2}}{1 + \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(-)}\right)}{2}} - 2\sum_{j=1}^{n-1} \phi_j, \tag{3.9}$$

where

$$\delta_n^{(-)} = \sum_{j=1}^{n-1} \gamma_{nj} = \sum_{j=1}^{n-1} \ln\left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2}\right]. \tag{3.10}$$

In the limit $t \to +\infty$, on the other hand, we see that $\xi_1, \xi_2, ..., \xi_{n-1} \to -\infty$, and $\xi_{n+1}, \xi_{n+2}, ..., \xi_N \to$

$+\infty$. Applying the similar analysis yields the asymptotic forms corresponding to (3.8) and (3.9)

$$u \sim \frac{k_n c_n \sinh \phi_n}{\cosh\left(\xi_n + \delta_n^{(+)}\right) + \cosh \phi_n}, \quad \rho \sim \rho_0 + \frac{k_n c_n \sin \psi_n}{\cosh\left(\xi_n + \delta_n^{(+)}\right) + \cos \psi_n}, \tag{3.11}$$

$$x - \tilde{c}_n t - x_{n0} \sim \frac{\xi_n}{\rho_0 k_n} + \ln \frac{1 - \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(+)}\right)}{2}}{1 + \tanh \frac{\phi_n}{2} \tanh \frac{\left(\xi_n + \delta_n^{(+)}\right)}{2}} - 2\sum_{j=1}^{n-1} \phi_j, \tag{3.12}$$

where

$$\delta_n^{(+)} = \sum_{j=n+1}^{N} \gamma_{nj} = \sum_{j=n+1}^{N} \ln\left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2}\right]. \tag{3.13}$$

These results show that as $t \to \pm\infty$, the N-soliton solution is a superposition of N independent solitons each of which has the form given by (3.4). The net effect of the collision of solitons appears as a phase shift. To see this, let x_{nc} be the center position of the nth soliton. It then follows from (3.9) and (3.12) that the trajectory of x_{nc} is given by

$$x_{nc} \sim \tilde{c}_n t - \frac{\delta_n^{(-)}}{\rho_0 k_n} - 2\sum_{j=1}^{n-1} \phi_j, \quad (t \to -\infty), \quad x_{nc} \sim \tilde{c}_n t - \frac{\delta_n^{(+)}}{\rho_0 k_n} - 2\sum_{j=n+1}^{N} \phi_j, \quad (t \to +\infty). \tag{3.14}$$

We define the phase shift of the nth soliton which propagates to the right by $\Delta_n^R = x_{nc}(t \to +\infty) - x_{nc}(t \to -\infty)$, and that propagates to the left by $\Delta_n^L = x_{nc}(t \to -\infty) - x_{nc}(t \to +\infty)$. Using (2.19*b*), (3.10), (3.13) and (3.14), we find that

$$\begin{aligned}\Delta_n^R = \frac{1}{\rho_0 k_n}&\left[\sum_{j=1}^{n-1} \ln\left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2}\right] - \sum_{j=n+1}^{N} \ln\left[\frac{(d_n - d_j)^2 + \rho_0^4 (k_n - k_j)^2}{(d_n - d_j)^2 + \rho_0^4 (k_n + k_j)^2}\right]\right] \\ &+ \sum_{j=n+1}^{N} \ln\left[\frac{(1 - \rho_0 k_j)\tilde{c}_j - \kappa^2}{(1 + \rho_0 k_j)\tilde{c}_j - \kappa^2}\right] - \sum_{j=1}^{n-1} \ln\left[\frac{(1 - \rho_0 k_j)\tilde{c}_j - \kappa^2}{(1 + \rho_0 k_j)\tilde{c}_j - \kappa^2}\right].\end{aligned} \tag{3.15}$$

The expression of Δ_n^L is equal to $-\Delta_n^R$.

3.3. Two-soliton solution

The two-soliton solution is the most fundamental element in understanding the dynamics of solitons since each soliton exhibits pair-wise interactions with every other soliton. There exist two types of interactions for the CH2 equation, i.e., the overtaking and head-on collisions.

