METRIZABILITY OF SPACES OF LIPSCHITZ FUNCTIONS

A. JIMÉNEZ-VARGAS

1. INTRODUCTION

Let $\text{Lip}_0(E)$ be the linear space of all scalar-valued Lipschitz functions vanishing at 0 on a normed space $E$. Let $\tau$ be a locally convex topology on $\text{Lip}_0(E)$ such that $\tau_0 \leq \tau \leq \tau_3$, where $\tau_0$ and $\tau_3$ denote the compact-open topology and the Nachbin–Couré topology on $\text{Lip}_0(E)$.

We prove in this note that $(\text{Lip}_0(E), \tau_0)$ is a metrizable space if and only if $E$ has finite dimension. Motivated by a positive answer in the setting of holomorphic mappings, the following question is raised: Is it true that $(\text{Lip}_0(E), \tau)$ is metrizable only if $E$ is finite-dimensional?

2. PRELIMINARIES

Let $E$ be a normed space and let $\text{Lip}_0(E)$ denote the linear space of all Lipschitz mappings $f$ from $E$ into $K$ for which $f(0) = 0$. We refer the reader to Weaver’s book [6] for the basic theory of $\text{Lip}_0(E)$.

Let $X$ be a topological space and let $C(X)$ be the linear space of all continuous mappings from $X$ into $K$. We recall the following topologies on $C(X)$.

The compact-open topology on $C(X)$ is the locally convex topology generated by the seminorms

$$
|f|_K = \sup_{x \in K} |f(x)|, \quad f \in C(X),
$$

where $K$ varies over the family of all compact subsets of $X$.

A seminorm $p$ on $C(X)$ is ported by the compact subset $K$ of $X$ if for every open neighborhood $V$ of $K$ in $X$, there is a constant $c > 0$ such that $p(f) \leq c\sup_{x \in V} |f(x)|$ for all $f \in C(X)$. The Nachbin topology on $C(X)$ is the locally convex topology generated by the seminorms on $C(X)$ which are ported by the compact subsets of $X$.

The Nachbin–Couré topology on $C(X)$ is the locally convex topology generated by the seminorms $p$ on $C(X)$ which satisfy the following property: for each increasing countable open cover $\{V_n\}_{n \in \mathbb{N}}$ of $X$, there are $m \in \mathbb{N}$ and $c_m > 0$ such that $p(f) \leq c_m\sup_{x \in V_m} |f(x)|$ for all $f \in C(X)$.

We will denote by $\tau_0$, $\tau_3$ and $\tau_5$ the compact-open topology, the Nachbin-ported topology and the Nachbin–Couré topology on $C(X)$, or on any linear subspace of $C(X)$.

Now we prove the following result.

**Theorem 2.1.** If $E$ is a Banach space, then $(\text{Lip}_0(E), \tau_0)$ is metrizable if and only if $E$ has finite dimension.

**Proof.** Suppose that $(\text{Lip}_0(E), \tau_0)$ is metrizable. Then there exists a sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets of $E$, containing the origin, such that the sequence of seminorms $|\cdot|_{K_n}$ defines the topology $\tau_0$ on $\text{Lip}_0(E)$. We claim that there exists a constant $c > 0$ such that $E$ is included in $\bigcup_{n \in \mathbb{N}} \text{cl}(K_n)$, where $\text{cl}(K_n)$ denotes the closed, convex, balanced hull of $K_n$ in $E$. Indeed, given $x \in E$, it is clear that $|\cdot|_x$ defined on $\text{Lip}_0(E)$ is a continuous seminorm on $(\text{Lip}_0(E), \tau_0)$, so there are $m \in \mathbb{N}$ and $c > 0$ such that $|f|_x \leq c|f|_{K_m}$ for all $f \in \text{Lip}_0(E)$. It follows that $|f(x)| \leq c|f|_{\overline{\text{cl}}(K_m)}$ for all $f \in \text{Lip}_0(E)$. Notice that each $\overline{\text{cl}}(K_n)$ is compact by the Mazur theorem. Since the dual space $E'$ is a subset of $\text{Lip}_0(E)$, we have $|f(x)| \leq c|f|_{\overline{\text{cl}}(K_m)}$ for all $f \in E'$. By the Hahn–Banach separation theorem, we infer that $x$ is in $\text{cl}(K_m)$ as we wanted. Since $E$ is a Baire space, our claim implies that there exists $p \in \mathbb{N}$ such that $\overline{\text{cl}}(K_p)$ has no empty interior in $E$. Hence there is a compact neighborhood of 0 in $E$ and therefore $E$ has finite dimension by the Riesz theorem.

Conversely, if $E$ is finite dimensional, then $(C(E), \tau_0)$ is metrizable (see the proof of [3, Theorem 16.9]) and therefore so is $(\text{Lip}_0(E), \tau_0)$. □

The results on the metrizability of spaces of holomorphic functions have an interesting history. In 1968, Alexander [1] proved the following theorem for Banach spaces with Schauder basis, which was generalized by Chae (see [3, Theorem 16.10]): If $U$ is an open subset of an infinite dimensional Banach space $E$ and
$\tau$ is a topology on the space $H(U)$ of all holomorphic functions on $U$ finer than the topology of pointwise convergence, then $(H(U), \tau)$ is not metrizable.

In 2007, this theorem probably motivated Ansemil and Ponte, whose paper [2] contains that if $U$ is an open subset of an infinite-dimensional complex metrizable locally convex space $E$, then $(H(U), \tau_\gamma)$ is not metrizable. This answered a question stated by Mujica in [5, Problem 11.9] thirty years ago. It is known that $\tau_0 \leq \tau_\gamma \leq \tau_\delta$ on $H(U)$.

In 2009, López-Salazar [4] improved this result showing that if $U$ is an open subset of a complex metrizable locally convex space $E$ and $\tau$ is a locally convex topology on $H(U)$ such that $\tau_0 \leq \tau \leq \tau_\delta$, then $(H(U), \tau)$ is a metrizable space if and only if $E$ has finite dimension.

Theorem 2.1 suggests to tackle the problem on the metrizability of $\text{Lip}_0(E)$ equipped with other topologies, with an approach similar to that described above for spaces of holomorphic functions.

REFERENCES


DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD DE ALMERÍA, 04120 ALMERÍA, SPAIN
E-mail address: ajimenez@ual.es