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Loxodromic Eisenstein Series for Cofinite Kleinian Groups

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Abstract

We introduce an Eisenstein series associated to a loxodromic element of cofinite Kleinian groups, named the loxodromic Eisenstein series, and study its fundamental properties. We also establish the precise spectral expansion associated to the Laplace-Beltrami operator and derive the analytic continuation with the location of the possible poles and their residues. In addition, we study the asymptotic behavior of the loxodromic Eisenstein series for a degenerating sequence of finite volume three-dimensional hyperbolic manifolds.

1 Introduction

1.1 What is the loxodromic Eisenstein series?

The loxodromic Eisenstein series is an analogue of the ordinary Eisenstein series defined for a cusp. It is defined for a pair of loxodromic fixed points, or equivalently a loxodromic element of cofinite Kleinian groups. It is also an analogue of the hyperbolic Eisenstein series for Fuchsian groups of the first kind, which is defined for a primitive hyperbolic element. There are many studies on the hyperbolic Eisenstein series. Thus we start by introducing some of their results.

1.2 Hyperbolic Eisenstein series for Fuchsian groups of the first kind

Form-valued hyperbolic Eisenstein series was first introduced by S. S. Kudla and J. J. Millson [10] in 1979 as an analogue of the ordinary Eisenstein series associated to a parabolic fixed point. They established an explicit construction of the harmonic 1-form dual to an oriented closed geodesic on an oriented Riemann surface M of genus greater than 1. After that scalar-valued hyperbolic Eisenstein series and elliptic Eisenstein series was introduced by J. Jorgenson and J. Klamar in 2004 (see [4] or [11]). Their definition of the hyperbolic Eisenstein series is given as follows.

Let $\mathbb{H}_2 := \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half plane and $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a Fuchsian group of the first kind. We use the coordinates $x = e^{\rho} \cos \varphi$, $y = e^{\rho} \sin \varphi$.

Definition 1 (J.Jorgenson and J.Kramer, 2004). Let $\gamma \in \Gamma$ be a primitive hyperbolic element($\Leftrightarrow$ $|\mathrm{tr}(\gamma)| > 2$ and primitive) and $\Gamma_\gamma = \langle\gamma\rangle$ be the centralizer of it in Γ. For $z \in \mathbb{H}_2$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$, the hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z, s)$ associated to γ is defined as follows.

$$E_{\mathrm{hyp},\gamma}(z, s) := \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sin \varphi(A\eta z)^s,$$

where $A \in \mathrm{PSL}(2, \mathbb{R})$ is the matrix such that $A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}$ for some $a(\gamma) \in \mathbb{R}$.

Let L_γ be the γ-invariant geodesic in $\mathbb{H}_2$ and $d_{\mathrm{hyp}}(z, L_\gamma)$ be the hyperbolic distance from z to the geodesic line L_γ. Then the hyperbolic Eisenstein series is written as

$$E_{\mathrm{hyp},\gamma}(z, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \cosh(d_{\mathrm{hyp}}(\eta z, L_\gamma))^{-s}.$$

The hyperbolic Eisenstein series $E_{\mathrm{hyp},\gamma}(z, s)$ converges absolutely and locally uniformly for any $z \in \mathbb{H}_2$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. It defines Γ-invariant function and satisfies the differential shift equation

$$(-\Delta + s(s-1))E_{\mathrm{hyp},\gamma}(z, s) = s^2 E_{\mathrm{hyp},\gamma}(z, s+2).$$

We introduce some more studies on the hyperbolic Eisenstein series.

- (J. Jorgenson, J. Kramer and A-M. v. Pippich, 2010)
  The hyperbolic Eisenstein series is in $L^2(\Gamma\backslash\mathbb{H}_2)$ and we can obtain the spectral expansion associated to Laplace-Beltrami operator Δ precisely. It has a meromorphic continuation to all $s \in \mathbb{C}$ and their possible pole and their residues are derived from the spectral expansion (see [8]).

- (D. Garbin, J. Jorgenson and M. Munn, 2008 and T. Falliero, 2007)
  The asymptotic behavior of hyperbolic Eisenstein series for degenerating families of finite volume hyperbolic Riemann surfaces is studied. It is known that in some cases hyperbolic Eisenstein series converges to the ordinary Eisenstein series on the limit surface (see [3] or [2]).