The tau-functions for the two-soliton solution are given by (2.17), (2.18) and (2.19) with $N = 2$. They read

$$f = 1 + e^{\xi_1 + \phi_1} + e^{\xi_2 + \phi_2} + \delta\, e^{\xi_1 + \xi_2 + \phi_1 + \phi_2}, \quad \tilde{f} = 1 + e^{\xi_1 - \phi_1} + e^{\xi_2 - \phi_2} + \delta\, e^{\xi_1 + \xi_2 - \phi_1 - \phi_2}, \tag{3.16}$$

$$g = 1 + e^{\xi_1 + i\psi_1} + e^{\xi_2 + i\psi_2} + \delta\, e^{\xi_1 + \xi_2 + i\psi_1 + i\psi_2}, \quad \tilde{g} = 1 + e^{\xi_1 - i\psi_1} + e^{\xi_2 - i\psi_2} + \delta\, e^{\xi_1 + \xi_2 - i\psi_1 - i\psi_2}, \tag{3.17}$$

where

$$\xi_j = k_j\,(y - c_j \tau - y_{j0}), \quad (j = 1, 2), \tag{3.18a}$$

$$\delta = e^{\gamma_{12}} = \frac{(d_1 - d_2)^2 + \rho_0^4 (k_1 - k_2)^2}{(d_1 - d_2)^2 + \rho_0^4 (k_1 + k_2)^2}, \tag{3.18b}$$

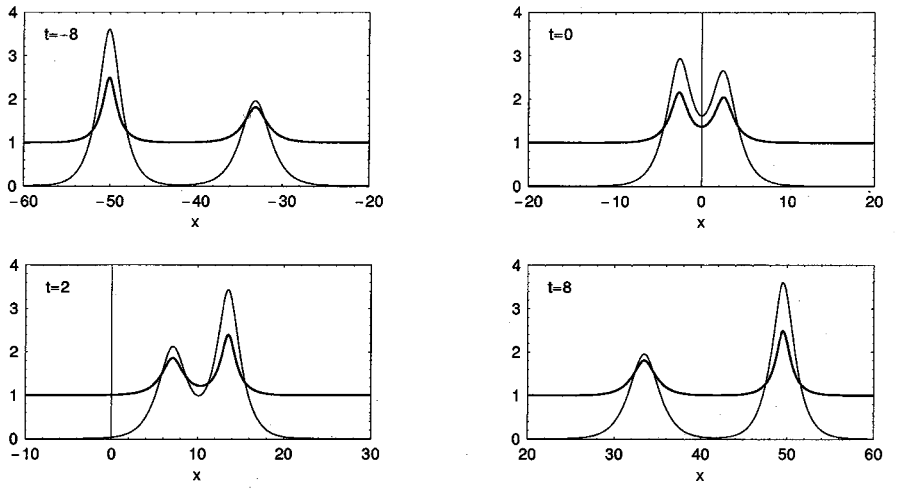

Figure 3. The overtaking collision of two solitons. u: thin solid curve, ρ: bold solid curve. $\kappa = 1, \rho_0 = 1, k_1 = 0.8, k_2 = 0.7, \tilde{c}_{1+} = 6.02, \tilde{c}_{2+} = 4.37$.

$$\mathrm{e}^{-\phi_j} = \sqrt{\frac{(1-\rho_0 k_j)c_j - \rho_0\kappa^2}{(1+\rho_0 k_j)c_j - \rho_0\kappa^2}}, \quad \mathrm{e}^{-\mathrm{i}\psi_j} = \sqrt{\frac{\left(\frac{\kappa^2}{\rho_0} - \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}{\left(\frac{\kappa^2}{\rho_0} + \mathrm{i}\rho_0 k_j\right)c_j + \rho_0^2}}, \quad (j=1,2). \tag{3.18c}$$

Recall from (2.21) that the velocity of jth soliton in (x,t) coodinate system is given by

$$\tilde{c}_j = c_j/\rho_0 = \frac{\rho_0^2}{d_j - \kappa^2}, \quad d_j = \epsilon_j\sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_j^2}, \quad (j=1,2). \tag{3.19}$$

Substituting (3.16) and (3.17) into (2.15), we obtain the parametric representation of the two-soliton solution. As seen from Figure 1, this solution describes both the overtaking and headon collisions, which are treated separately.

