数理解析研究所講究録
第2035巻 2017年 150-155

THE DETERMINANT OF A ROW-FACTORIZATION MATRIX IN A NUMERICAL SEMIGROUP

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1. Notations and Definitions

In this paper, we study the determinant of row-factrization matrix in a numerical semigrooup. Row-factrization matrices, in short RF-matrices, for pseudo-Frobenius numbers f in a numerical semigroup S are defined by Moscariello in [5], to prove the type of almost symmetric semigroups generated by four elements is less than or equal to three. Their determinants, in general, is multiples of f. If its absolute value is f, then we get a basis of the kernel space defined by S, from the RF-matrix. Then we can also get a generating system of a defining ideal of the semigroup associated with S. Hence, it is important to investigate the determinants of RF-matrices.

First, we give notations and definitions. Let $\mathbb{Z}$ be the ring of integers, and $\mathbb{N}$ the set of non negative integers. Let S be a non empty subset in $\mathbb{N}$. We say that S is a *semigroup* in $\mathbb{N}$, if

(1) $0 \in S$,
(2) $a + b \in S$, if $a, b \in S$.

Let S be a semigroup in $\mathbb{N}$ and $n_1, \dots, n_s \in \mathbb{N}$. We say that S is *generated* by $n_1, \dots, n_s$ if

$$S = \{a_1 n_1 + \cdots + a_s n_s : a_1, \dots, a_s \in \mathbb{N}\}.$$

We also say that S is *minimally* generated by $n_1, \dots, n_s$, if any proper subset of $\{n_1, \dots, n_s\}$ does not generate S. Then we denote S by $\langle n_1, \dots, n_s \rangle$ and call s the *embedding dimension* of S.

If $\mathbb{N} - S$ is finite, we say that S is *numerical.* We note that S is numerical if and only if the general common divisor of $n_1, \dots, n_s$ is one.

Example 1.

$$\langle 3, 5 \rangle = \{0, 3, 5, 6, 8, 9, 10, \dots\}$$

is a numerical semigroup generated by 3 and 5.

From now, all semigroups are assumed to be numerical semigroups in $\mathbb{N}$. Let S be a semigroup. The number

$$\mathrm{F}(S) = \max\{a \in \mathbb{Z} : a \notin S\}$$

is called the *Frobenius number* of S. We also define

$$\mathrm{PF}(S) = \{a \in \mathbb{Z} : a + x \in S \text{ if } x \in S \text{ and } x \neq 0\}$$

and an element in $\mathrm{PF}(S)$ is called a *pseudo-Frobenius number*. Obviously, $\mathrm{F}(S) \in \mathrm{PF}(S)$. We say that the number of $\mathrm{PF}(S)$ is the *type* of S, denoted by $\mathrm{t}(S)$. For $d \in S$, We define the *Apery set* $\mathrm{Ap}(S, d)$ as follows:

$$\mathrm{Ap}(S, d) = \{x \in S : x - d \notin S\}.$$

Note $|\mathrm{Ap}(S, d)| = d$ and, for any $a \in \{0, 1, \dots, d-1\}$, there is $x \in \mathrm{Ap}(S, d)$ with $a \equiv x \mod d$.

Next, we define RF-matrices. Let $S = \langle n_1, \dots, n_s \rangle$ be a semigroup and $f \in \mathbb{Z} - S$. For each i, there is $a_{ii} < 0$ with $f - a_{ii}n_i \in \mathrm{Ap}(S, n_i)$. Then there are $a_{ij} \geq 0$ for $j \neq i$ satisfying $f - a_{ii}n_i = \sum_{j \neq i} a_{ij}n_j$. We say that the matrix $\mathrm{RF}(f) = (a_{ij})$ is an RF-*matrix* (row-factorization matrix) for f in S. We denote it by $\mathrm{RF}(f)$.

Example 2. (Examples of RF-matrices)

(1) Let $S = \langle 3, 4, 5 \rangle$ and $f = 2 \notin S$. Then

$$\mathrm{RF}(2) = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

(2) Let $S = \langle 4, 5, 6 \rangle$ and $f = 7 \notin S$. Then

$$\mathrm{RF}(7) = \begin{pmatrix} -1 & 1 & 1 \\ 3 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 1 & -1 \end{pmatrix}.$$

From above example, it follows that an RF-matrix for pseudo-Frobenius number is not unique in general.

2. The determinants of RF-matrices

In this section, we consider the following question: Let S be a numerical semigroup with embedding dimension s and $f \in \mathrm{PF}(S)$. Then, does the equation

$$(*) \qquad \det \mathrm{RF}(f) = (-1)^{s+1} f$$

hold?

Theorem 1. If $s = 2$ or 3, then $(*)$ holds.