2 Preliminaries

2.1 Notation

Let $\mathbb{H}_3 := \mathbb{C} \times (0, \infty) = \{(z, r) \mid z \in \mathbb{C}, r > 0\}$ be the three-dimensional hyperbolic space. For $P \in \mathbb{H}^3$, we use the notation $P = (z, r) = (x, y, r) = z + rj \in \mathbb{H}_3$, where $z = x + iy, \quad j = (0, 0, 1)$. The hyperbolic line element $d\sigma^2$ and the hyperbolic volume element dv are given by

$$d\sigma^2 := \frac{dx^2 + dy^2 + dr^2}{r^2}, \qquad dv := \frac{dxdydr}{r^3}.$$

The hyperbolic Laplace-Beltrami operator Δ associated with $d\sigma^2$ is given by

$$\Delta = r^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2}\Big) - r\frac{\partial}{\partial r}.$$

The group $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$ acts on $\mathbb{H}_3$ by fractional linear transformation. It is defined as follows.

For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$ and $P = z + rj \in \mathbb{H}_3$,

$$P \mapsto MP := M(P) := (aP + b)(cP + d)^{-1},$$

where the inverse is taken in the skew field of quaternions. More explicitly,

$$M(P) = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2} + \frac{r}{|cz + d|^2 + |c|^2r^2}j.$$

An element $\gamma \in \mathrm{SL}(2,\mathbb{C}), \gamma \neq \pm I$ is called

$$\begin{aligned} \textit{elliptic} \ \text{ if } \ |\mathrm{tr}(\gamma)| < 2 \ \text{ and } \ \mathrm{tr}(\gamma) \in \mathbb{R}, \\ \textit{parabolic} \ \text{ if } \ |\mathrm{tr}(\gamma)| = 2 \ \text{ and } \ \mathrm{tr}(\gamma) \in \mathbb{R}, \\ \textit{loxodromic} \ \text{ if } \ |\mathrm{tr}(\gamma)| > 2 \ \text{ and } \ \mathrm{tr}(\gamma) \in \mathbb{R}, \\ \text{or } \ \mathrm{tr}(\gamma) \notin \mathbb{R}. \end{aligned}$$

Let $\gamma \in \mathrm{PSL}(2,\mathbb{C})$ be the loxodromic element. Then γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to an element

$$D(\gamma) = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix},$$

where $D(\gamma)$ is uniquely determined by the condition $|a(\gamma)| > 1$ and $N(\gamma) := |a(\gamma)|^2 > 1$ is called the norm of γ. Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ be a cofinite Kleinian group i.e. Γ is discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ with finite volume fundamental domain.

2.2 Parabolic Eisenstein series for cofinite Kleinian groups

The ordinary Eisenstein series are defined for a cusp or equivalently a parabolic element of Γ. We call it "parabolic Eisenstein series" in order to distinguish from the loxodromic Eisenstein series.

Definition 2. Let $\gamma \in \Gamma$ be a parabolic element and $\nu = A^{-1}\infty \in \mathbb{P}^1\mathbb{C}$ be the cusp corresponding to γ. Then the parabolic Eisenstein series associated to γ is defined for $P \in \mathbb{H}_3$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$ as follows.

$$E_{par,\gamma}(P,s) := E_\nu(P,s) = \frac{1}{[\Gamma_\nu : \Gamma'_\nu]} \sum_{\eta \in \Gamma'_\nu \backslash \Gamma} r(A\eta P)^s,$$

where Γ_ν denotes the stabilizer subgroup of ν in Γ and Γ'_ν the maximal unipotent subgroup of Γ_ν.

Let $r_0 > 0$ be sufficiently large such that it satisfies $r_0 > r(AMP)$ for any $M \in \Gamma$ and S_{r_0} be the horosphere in $\mathbb{H}_3$ defined by $\{r(P) = r_0\}$. Then we have

$$r(P)^s = r_0^s \exp\bigl(-s \cdot d_{\mathrm{hyp}}(P, S_{r_0})\bigr).$$

We define parabolic counting function $N_{\mathrm{par},\zeta}(T; P, S_{r_0})$ as follows.