(a) Overtaking collision

We consider the case $c_j = c_{j+}, 0 < \rho_0 k_j < 1$ so that $0 < \tilde{c}_{2+} < \tilde{c}_{1+}$. Figure 3 illustrates the overtaking collision of two solution for four distinct values of t. The solitonic feature of the solution is obvious from the figure which confirms an asymptotic analysis presented in §3.1. The phase shift of each soliton is given by (3.15). Explicitly,

$$\Delta_1^R = -\frac{1}{\rho_0 k_1}\ln\left[\frac{(d_1-d_2)^2 + \rho_0^4(k_1-k_2)^2}{(d_1-d_2)^2 + \rho_0^4(k_1+k_2)^2}\right] + \ln\left[\frac{(1-\rho_0 k_2)\tilde{c}_2 - \kappa^2}{(1+\rho_0 k_2)\tilde{c}_2 - \kappa^2}\right], \tag{3.20a}$$

$$\Delta_2^R = \frac{1}{\rho_0 k_2}\ln\left[\frac{(d_1-d_2)^2 + \rho_0^4(k_1-k_2)^2}{(d_1-d_2)^2 + \rho_0^4(k_1+k_2)^2}\right] - \ln\left[\frac{(1-\rho_0 k_1)\tilde{c}_1 - \kappa^2}{(1+\rho_0 k_1)\tilde{c}_1 - \kappa^2}\right], \tag{3.20b}$$

with

$$d_1 = \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_1^2}, \quad d_2 = \sqrt{\kappa^4 + \rho_0^2 - \rho_0^4 k_2^2}. \tag{3.20c}$$

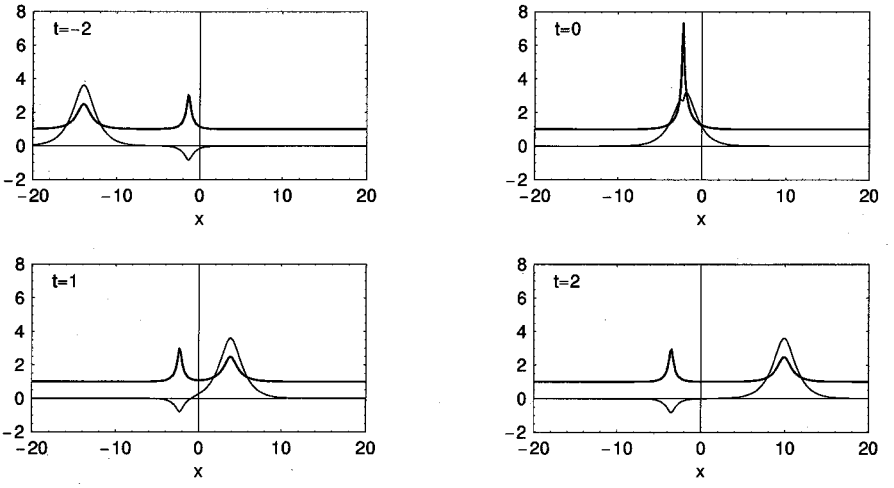

Figure 4. The head-on collision of two solitons. u: thin solid curve, ρ: bold solid curve. $\kappa = 1, \rho_0 = 1, k_1 = 0.8, k_2 = 0.7, \tilde{c}_{1+} = 6.02, \tilde{c}_{2+} = -1.25$

(b) Head-on collision

An example of the head-on collision is shown in Figure 4, where the velocity of each soliton is chosen as $c_{2+} < 0 < c_{1+}$. The formula of the phase shift for the right-running soliton is the same as (3.20a) whereas that of the left-running soliton is given by $\Delta_2^L = -\Delta_2^R$.