Proof. Assume $s = 2$ and let $S = \langle n_1, n_2 \rangle$. Then $\mathrm{F}(S) = n_1n_2 - n_1 - n_2$ is a unique pseudo-Frobenius number and

$$\mathrm{RF}(\mathrm{F}(S)) = \begin{pmatrix} -1 & n_1 - 1 \\ n_2 - 1 & -1 \end{pmatrix},$$

thus $\det \mathrm{RF}(\mathrm{F}(S)) = -\mathrm{F}(S)$.

Assume $s = 3$ and let $S = \langle n_1, n_2, n_3 \rangle$. If $\mathrm{t}(S) = 1$, then we may assume $dn_3 \in \langle n_1, n_2 \rangle$ where $d = \gcd(n_1, n_2)$. Then $\mathrm{F}(S) = n_1n_2/d - n_1 - n_2 + (d-1)n_3$ is a unique pseudo-Frobenius number and

$$\mathrm{RF}(\mathrm{F}(S)) = \begin{pmatrix} -1 & n_1/d - 1 & d - 1 \\ n_2/d - 1 & -1 & d - 1 \\ a_{31} & a_{32} & -1 \end{pmatrix},$$

where $dn_3 = (a_{31} + 1 - n_2/d)n_1 + (a_{32} + 1)n_2$ or $(a_{31} + 1)n_1 + (a_{32} + 1 - n_1/d)n_2$. Then $\det \mathrm{RF}(\mathrm{F}(S)) = \mathrm{F}(S)$.

The rest case is that of $s = 3$ and $\mathrm{t}(S) = 2$. Let $\mathrm{PF}(S) = \{f_1, f_2\}$ and put $\mathrm{RF}(f_1) = (a_{ij})$ and $\mathrm{RF}(f_1) = (b_{ij})$. By classical result, they are unique and we may assume

$$\begin{aligned} a_{12} &= b_{32} = a_{32} + b_{12} + 1, \\ a_{23} &= b_{13} = a_{13} + b_{23} + 1, \\ a_{31} &= b_{21} = a_{21} + b_{31} + 1 \end{aligned}$$

Since

$$\begin{aligned} n_1 &= (a_{12} + 1)(a_{13} + 1) + (b_{12} + 1)(b_{23} + 1), \\ n_2 &= (a_{23} + 1)(a_{21} + 1) + (b_{23} + 1)(b_{31} + 1), \\ n_3 &= (a_{31} + 1)(a_{32} + 1) + (b_{31} + 1)(b_{12} + 1), \end{aligned}$$

we have $\det \mathrm{RF}(f_i) = f_i$ for $i = 1, 2$. □

Definition. Let S_1, S_2 be numerical semigroups and $d_1 \in S_2$ and $d_2 \in S_1$. If d_1 and d_2 are coprime, then

$$S = d_1S_1 + d_2S_2 = \{d_1x + d_2y : x \in S_1, y \in S_2\}$$

is a numerical semigroup. We say that S is *glued* by S_1 and S_2.

Definition. We say that S is *completely glued* if one of the following is satisfied:

(1) $S = \langle 1 \rangle$,
(2) S is glued by completely glued semigroups.

If the embedding dimension of S is 2, then S is completely glued. If S is completely glued, then its type is one.

Theorem 2. If S is completely glued, then there is an RF-matrix of $\mathrm{F}(S)$ which satisfies $(*)$.

Proof. If $S = \langle 1 \rangle$, then the assertion is clear. Assume $S = d_1 S_1 + d_2 S_2$ where $d_1 \in S_2$, $d_2 \in S_1$ and both S_1 and S_2 are completely glued. Then

$$\mathrm{F}(S) = d_1 \mathrm{F}(S_1) + d_2 \mathrm{F}(S_2) + d_1 d_2$$

and there is an RF-matrix M_1 (resp. M_2) for $\mathrm{F}(S_1)$ (resp. $\mathrm{F}(S_2)$) in S_1 (resp. (S_2)) satisfying $\det M_1 = (-1)^{s_1} \mathrm{F}(S_1)$ (resp. $\det M_2 = (-1)^{s_2} \mathrm{F}(S_2)$) where s_1 (resp. s_2) is the embedding dimension of M_1 (resp. M_2). Since $\mathrm{F}(S_1) + d_2 \in S_1$ (resp. $\mathrm{F}(S_2) + d_1 \in S_2$), we may write $\mathrm{F}(S_1) + d_2 = \sum_i a_i n_i$ (resp. $\mathrm{F}(S_2) + d_1 = \sum_i a'_i n'_i$) where $S_1 = \langle n_1, \ldots, n_{s_1} \rangle$ (resp. $S_2 = \langle n'_1, \ldots, n'_{s_2} \rangle$) and $a_i \geq 0$ (resp. $a'_i \geq 0$) for each i. Let N_1 (resp. N_2) be an $s_2 \times s_1$-matrix (resp. $s_1 \times s_2$-matrix) whose ij-entry is a_i (resp. a'_i) for each i, j. And put

$$M = \begin{pmatrix} M_1 & N_2 \\ N_1 & M_2 \end{pmatrix}.$$

Then M is an RF-matrix for $\mathrm{F}(S)$ in S and $\det M = (-1)^{s_1+s_2} \mathrm{F}(S)$. □

Theorem 3. Assume $s = 4$. If the type of S is one, or if S is pseudo-symmetric, then there is an RF-matrix of $\mathrm{F}(S)$ which satisfies $(*)$. We say that S is *pseudo-symmetric* if $\mathrm{PF}(S) = \{\mathrm{F}(S)/2, \mathrm{F}(S)\}$.