$$N_{\mathrm{par},\zeta}(T; P, S_{r_0}) := \sharp\{\eta \in \Gamma_\zeta \backslash \Gamma \mid d_{\mathrm{hyp}}(\eta P, S_{r_0}) < T\}, \tag{1}$$

where $\sharp$ denotes the cardinality of the set. By using the counting function $N_{\mathrm{par},\zeta}(T; P, S_{r_0})$ we can express the parabolic Eisenstein series as the Stieltjes integral

$$E_{\mathrm{par},\zeta}(P, s) = r_0^s \int_0^\infty e^{-su} dN_{\mathrm{par},\zeta}(u; P, S_{r_0}). \tag{2}$$

$E_{par,\gamma}(P, s)$ converges locally uniformly and absolutely for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$. It has the meropmorphic continuation to all of $\mathbb{C}$ and satisfies the following functional equation

$$(-\Delta + s(s-2))E_{par,\gamma}(P, s) = 0.$$

It has no poles in $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$ except possibly finitely many points in the semi-open interval $(1, 2]$ on the real line.

3 Loxodromic Eisenstein series

Let $\gamma \in \Gamma$ be a loxodromic element ($\Leftrightarrow$ $\mathrm{tr}(\gamma) \in \mathbb{R}$ and $|\mathrm{tr}(\gamma)| > 2$, or $\mathrm{tr}(\gamma) \notin \mathbb{R}$). Then there exists $A \in \mathrm{PSL}(2, \mathbb{C})$ such that $A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}$, $|a(\gamma)| > 1$. We use the following change of the coordinates $x = e^\rho \cos\varphi \cos\theta$, $y = e^\rho \cos\varphi \sin\theta$, $r = e^\rho \sin\varphi$.

Definition 3 (loxodromic Eisenstein series). Let $\gamma \in \Gamma$ be a loxodromic element and Γ_γ be the centralizer of γ in Γ. Then the loxodromic Eisenstein series $E_{\mathrm{lox},\gamma}(P, s)$ associated to γ is defined for $P \in \mathbb{H}_3$ and $s \in \mathbb{C}$ with sufficiently large $\mathrm{Re}(s)$ by

$$E_{\mathrm{lox},\gamma}(P, s) := \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \sin\varphi(A\eta P)^s, \tag{3}$$

where $A \in \mathrm{PSL}(2, \mathbb{C})$ is the matrix such that $A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}$ and $|a(\gamma)| > 1$.

Let L_γ be the γ-invariant geodesic in $\mathbb{H}_3$ and L_0 be the positive r-axis. Then $L_\gamma = A^{-1}L_0$. The hyperbolic distance $d_{\text{hyp}}(P, L_0)$ from P to the geodesic line L_0 holds the following formula

$$\sin(\varphi(P))\cosh(d_{\text{hyp}}(P, L_0)) = 1.$$

Using this formula, we can rewrite the loxodromic Eisenstein series as

$$E_{\text{lox},\gamma}(P, s) = \sum_{\eta \in \Gamma_\gamma \backslash \Gamma} \cosh(d_{\text{hyp}}(\eta P, L_\gamma))^{-s}. \tag{4}$$

For $T > 0$, we define the counting function $N_{\text{lox},\gamma}(T; P, L_\gamma)$ as follows.

$$N_{\text{lox},\gamma}(T; P, L_\gamma) := \sharp\{\eta \in \Gamma_\gamma \backslash \Gamma | d_{\text{hyp}}(\eta P, L_\gamma) < T\}, \tag{5}$$

where $\sharp$ denotes the cardinality of the set. Then we can express the loxodromic Eisenstein series (1) as a Stieltjes integral

$$E_{\text{lox},\gamma}(P, s) = \int_0^\infty \cosh(u)^{-s} dN_{\text{lox},\gamma}(u; P, L_\gamma). \tag{6}$$

Theorem 1. The loxodromic Eisenstein series (3) converges absolutely and locally uniformly for any $P \in \mathbb{H}_3$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 2$. It defines Γ-invariant function where it converges and satisfies the differential shift equation

$$(-\Delta + s(s-2))E_{\text{lox},\gamma}(P, s) = s^2 E_{\text{lox},\gamma}(P, s+2). \tag{7}$$

The outline of the proof of Theorem 1 is as follows. When $\text{Re}(s) > 2$, by using the counting function (5), we can show that for any $\varepsilon > 0$ there exists $T_0 > 0$ such that for any $T > T_0$

$$\left| \int_T^\infty (\cosh u)^{-s} dN_{\text{lox},\gamma}(u; P) \right| < \varepsilon.$$

The Γ-invariance and differential equation (7) follow from direct calculation.