4. Reductions to the Camassa-Holm and two-component Hunter-Saxton equations

In this section, we first show that the CH2 equation and its N-soliton solution reduce to those of the CH equation by means of an appropriate limiting procedure. Then, we demonstrate that the short-wave limit of the CH2 equation yields the two-component Hunter-Saxton equation.

4.1. Reduction to the Camassa-Holm equation

The CH equation (1.3) is derived simply from the CH2 equation by putting $\rho = 0$. In this setting, one must impose the boundary condition $\rho_0 = 0$. The N-soliton solution of the CH equation is reduced from that of the CH2 equation by taking the limit $\rho_0 \to 0$. To show this, we introduce tha following scaling variables

$$u = \bar{u},\ \rho = \rho_0\bar{\rho},\ m = \bar{m},\ x = \bar{x},\ y = \frac{\rho_0}{\kappa}\bar{y},\ t = \bar{t},\ \tau = \bar{\tau},\ d = \bar{d},$$

$$k_j = \frac{\kappa}{\rho_0}\bar{k}_j,\ c_j = \frac{\rho_0}{\kappa}\bar{c}_j,\ y_{j0} = \frac{\rho_0}{\kappa}\bar{y}_{j0},\ (j = 1, 2, ..., N). \tag{4.1}$$

Then, the leading-order asymptotics of c_j from (2.21) and ϕ_j, ψ_j and γ_{jl} from (2.19b, c) are found to be

$$c_j \sim \frac{2\rho_0\kappa^2}{1 - (\kappa\bar{k}_j)^2},\ (j = 1, 2, ..., N), \tag{4.2a}$$

$$e^{-\phi_j} \sim \frac{1 - \kappa\bar{k}_j}{1 + \kappa\bar{k}_j} \equiv e^{-\bar{\phi}_j}, \quad e^{-i\psi_j} \sim 1 - i\frac{\rho_0}{\kappa}\bar{k}_j,\ (j = 1, 2, ..., N), \tag{4.2b}$$

$$e^{\gamma_{jl}} = \left(\frac{\bar{k}_j - \bar{k}_l}{\bar{k}_j + \bar{k}_l}\right)^2 \equiv e^{\bar{\gamma}_{jl}}, \ (j, l = 1, 2, ..., N; j \neq l). \tag{4.2c}$$

We note that a limiting form $\bar{c}_j \sim -\rho_0^2/(2\kappa)$ which arises from (2.21) with $\epsilon_j = -1 \, (j = 1, 2)$ is not relevant since this expression degenerates to zero as $\rho_0 \to 0$.

The asymptotics of the tau-functions f and $\tilde{f}$ from (2.17) and g and $\tilde{g}$ from (2.18) become

$$f \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j + \bar{\phi}_j\right) + \sum_{1\leq j<l\leq N} \mu_j \mu_l \bar{\gamma}_{jl}\right] \equiv \bar{f}, \tag{4.3a}$$

$$\tilde{f} \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \left(\bar{\xi}_j - \bar{\phi}_j\right) + \sum_{1\leq j<l\leq N} \mu_j \mu_l \bar{\gamma}_{jl}\right] \equiv \bar{\tilde{f}}, \tag{4.3b}$$

$$g = \bar{f}_0 + \mathrm{i}\frac{\rho_0}{\kappa} \bar{f}_{0,\bar{y}} + O(\rho_0^2), \quad \tilde{g} = \bar{f}_0 - \mathrm{i}\frac{\rho_0}{\kappa} \bar{f}_{0,\bar{y}} + O(\rho_0^2), \tag{4.4}$$

where

$$\bar{f}_0 = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j \bar{\xi}_j + \sum_{1\leq j<l\leq N} \mu_j \mu_l \bar{\gamma}_{jl}\right], \tag{4.5a}$$

$$\bar{\xi}_j = \bar{k}_j \left(\bar{y} - \bar{c}_j \bar{\tau} - \bar{y}_{j0}\right), \quad \bar{c}_j = \frac{2\kappa^3}{1 - (\kappa \bar{k}_j)^2}, \quad (j = 1, 2, ..., N). \tag{4.5b}$$

