数理解析研究所講究録
第2035巻 2017年 173-178

PECULIAR HOMOMORPHISMS ON COMMUTATIVE BANACH ALGEBRAS OF VECTOR-VALUED FUNCTIONS

OSAMU HATORI, SHIHO OI, AND HIROYUKI TAKAGI

The unital commutative Banach algebra of all complex-valued continuous functions on a compact Hausdorff space K_j is denoted by $C(K_j)$. A map $\psi : C(K_1) \to C(K_2)$ is a unital homomorphism (a homomorphism which preserves identity) if and only if there is a continuous map $\varphi : K_2 \to K_1$ such that $\psi(f) = f\circ\varphi$ for every $f \in C(K_1)$. In general Gelfand theory asserts that a unital homomorphism between unital semisimple commutative Banach algebras is represented by a composition operator induced by the associated continuous map between maximal ideal spaces. The converse assertion is not always true; there is a restriction on the continuous map between maximal ideal spaces which defines a unital homomorphism. There can be a continuous map whose composition does not even define a map between underlying algebras. For example a map $\varphi : \bar{D} \to \bar{D}$ from the closed unit disk $\bar{D}$ into itself define a unital homomorphism (composition operator) from the disk algebra into itself if and only if the map φ is analytic on the open disk.

Let K be a compact metric space and E a unital commutative Banach algebra. We say that a map $F : K \to E$ is a Lipschitz map from K into E if the Lipschitz constant is finite; $L(F) = \sup_{x\neq y} \frac{\|F(x)-F(y)\|_E}{d(x,y)} < \infty$, where $d(\cdot,\cdot)$ denotes the metric on K. The algebra of all Lipschitz maps from K into E is denoted by $\mathrm{Lip}(K,E)$. Then $\mathrm{Lip}(K,E)$ is a unital commutative Banach algebra with the norm $\|\cdot\|_L = L(\cdot) + \|\cdot\|_{\infty(K)}$. In this paper we study Banach algebras between which a unital homomorphism always has a special form. In particular, we study the case of the algebras of Lipschitz maps from compact metric spaces into unital semisimple commutataive Banach algebras. The maximal ideal space of $\mathrm{Lip}(K_j,E_j)$ is homeomorphic to $K_j \times M(E_j)$, where $M(E_j)$ is the maximal ideal space of E_j. If E_j is semisimple, then $\mathrm{Lip}(K_j,E_j)$ is semisimple, and we may suppose that

$$\mathrm{Lip}(K_j,E_j) \subset C(K_j,E_j).$$

Suppose that

$$\psi : \mathrm{Lip}(K_1,E_1) \to \mathrm{Lip}(K_2,E_2)$$

is a unital homomorphism. Then there exists a continuous map $\Phi : K_2 \times M(E_2) \to K_1 \times M(E_1)$ denoted by $\Phi(x,\phi) = (\varphi_1(x,\phi), \varphi_2(x,\phi))$ such that

$$\psi(F)(x,\phi) = F(\varphi_1(x,\phi), \varphi_2(x,\phi)), \quad \forall (x,\phi) \in K_2 \times M(E_2).$$

In the case of $E_j = \mathrm{Lip}(L_j,\mathbb{C})$ for a compact metric space L_j, the maximal ideal space of $\mathrm{Lip}(K_j,E_j)$ is homeomorphic to $K_j \times L_j$ and the induced composition operator defined by *any* Lipschitz map from $K_2 \times L_2$ into $K_1 \times L_1$ is a unital homomorphism from $\mathrm{Lip}(K_1,E_1)$ into $\mathrm{Lip}(K_2,E_2)$. On the other hand, an interesting observation was exhibited

Key words and phrases. vector-valued functions, vector-valued Lipschitz algebras, algebra homomorphisms, algebra isomorphisms.

by Botelho and Jamison [1]; if K_2 is connected and E_j is the algebra of convergent sequences or the algebra of bounded sequences, then φ_2 depends only on $M(E_2)$, not on K_2. Oi [6] generalized their result by proving that it is the case where E_j is a unital commutative C^*-algebra. We call a unital homomorphism represented by the composition operator induced by a continuous map $\Phi(x,\phi) = (\varphi_1(x,\phi), \varphi_2(\phi))$ is of type BJ. Oi [6] in fact proved that any unital homomorphisms between algebras of Lipsichtz maps on connected compact metric spaces into unital commutative C^*-algebras are of type BJ; a unital homomorphism $\psi : \mathrm{Lip}(K_1, C(K_1)) \to \mathrm{Lip}(K_2, C(K_2))$ is represented by the form of

$$\psi(F)(x,\phi) = F(\varphi_1(x,\phi), \varphi_2(\phi)), \quad \forall(x,\phi) \in K_2 \times K_2$$

if K_2 is connected. In this paper, we further study homomorphisms of type BJ. It is interesting to note that certain isometries between Banach algebras of vector-valued Lipschitz maps $\mathrm{Lip}(K,E)$ is of type BJ unless K is connected. We give a sufficient condition for *admissible quadruples* between which unital homomorphisms are *always* of type BJ.