4 Spectral expansion and meromorphic continuation

4.1 Spectral expansion

Lemma 1. For any $s \in \mathbb{C}$ with $\text{Re}(s) > 2$, the loxodromic Eisenstein series $E_{\text{lox},\gamma}(P, s)$ is bounded as a function of $P \in \Gamma \backslash \mathbb{H}_3$. If Γ is not cocompact and ν is a cusp such that $\nu = A(j\infty)$ for some $A \in \mathrm{PSL}(2,\mathbb{C})$, we have the estimate

$$|E_{\mathrm{lox},\gamma}(P,s)| = O(r(A^{-1}P)^{-\mathrm{Re}(s)})$$

as $P \to \nu$. In particular, the loxodromic Eisenstein series is square integrable i.e.

$$E_{\mathrm{lox},\gamma}(P,s) \in L^2(\Gamma\backslash\mathbb{H}_3).$$

Lemma 2. Let $\langle\cdot,\cdot\rangle$ be the inner product in $L^2(\Gamma\backslash\mathbb{H}_3)$ and ψ be the real-valued, smooth, bounded function on a fundamental domain $\mathcal{F}_\Gamma = \Gamma\backslash\mathbb{H}_3$. Let $(\Gamma_\gamma)_{\mathrm{tor}}$ be the torsion subgroup of the Γ_γ and $\widetilde{L}_\gamma$ be the geodesic line associated to γ on $\mathcal{F}_\Gamma$. Assume that $\varepsilon > 0$ is sufficiently small. Then we have the following estimate.

$$\langle E_{\mathrm{lox},\gamma}(P,s), \psi\rangle = \frac{2\pi}{|(\Gamma_\gamma)_{\mathrm{tor}}|}\cdot\frac{1}{s-2}\cdot\int_{\widetilde{L}_\gamma}\psi(P)d\sigma + O\Big(\frac{\varepsilon}{s-2}\Big)$$

as $s \to \infty$.

Theorem 2. For any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$, the loxodromic Eisenstein series $E_{\mathrm{lox},\gamma}(P,s)$ associated to $\gamma \in \Gamma$ admits the following spectral expansion.

$$\begin{aligned} E_{\mathrm{lox},\gamma}(P,s) &= \sum_{m\in\mathcal{D}} a_{m,\gamma}(s)e_m(P) \\ &\quad + \frac{1}{4\pi}\sum_{j=1}^{h}\frac{[\Gamma_{\nu_j}:\Gamma'_{\nu_j}]}{|\Lambda_{\nu_j}|}\int_{-\infty}^{\infty} a_{1+i\mu,\gamma}(s)E_{\nu_j}(P,1+i\mu)d\mu, \qquad (8)\end{aligned}$$

where e_m is the eigenfunction of $-\Delta$ and $|\Lambda_{\nu_j}|$ is the Euclidean area of a period parallelogram for lattice Λ_{ν_j}. The coefficients $a_{m,\gamma}(s)$ and $a_{1+i\mu,\gamma}(s)$ are given by

$$\begin{aligned} a_{m,\gamma} = \frac{1}{|(\Gamma_\gamma)_{\mathrm{tor}}|}\cdot\frac{\pi}{e}\cdot\frac{\Gamma((s-1+\mu_m)/2)\Gamma((s-1-\mu_m)/2)}{\Gamma(s/2)^2} \\ \times\int_{\widetilde{L}_\gamma} e_m d\sigma, \qquad (9)\end{aligned}$$

$$a_{1+i\mu,\gamma} = \frac{1}{|(\Gamma_\gamma)_{\mathrm{tor}}|} \cdot \frac{\pi}{e} \cdot \frac{\Gamma((s-1+i\mu)/2)\Gamma((s-1-i\mu)/2)}{\Gamma(s/2)^2} \times \int_{\widetilde{L}_\gamma} E_{\nu_j}(P, 1+i\mu)d\sigma, \quad (10)$$

where $\mu_m^2 = 1 - \lambda_m$ and λ_m is the eigenvalue of the eigenfunction e_m.

The outline of the proof of Theorem 2 is as follows. From Lemma 1, the loxodromic Eisenstein series is in $L^2(\Gamma\backslash\mathbb{H}_3)$. The existence of the spectral expansion (8) follows from this. In order to give the coefficients $a_{m,\gamma}(s)$, we calculate the inner product $\langle E_{\mathrm{lox},\gamma}, e_m\rangle$ and compare its order with

$$\frac{\Gamma((s-1+\mu_m)/2)\Gamma((s-1-\mu_m)/2)}{\Gamma(s/2)^2}$$

by using Lemma 2 and Stirling's asymptotic formula.