Introducing (4.3) and (4.4) into (2.15), we obtain the limiting forms of u, ρ and x

$$\bar{u} = \left(\ln \frac{\bar{\tilde{f}}}{\bar{f}}\right)_{\bar{\tau}}, \quad \rho \sim \rho_0 \left(1 - \frac{2}{\kappa} (\ln \bar{f}_0)_{\bar{y}\bar{\tau}}\right) \equiv \rho_0 \bar{\rho}, \tag{4.6a}$$

$$\bar{x} = \frac{\bar{y}}{\kappa} + \ln \frac{\bar{\tilde{f}}}{\bar{f}} + \bar{d}. \tag{4.6b}$$

The parametric representation of the N-soliton solution given by (4.6) with the tau-functions (4.3) coincides perfectly with that of the CH equation presented in [6]. In particular, the one-soliton solution (3.4) reduces to

$$\bar{u} = \frac{2\kappa \bar{c} \bar{k}^2}{1 + \kappa^2 \bar{k}^2 + (1 - \kappa^2 \bar{k}^2) \cosh \bar{\xi}}, \tag{4.7a}$$

$$\bar{X} = \bar{x} - \bar{\tilde{c}} - \bar{x}_0 = \frac{\bar{\xi}}{\kappa \bar{k}} + \ln \frac{(1 - \kappa \bar{k}) \, e^{\bar{\xi}} + 1 + \kappa \bar{k}}{(1 + \kappa \bar{k}) \, e^{\bar{\xi}} + 1 - \kappa \bar{k}}, \tag{4.7b}$$

with

$$\bar{\xi} = \bar{k}(\bar{y} - \bar{c}\bar{\tau} - \bar{y}_0) \quad \bar{c} = \frac{2\kappa^3}{1 - (\kappa \bar{k})^2}, \quad \bar{\tilde{c}} = \bar{c}/\kappa, \tag{4.7c}$$

reproducing the one-soliton solution of the CH equation.

The limiting form of the phase shift which is denoted by $\bar{\Delta}_n^R$ can be obtained by applying the scalings (4.1) to (3.15) and using (4.1) and (4.2a), resulting in

$$\bar{\Delta}_n^R = \frac{1}{\kappa \bar{k}_n} \left[\sum_{j=1}^{n-1} \ln \left(\frac{\bar{k}_n - \bar{k}_j}{\bar{k}_n + \bar{k}_j}\right)^2 - \sum_{j=n+1}^{N} \ln \left(\frac{\bar{k}_n - \bar{k}_j}{\bar{k}_n + \bar{k}_j}\right)^2\right]$$

$$+\sum_{j=n+1}^{N}\ln\left(\frac{1-\kappa k_j}{1+\kappa k_j}\right)^2-\sum_{j=1}^{n-1}\ln\left(\frac{1-\kappa k_j}{1+\kappa k_j}\right)^2, \quad (n=1,2,...,N). \tag{4.8}$$

This coincides with the formula for the phase shift of the nth soliton which has been derive for the N-soliton solution of the CH equation [6].

Remark 4.1.

If we put $\bar{r}=\kappa-2(\ln\bar{f}_0)_{\bar{y}\bar{\tau}}$, then

$$\bar{m}=\bar{r}^2, \quad \bar{\rho}=\bar{r}/\kappa. \tag{4.9}$$

The reciprocal transformation (2.1a) reproduces the corresponding one for the CH equation

$$d\bar{y}=\bar{r}d\bar{x}-\bar{r}\bar{u}d\bar{t}, \quad d\bar{\tau}=d\bar{t}. \tag{4.10}$$