Proof. Let $S = \langle n_1, n_2, n_3, n_4 \rangle$. Assume $\mathrm{t}(S) = 1$. Further, we may assume that S is not completely glued. Then, by [1], after suitable renumbering, we have

$$\mathrm{RF}(\mathrm{F}(S)) = \begin{pmatrix} -1 & \alpha_2 - 1 & \alpha_3 - 1 & a_{14} \\ a_{21} & -1 & \alpha_3 - 1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{32} & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & \alpha_2 - 1 & a_{43} & -1 \end{pmatrix},$$

where α_i is the minimal positive number satisfying that $(\alpha_i - 1)n_i$ has the unique factorization by $n_1, \ldots, n_4$ for each i and $0 < a_{21} < \alpha_1$, $0 < a_{32} < \alpha_2$, $0 < a_{43} < \alpha_3$, and $0 < a_{14} < \alpha_4$. From this, we have $\det \mathrm{RF}(\mathrm{F}(S)) = -\mathrm{F}(S)$.

Assume that S is pseudo-symmetric. By [3], after suitable renumbering, we also have

$$\mathrm{RF}(\mathrm{F}(S)) = \begin{pmatrix} -1 & \alpha_2 - 2 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & -1 & \alpha_3 - 2 & \alpha_4 - 1 \\ \alpha_1 - 2 & \alpha_2 - 1 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} - 1 & \alpha_3 - 1 & -1 \end{pmatrix},$$

$$\mathrm{RF}(\mathrm{F}(S)/2) = \begin{pmatrix} -1 & \alpha_2 - 1 & 0 & 0 \\ 0 & -1 & \alpha_3 - 1 & 0 \\ \alpha_1 - 1 & 0 & -1 & \alpha_4 - 1 \\ \alpha_1 - 1 & a_{42} & 0 & -1 \end{pmatrix},$$

where α_i is defined above and $0 < a_{42} < \alpha_2$. From this, we also have $\det \mathrm{RF}(\mathrm{F}(S)) = -\mathrm{F}(S)$. □

Finally, we give some examples of RF-matrices in an almost symmetric semigroup.

Definition. Let S be a semigroup. For any $f \in \mathrm{PF}(S)$ with $f \neq \mathrm{F}(S)$, if $\mathrm{F}(S) - f \in \mathrm{PF}(S)$, we say that S is *almost symmetric.*

Example 3 (Watanabe's example). Let $S = \langle 22, 46, 9, 57\rangle$. Then S is almost symmetric of type 3 and $\mathrm{PF}(S) = \{35, 70, 105\}$. We also have

$$\mathrm{RF}(70) = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 2 & -1 & 8 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 9 & -1 \end{pmatrix}, \mathrm{RF}(35) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 9 & 0 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}.$$

$\det \mathrm{RF}(70) = 0$, $\det \mathrm{RF}(35) = -35$.

Example 4. Let $S = \langle 22, 26, 79, 83\rangle$. Then S is almost symmetric of type 3 and $\mathrm{PF}(S) = \{57, 238, 295\}$. We also have

$$\mathrm{RF}(57) = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 5 & 1 & -1 & 0 \\ 4 & 2 & 0 & -1 \end{pmatrix}, \mathrm{RF}(238) = \begin{pmatrix} -1 & 10 & 0 & 0 \\ 12 & -1 & 0 & 0 \\ 0 & 9 & -1 & 1 \\ 11 & 0 & 1 & -1 \end{pmatrix}$$

and

$$\mathrm{RF}(295) = \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 0 & 8 & -1 & 2 \\ 10 & 0 & 2 & -1 \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 9 & 0 & 1 \\ 11 & -1 & 1 & 0 \\ 4 & 11 & -1 & 0 \\ 10 & 0 & 2 & -1 \end{pmatrix}.$$

$\det \mathrm{RF}(57) = \det \mathrm{RF}(238) = 0$. We note that the determinant of the former RF-matrix for 295 is zero, and that of the latter one is -295.

From above examples, it follows that the condition $(*)$ does not hold for all RF-matrices for pseudo-Frobenius numbers. Hence we modify the question as follows:

Question. Let S be a semigroup with embedding dimension s. Then, does the equation

$$(*) \qquad \det \mathrm{RF}(\mathrm{F}(S)) = (-1)^{s+1}\,\mathrm{F}(S)$$

hold for some RF-matrix for $\mathrm{F}(S)$?

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