1. Admissible quadruples

An admissible quadruple is defined by Nikou and O'Farrell in [5]. For a given Banach algebra of complex-valued continuous functions, the corresponding admissible quadruple is a Banach algebra of vector-valued continuous maps of the same kind as complex-valued continous functions in the give Banach algebra. Prior to define an admissible quadruple, we define a vector-valued function algebra.

Definition 1. We say that A is a E-valued function algebra on a compact Hausdorff space X *in the strong sense* if A is a subalgebra of $C(X,E)$ for a unital commutative Banach algebra E such that the following conditions are satisfied.

(1) A is a Banach algebra with some norm $\|\cdot\|_A$,
(2) For every $a \in E$ the constant map on X defined by $x \mapsto a$ is in A,
(3) A separates the points of X, that is, for every pair x and y of different points in X, there exists f in A such that $f(x) \neq f(y)$,
(4) for every $x \in X$ the evaluation map $e_x : A \to E$ defined by $f \mapsto f(x)$ is continuous.

Note that a $\mathbb{C}$-valued function algebra in the strong sense is a $\mathbb{C}$-valued function algebra in the sense of Nikou and O'Farrell. But E-valued function algebra in the sense of Nikou and O'Farrell need not be in the strong sense when E is of dimension 2 or more. Note also that if E is semisimple, then the evaluation map $e_x : A \to E$ defined by $e_x(f) = f(x)$ for $f \in E$ is automatically continuous for every $x \in X$ by a theorem of Šilov (cf. [7, Theorem 3.1.11]). The algebra $C(X,E)$ is a E-valued function algebra on X in the strong sense with the supremum norm; $\|f\|_{\infty(X)} = \sup\{\|f(x)\|_E : x \in X\}$. We call a $\mathbb{C}$-valued function algebra A in the strong sense is natural if the map from X into $M(A)$ defined by $x \mapsto e_x$ is surjective, to say simply $X = M(A)$.

Let A be a $\mathbb{C}$-valued function algebra on a compact Hausdorff space in the strong sense and E a unital commutative Banach algebra. For $f \in A$ and $b \in E$, $f \otimes b$ denotes the map in $C(X,E)$ such that $(f \otimes b)(x) = f(x)b$ for $x \in X$. We denote

$$A \otimes E = \left\{ \sum_{j=1}^{n} f_j \otimes b_j : n \in \mathbb{N},\ f_j \in A,\ b_j \in E\ (j = 1,2,\ldots,n) \right\},$$

where $\mathbb{N}$ is the set of all positive integers. We say that $\mathbb{C}$-valued function algebra on X in the strong sense is a uniform algebra on X if it is uniformly closed. See [2] for general theory of uniform algebras. Note that the terminology "a function algebra" in [2] means a uniform algebra. An admissible quadruple is a vector-valued version of a given function algebra. It was defined by Nikou and O'Farrell in [5]. The following is an essentially the same definition as the one given in [5].

Definition 2. By an admissible quadruple we mean a quadruple $(X, E, B, \widetilde{B})$, where

(1) X is a compact Hausdorff space,
(2) E is a unital commutative Banach algebra,
(3) $B \subset C(X)$ is a natural $\mathbb{C}$-valued function algebra on X,
(4) $\widetilde{B} \subset C(X, E)$ is an E-valued function algebra on X in the strong sense,
(5) $B \otimes E \subset \widetilde{B}$ and
(6) $\{\lambda \circ f : f \in \widetilde{B}, \lambda \in M(E)\} \subset B$.

For a compact metric space K and a unital commutative Banach algebra E, $(K, E, \mathrm{Lip}(K), \mathrm{Lip}(K, E))$ is an admissible quadruple.