The bilinear equations (2.10)-(2.12) reduce, in the scaling limit, to the bilinear equations

$$\kappa D_{\bar{y}}\bar{\bar{f}}\cdot\bar{f}+\bar{\bar{f}}\bar{f}-\bar{f}_0^2=0, \tag{4.11}$$

$$D_{\bar{\tau}}D_{\bar{y}}\bar{f}_0\cdot\bar{f}_0+\kappa(\bar{\bar{f}}\bar{f}-\bar{f}_0^2)=0, \tag{4.12}$$

$$\kappa D_{\bar{\tau}}D_{\bar{y}}\bar{\bar{f}}\cdot\bar{f}+D_{\bar{\tau}}\bar{\bar{f}}\cdot\bar{f}+\kappa^3D_{\bar{y}}\bar{\bar{f}}\cdot\bar{f}=0, \tag{4.13}$$

whereas the scaling limit of (2.13) is shown to coincide with (4.11). One can show that the tau-functions $\bar{f}$ and $\bar{\bar{f}}$ from (4.3) and $\bar{f}_0$ from (4.5) solve the above bilinear equations.

4.2. Reduction to the two-component Hunter-Saxton equation

The two-component Hunter-Saxton (HS2) equation stems from the short-wave limit of the CH2 equation. To show this, we introduce the scaling variables

$$u=\epsilon^2\hat{u},\ \rho=\epsilon\hat{\rho},\ m=\hat{m},\ x=\epsilon\hat{x},\ y=\epsilon^2\hat{y},\ t=\frac{\hat{t}}{\epsilon},\ \tau=\frac{\hat{\tau}}{\epsilon}. \tag{4.14}$$

Rescaling the CH2 equation (1.1) by (4.14) and taking the limit $\epsilon\to 0$, we obtain the HS2 equation

$$\hat{m}_{\hat{t}}+\hat{u}\hat{m}_{\hat{x}}+2\hat{m}\hat{u}_{\hat{x}}+\hat{\rho}\hat{\rho}_{\hat{x}}=0, \quad \hat{\rho}_{\hat{t}}+(\hat{\rho}\hat{u})_{\hat{x}}=0, \tag{4.15}$$

where $\hat{m}=-\hat{u}_{\hat{x}\hat{x}}+\kappa^2$. The N-soliton solution of the HS2 equation can be reduced from that of the CH2 equation by means of a limiting procedure. Th appropriate scaling variable are found to be

$$k_j=\frac{\hat{k}_j}{\epsilon^2},\ c_j=\epsilon^3\hat{c}_j,\ y_{j0}=\epsilon^2\bar{y}_{j0},\ (j=1,2,...,N),\ \rho_0=\epsilon\hat{\rho}_0,\ d=\epsilon\hat{d}. \tag{4.16}$$

In the limit $\epsilon\to 0$, the soliton parameters corresponding to those given by (4.2) have the leading-order asymptotics

$$c_j\sim-\frac{\epsilon^3}{\hat{\rho}_0\hat{k}_j^2}(\kappa^2+\hat{d}_j), \quad \hat{d}_j=\epsilon_j\sqrt{\kappa^4-\hat{\rho}_0^4\hat{k}_j^2}, \quad (j=1,2,...,N), \tag{4.17a}$$

$$e^{-\phi_j}\sim 1+\epsilon\frac{\hat{k}_j\hat{c}_j}{\kappa^2}, \quad e^{-i\psi_j}\sim\sqrt{\frac{\left(\frac{\kappa^2}{\hat{\rho}_0}-i\hat{\rho}_0\hat{k}_j\right)\hat{c}_j+\hat{\rho}_0^2}{\left(\frac{\kappa^2}{\hat{\rho}_0}+i\hat{\rho}_0\hat{k}_j\right)\hat{c}_j+\hat{\rho}_0^2}}\equiv e^{-i\hat{\psi}_j}, \quad (j=1,2,...,N), \tag{4.17b}$$

$$e^{\gamma_{jl}}\sim\frac{(\hat{d}_j-\hat{d}_l)+\hat{\rho}_0^4(\hat{k}_j-\hat{k}_j)^2}{(\hat{d}_j-\hat{d}_l)+\hat{\rho}_0^4(\hat{k}_j+\hat{k}_j)^2}\equiv e^{\hat{\gamma}_{jl}},\ (j,l=1,2,...,N;j\neq l). \tag{4.17c}$$