4.2 Meromorphic continuation

As a consequence of Theorem 2, we can derive the meromorphic continuation and obtain the location of the possible poles and their residues.

Theorem 3. The loxodromic Eisenstein series $E_{\mathrm{lox},\gamma}(P,s)$ have a meromorphic continuation to all complex numbers $s \in \mathbb{C}$. The possible poles of the continued function are located at the following points.

(a) $s = 1 \pm \mu_m - 2n$, where $n \in \mathbb{N}$ and $\mu_m^2 = 1 - \lambda_m$ for the eigenvalue λ_m, with residues

$$\mathrm{Res}_{s=1\pm\mu_m-2n}\Big[E_{\mathrm{lox},\gamma}(P,s)\Big] = \frac{1}{|(\Gamma_\gamma)_{\mathrm{tor}}|} \cdot \frac{\pi}{e} \cdot \frac{(-1)^n\Gamma(\pm\mu_m - n)}{n! \cdot \Gamma((1\pm\mu_m-2n)/2)^2} \cdot e_m(P) \cdot \int_{\widetilde{L}_\gamma} e_m(P)d\sigma.$$

(b) $s = \rho_\nu - 2n$, where $n \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series $E_\nu(P,\omega)$ with $\mathrm{Re}(\rho_\nu) < 1$, with residues

$$\mathrm{Res}_{s=\rho_\nu-2n}\Big[E_{\mathrm{lox},\gamma}(P,s)\Big] = \frac{1}{|(\Gamma_\gamma)_{\mathrm{tor}}|} \cdot \frac{4\pi^2 i}{e} \cdot \sum_{k=0}^{m} \frac{(-1)^k\Gamma(\rho_\nu - 2n + k - 1)}{k! \cdot \Gamma((\rho_\nu - 2n)/2)^2}$$

$$\times \sum_{\nu=1}^{h} \Big[\mathrm{CT}_{\omega=\rho_\nu-2n+2k} E_\nu(P,\omega) \cdot \int_{\widetilde{L}_\gamma} \mathrm{Res}_{\omega=\rho_\nu-2n+2k} E_\nu(P,\omega) d\sigma$$

$$+ \mathrm{Res}_{\omega=\rho_\nu-2n+2k} E_\nu(P,\omega) \cdot \int_{\widetilde{L}_\gamma} \mathrm{CT}_{\omega=\rho_\nu-2n+2k} E_\nu(P,\omega) d\sigma \Big].$$

(c) $s = 2 - \rho_\nu - 2n$, where $n \in \mathbb{N}$ and $\omega = \rho_\nu$ is a pole of the Eisenstein series $E_\nu(P,\omega)$ with $\mathrm{Re}(\rho_\nu) \in (1,2]$, with residues

$$\mathrm{Res}_{s=2-\rho_\nu-2n} \Big[E_{\mathrm{lox},\gamma}(P,s) \Big]$$

$$= \frac{1}{|(\Gamma_\gamma)_{\mathrm{tor}}|} \cdot \frac{4\pi^2 i}{e} \cdot \frac{(-1)^n \Gamma(1-\rho_\nu-n)}{n! \cdot \Gamma((2-\rho_\nu-2n)/2)^2}$$

$$\times \sum_{\nu=1}^{h} \Big[\mathrm{CT}_{\omega=\rho_\nu} E_\nu(P,\omega) \cdot \int_{\widetilde{L}_\gamma} \mathrm{Res}_{\omega=\rho_\nu} E_\nu(P,\omega) d\sigma$$

$$+ \mathrm{Res}_{\omega=\rho_\nu} E_\nu(P,\omega) \cdot \int_{\widetilde{L}_\gamma} \mathrm{CT}_{\omega=\rho_\nu} E_\nu(P,\omega) d\sigma \Big].$$

5 Asymptotic behavior through degeneration

We consider degeneration of hyperbolic three-manifolds and study the asymptotic behavior of the Loxodromic Eisenstein series.