Definition 3. Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Let $\pi : X \times M(E) \to M(\widetilde{B})$ be given by $\pi(x, \phi) = \phi \circ e_x$, where $\phi \circ e_x(F) = \phi(F(x))$ for every $F \in \widetilde{B}$. Then by a routine argument π is a continuous injection. We say that an admissible quadruple $(X, E, B, \widetilde{B})$ is natural if the associated map π is bijective. In this case π is a homeomorphsims from $X \times M(E)$ onto $\{\phi \circ e_x : (x, \phi) \in X \times M(E)\} = M(\widetilde{B})$.

Suppose that $(X, E, B, \widetilde{B})$ is semisimple and natural; $\pi : X \times M(E) \to M(\widetilde{B})$ is surjection. Then we may suppose that

$$\widetilde{B} \subset C(X \times M(E)). \tag{1.1}$$

Proposition 4. *Let $(X, E, B, \widetilde{B})$ be an admissible quadruple. Suppose that B is dense in $C(X)$. Suppose also that $\widetilde{B}$ is inverse-closed; $F \in \widetilde{B}$ with $\Gamma_{\widetilde{B}}(F)(\phi \circ e_x) \neq 0$ for every pair $x \in X$ and $\phi \in M(E)$ implies $F^{-1} \in \widetilde{B}$. Then $(X, E, B, \widetilde{B})$ is natural.*

By Proposition 4 we easily see that $(K, E, \mathrm{Lip}(K, \mathbb{C}), \mathrm{Lip}(K, E))$ is a natural admissible quadruple.

Proposition 5. *An admissible quadruple $(X, E, B, \widetilde{B})$ is semisimple if and only if E is semisimple.*

If E is semisimple, then $(K, E, \mathrm{Lip}(K, \mathbb{C}), \mathrm{Lip}(K, E))$ is semisimple and natural. Hence we may suppose that

$$\mathrm{Lip}(K, E) \subset C(K \times M(E)) \tag{1.2}$$

2. Algebra homomorphisms

In this section we show that a unital homomorphism between admissible quadruples has a peculiar form under certain topological assumptions on maximal ideal spaces. Just for simplicity we assme that a commutative Banach algebra E_j is semisimple; see [3] for a genral case. We omit proofs of Theorems 6 and 7; precise proofs are given in [3].

Theorem 6. *Suppose that E_j is semisimple and $(X_j, E_j, B_j, \widetilde{B_j})$ is natural. Suppose that $\widetilde{B_1} \subset \overline{B_1 \otimes E_1}$, where $\bar{\cdot}$ denotes the uniform closure on $M(\widetilde{B_1})$. Suppose that X_2 is connected with respect to the relative topology induced by the metric inherited from the dual space of B_2 and that $M(E_1)$ is totally disconnected with respect to the relative topology induced by the metric inherited from the dual space of E_1. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Then there exists a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : X_2 \times M(E_2) \to X_1$ which satisfies that*

$$\psi(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times M(E_2)$$

for every $F \in \widetilde{B_1}$; ψ is of type BJ.

Theorem 7. *Suppose that E_j is semisimple and $(X_j, E_j, B_j, \widetilde{B_j})$ is natural. Suppose that $\widetilde{B_1} \subset \overline{B_1 \otimes E_1}$, where $\bar{\cdot}$ denotes the uniform closure on $M(\widetilde{B_1})$. Suppose that X_2 is connected and $M(E_1)$ is totally disconnected. Let $\psi : \widetilde{B_1} \to \widetilde{B_2}$ be a unital homomorphism. Then there exists a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : X_2 \times M(E_2) \to X_1$ which satisfies that*

$$\psi(F)(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in X_2 \times M(E_2)$$

for every $F \in \widetilde{B_1}$; ψ is of type BJ.

3. The case of algebras of vector valued Lipschitz maps

If E is semisimple we have

$$\mathrm{Lip}(K, \mathbb{C}) \otimes E \subset \mathrm{Lip}(K, E) \subset \overline{\mathrm{Lip}(K, \mathbb{C}) \otimes E}.$$

Hence we have the following as a corollary of Theorems 6 and 7. Note that the original topology on K, the Gelfand topology induced by $\mathrm{Lip}(K, \mathbb{C})$, and the relative topology induced by the metric induced by operator norm topology on the dual space of $\mathrm{Lip}(K, \mathbb{C})$ all coincide with each other