The tau-functions (2.17) and (2.18) have the leading-order asymptotics

$$f \sim \hat{f} + \frac{\epsilon}{\kappa^2}\hat{f}_{\hat{\tau}}, \quad \tilde{f} \sim \hat{f} - \frac{\epsilon}{\kappa^2}\hat{f}_{\hat{\tau}}, \tag{4.18}$$

$$g \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\hat{\xi}_j + \mathrm{i}\hat{\psi}_j\right) + \sum_{1\le j<l\le N} \mu_j\mu_l\hat{\gamma}_{jl}\right] \equiv \hat{g}, \tag{4.19a}$$

$$\tilde{g} \sim \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\left(\hat{\xi}_j - \mathrm{i}\hat{\psi}_j\right) + \sum_{1\le j<l\le N} \mu_j\mu_l\hat{\gamma}_{jl}\right] \equiv \hat{\tilde{g}}, \tag{4.19b}$$

where

$$\hat{f} = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\hat{\xi}_j + \sum_{1\le j<l\le N} \mu_j\mu_l\hat{\gamma}_{jl}\right], \tag{4.20a}$$

$$\hat{\xi}_j = \hat{k}_j(\hat{y} - \hat{c}_j\hat{\tau} - \hat{y}_{j0}), \quad \hat{c}_j = -\frac{1}{\hat{\rho}_0\hat{k}_j^2}(\kappa^2 + \epsilon_j\sqrt{\kappa^4 - \hat{\rho}_0^4\hat{k}_j^2}), \quad (j = 1, 2, ..., N), \tag{4.20b}$$

The parametric representation for the N-soliton solution of the HS2 equation follows by introducing (4.18) and (4.19) into (2.15) and taking the limit $\epsilon \to 0$. Explicitly, it is given by

$$\hat{u} = -\frac{2}{\kappa^2}(\ln \hat{f})_{\hat{\tau}\hat{\tau}}, \quad \hat{\rho} = \hat{\rho}_0 - \frac{2}{\kappa^2}\mathrm{i}\left(\frac{\hat{\tilde{g}}}{\hat{g}}\right)_{\hat{\tau}}, \tag{4.21a}$$

$$\hat{x} = \frac{\hat{y}}{\hat{\rho}_0} - \frac{2}{\kappa^2}(\ln \hat{f})_{\hat{\tau}} + \hat{d}. \tag{4.21b}$$

The limiting forms of $1/\rho$ and m/ρ^2 from (2.16) read

$$\frac{1}{\hat{\rho}} = \frac{1}{\hat{\rho}_0} - \frac{2}{\kappa^2}(\ln \hat{f})_{\hat{\tau}\hat{y}}, \quad \frac{\hat{m}}{\hat{\rho}^2} = \frac{\kappa^2}{\hat{\rho}_0^2} + \mathrm{i}\left(\frac{\hat{\tilde{g}}}{\hat{g}}\right)_{\hat{y}}. \tag{4.22}$$

We write the one-soliton solution for reference:

$$\hat{u} = -\frac{1}{2\kappa^2}\frac{(\hat{k}\hat{c})^2}{\cosh^2\frac{\hat{\xi}}{2}}, \quad \hat{\rho} = \frac{1}{\frac{1}{\hat{\rho}_0} + \frac{\hat{k}^2\hat{c}}{2\kappa^2}\frac{1}{\cosh^2\frac{\hat{\xi}}{2}}}, \tag{4.23a}$$

$$\hat{X} = \hat{x} - \hat{\tilde{c}}\hat{t} - \hat{x}_0 = \frac{\hat{\xi}}{\hat{\rho}_0} + \frac{\hat{k}\hat{c}}{\kappa^2}\tanh\frac{\hat{\xi}}{2}, \tag{4.23b}$$

with

$$\hat{\xi} = \hat{k}(\hat{y} - \hat{c}\hat{\tau} - \hat{y}_0), \quad \hat{c} = -\frac{1}{\hat{\rho}_0\hat{k}^2}\left(\kappa^2 \pm \sqrt{\kappa^4 - \hat{\rho}_0^4\hat{k}^2}\right), \quad \hat{\tilde{c}} = \hat{c}/\hat{\rho}_0. \tag{4.23c}$$

Note that the velovity $\hat{\tilde{c}}$ from (4.23c) is always negative so that the soliton propagates to the left as opposed to the propagation characteristic of the CH2 solitons.

Remark 4.2.

Under the scaling (4.14), the reciprocal transformation (2.1) and equations (2.2)-(2.5) remain the same form. The bilinear equations (2.10), (2.11) and (2.13) reduce respectively to

$$D_{\hat{\tau}}D_{\hat{y}}\hat{f}\cdot\hat{f}-\frac{\kappa^2}{\hat{\rho}_0^2}(\hat{f}^2-\hat{\bar{g}}\hat{g})=0, \tag{4.24}$$

$$\mathrm{i}D_{\hat{\tau}}\hat{\bar{g}}\cdot\hat{g}+\hat{\rho}_0(\hat{f}^2-\hat{\bar{g}}\hat{g})=0, \tag{4.25}$$

$$D_{\hat{\tau}}D_{\hat{y}}\hat{\bar{g}}\cdot\hat{g}-\mathrm{i}\frac{\kappa^2}{\hat{\rho}_0^2}D_{\hat{\tau}}\hat{\bar{g}}\cdot\hat{g}+\mathrm{i}\hat{\rho}_0 D_{\hat{y}}\hat{\bar{g}}\cdot\hat{g}=0, \tag{4.26}$$

whereas the bilinear equation (2.12) rduces to (4.24) when coupled with (2.10).

5. Discussion

We have constructed the multisoliton solutions of the CH2 equation by a direct method combined with the reciprocal transformation. Subsequently, we have shown that the mulitisoliton solutions of the CH and HS2 equations are reduced from those of the CH2 equation by means of appropriate scaling limits. We note that the CH2 equation does not exhibit peakons as opposed to the CH equation. This fact can be confirmed by taking the zero dispersion limit $\kappa \to 0$ for the one-soliton solution (3.4). On the other-hand, the one-soliton solution (4.7) of the CH equation yields the peakon solution in the limit $\kappa \to 0$ [9, 10]. It has also been pointed out that the HS2 equation (4.15) does not support peakons when $\kappa = 0$ and $\rho_0 \neq 0$. Nevertheless, if one imposes the boundary condition $\rho \to 0$ as $|x| \to \infty$, then the HS2 eqaution has multipeakon solutions [1]. It is an interesting problem for the HS2 equation to recover the peakon solutions from the smooth soliton solutions.

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References

[1] A. Constantin and R. I. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lewtt. A **372** (2009) 7129-7132.

[2] R.I. Ivanov, *Two-component integrable systems modelling shallow water waves: The constant vorticity case*, Wave Motion **46** (2009) 389-396

[3] R. Camassa and D.D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71** (1993) 1661-1664.

[4] Y. Matsuno, Bilinear Transformation Method, 1884, Academic Press, New York.

[5] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University Press, Cambridge, 2004.

[6] Y. Matsuno, *Parametric representation for the multisoliton solution of the Camassa-Holm equation*, J. Phys. Soc. Jpn. **74** (2005) 1983-1987.

[7] Y. Matsuno, *Bäcklund transformation and smooth multisoliton solutions for a modified Camassa-Holm equation with cubic nonlinearity*, J. Math. Phys. **54** (2013) 051504.

[8] Y. Matsuno, *Smooth and singular multisoliton solutions of a modified Camassa-Holm equation with cubic nonlinearity and linear dispersion*, J. Phys. A: Mat. Theor. **47** (2014) 125203.

[9] A. Parker and Y. Matsuno, *The peakon limits of soliton solutions of the Camassa-Holm equation*, J. Phys. Soc. Jpn. **75** (2006) 124001.

[10] Y. Matsuno, *The peakon limit of the N-soliton solution of the Camassa-Holm equation*, J. Phys. Soc. Jpn. **76** (2007) 034003.