Corollary 8. *Let K_j be a compact metric space and E_j a unital semisimple commutative Banach algebra for $j = 1, 2$. Suppose that K_2 is connected. Suppose that $M(E_1)$ is totally disconnected with respect to either the Gelfand topology (the original topology as the maximal ideal space) or the relative topology induced by the metric inherited from the dual space of E_1. Let $\psi : \mathrm{Lip}(K_1, E_1) \to \mathrm{Lip}(K_2, E_2)$ be a unital homomorphism. Then there exists a continuous map $\tau : M(E_2) \to M(E_1)$ and a continuous map $\varphi : K_2 \times M(E_2) \to K_1$ such that the map $\varphi(\cdot, \phi) : K_2 \to K_1$ is a Lipschitz map for each $\phi \in M(E_2)$, which satisfies that*

$$(\psi(F))(x, \phi) = F(\varphi(x, \phi), \tau(\phi)), \quad (x, \phi) \in K_2 \times M(E_2)$$

for every $F \in \mathrm{Lip}(K_1, E_1)$; ψ is of type BJ.

We show several examples of unital semisimple commutative Banach algebras E such that the maximal ideal spaces are totally disconnecte with respect to corresponding topologies desicribed in Cororally 8.

Example 9 (cf. [3]). (1) Let M be a compact Hausdorff space. The Banach algebra $C(M)$ of all complex-valued continuous functions on M. Then M is homeomorphic to the maximal ideal space of $C(M)$. By the Urysohn's lemma we infer that M is discrete with respect to the relative topology induced by the metric inherited from the dual space of $C(M)$.

(2) Let $\mathbb{T}$ be the unit circle in the complex plane. Recall that the Wiener algebra is the algebra of all complex-valued continuous functions on $\mathbb{T}$ which have absolute converging Fourier series; $W(\mathbb{T}) = \{f \in C(\mathbb{T}) : \sum |\hat{f}(n)| < \infty\}$ with the norm $\|f\|_W = \sum_m |\hat{f}(m)|$ for $f \in W(\mathbb{T})$. The maximal ideal space of $W(\mathbb{T})$ is homeomorphic to $\mathbb{T}$. By a simple calculation we see that $\mathbb{T}$ is discrete with respect to the relative topology induced by the metric inherited from the dual space of $W(\mathbb{T})$.

(3) Let A be a uniform algebra such that the maximal ideal space coincides with the Choquet boundary. The Choquet boundary for a uniform algebra A is discrete with respect to the relative topology induced by the metric inherited from the dual space of A. It is known as the Cole's counter example to the peak point conjecture [2] that such a uniform algebra which is not a C^*-algebra exists.

(4) Let G be a compact Abelian group and Γ its dual group. Suppose that Γ is a discrete group of bounded order. Then G is a totally disconnected compact Abelian group [8, Example 2.5.7. (iii)]. The group algebra $A(G)$ of all Fourier transforms of functions in $L^1(\Gamma)$ is a unital semisimple commutative Banach algebra whose maximal ideal space is G. See the paper of Katznelson and Rudin [4] and a book of Rudin [8] for further examples and informations.

References

[1] F. Botelho and J. Jamison, *Homomorphisms on a class of commutative Banach algebras*, Rocky Mountain J. Math., **43** (2013), 395–416

[2] A. Browder, *Introduction to Function Algebras*, W. A. Benjamin, Inc., New York-Amsterdam 1969

[3] O. Hatori, S. Oi and H. Takagi, *Peculiar homomorphisms on algebras of vector-valued functions*, preprint

[4] Y. Katznelson and W. Rudin, *The Stone-Weierstrass property in Banach algebras*, Pacific J. Math., **11** (1961), 253–265

[5] A. Nikou and A. G. O'Farrell, *Banach algebras of vector-valued functions*, Glasgow Math. J., **56** (2014), 419–426

[6] S. Oi, *Homomorphisms between algebras of Lipschitz functions with the values in function algebras*, Jour. Math. Anal. Appl., **444** (2016), 210–229

[7] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras. Vol. I. Algebras and Banach algebras*, Encyclopedia of Mathematics and its Applications, 49. Cambridge University Press, Cambridge, 1994.

[8] W. Rudin, *Fourier analysis on gropus*, Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.

Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
E-mail address: hatori@math.sc.niigata-u.ac.jp

Niigata Prefectural Nagaoka High School, 3-14-1 Gakko-cho, Nagaoka City, Niigata Prefecture 940-0041, Japan.
E-mail address: shiho.oi.pmfn20@gmail.com

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto 390-8621, Japan
E-mail address: takagi@math.shinshu-u.ac.jp