Theorem (W. Thurston). Let M be a complete orientable hyperbolic three-manifold of finite volume which has $p+q$ cusps. Then there is a convergent sequence of hyperbolic three-manifolds $\{M_i\}_{i=1}^{\infty}$ such that $M_i \to M$, $(i \to \infty)$ and each M_i has exactly p cusps and q short geodesics (see [13] or [5]).

Then there is a positive sequence $\varepsilon_i \to 0$, $(i \to \infty)$ such that each of q short geodesics of M_i has the length $\leq \varepsilon_i$. In particular, any complete non-compact hyperbolic three-manifold of finite volume is a limit of a sequence of compact hyperbolic three-manifolds. The sequence $\{M_i\}_{i=0}^{\infty}$ is called the degenerating sequence with limit manifold M.

Theorem (D. Kazhdan and G. Margulis). There exists a positive number μ such that for each orientable hyperbolic manifold M and each $x \in M$ the loops based at x of length $\leq 2\mu$ generate a free Abelian group of rank at most two in $\pi_1(M,x)$ (see [9]).

The universal constant μ is called Kazhdan-Margulis constant.

Definition 4. For complete three-dimensional hyperbolic manifold M and $\varepsilon > 0$, we define $M_{(0,\varepsilon)}$ as the set of $x \in M$ such that there exists a non-contractible loop at x of length $\leq \varepsilon$ and $M_{(\varepsilon,\infty)}$ as $M \backslash M_{(0,\varepsilon)}$.

$M_{(0,\varepsilon)}$ and $M_{(\varepsilon,\infty)}$ are called ε-thin part and ε-thick part of M respectively. Let $c = c(\varepsilon) > 0$ be the positive real number such that the hyperbolic metric from $(0, 0, c)$ to $(1, 0, c)$ equals to ε. If $\varepsilon < 2\mu$, then a connected component of ε-thin part of M can be classified as following three types.

- cusp tube ($\mathbb{Z}$-cusp)
  It is isometric to the quotient space $\langle z \mapsto z + 1\rangle \backslash H_c$, where $H_c := \{P \in \mathbb{H}^3 \mid r(P) > c\}$.
- cusp torus ($\mathbb{Z} \times \mathbb{Z}$-cusp)
  It is isometric to the quotient space $\langle z \mapsto z + 1, z \mapsto z + \tau\rangle \backslash H_c$, $(\mathrm{Im}(\tau) > 0,\ |\tau| > 1)$.
- Margulis torus (infinite tube)
  It is isometric to the ε-thin part of $\langle\gamma\rangle \backslash U$, where γ is a loxodromic transformation and U is the tubular neighborhood of the axis of γ.

Remark. Let $M_{(0,\varepsilon),L_\gamma}$ and $M_{(0,\varepsilon),\zeta}$ be the connected component of the ε-thin part $M_{(0,\varepsilon)}$ containing the short geodesic L_γ and the cusp ζ respectively. If M is a complete hyperbolic three-manifold of finite volume, then $M_{(0,\varepsilon),L_\gamma}$ is the Margulis torus and $M_{(0,\varepsilon),\zeta}$ is the cusp torus.

Let $M_i \to M$ be a degenerating sequence of hyperbolic three-manifolds of finite volume with limit manifold M. Let L_{γ_i} be a short geodesic of M_i of which the length $l_{\gamma_i} \to 0$ as $i \to \infty$ and result new cusp ζ on M. Then for sufficiently large positive real number $r_0 > 0$ and l_{γ_i}, we define $g(r_0, l_{\gamma_i})$ as follows.

$$g(r_0, l_{\gamma_i}) = \int_{\cot^{-1}\left(\sqrt{\frac{|\mathcal{P}|}{2\pi l_{\gamma_i} r_0^2}}\right)}^{\frac{\pi}{2}} \frac{d\phi}{\sin\phi}, \tag{11}$$

where $|\mathcal{P}|$ denotes the Euclidean area determined by the boundary torus of the connected component $M_{i,(0,\varepsilon),L_{\gamma_i}}$.

Lemma 3. Under the above setting, the following equation holds.

$$\lim_{i\to\infty} N_{\mathrm{lox},M_i,\gamma_i}(T+g(r_0,l_{\gamma_i});P,L_{\gamma_i}) = N_{\mathrm{par},M,\zeta}(T,P,S_{r_0}),$$

where $N_{\mathrm{lox},M_i,\gamma_i}(T;P,L_{\gamma_i})$ and $N_{\mathrm{par},M,\zeta}(T;P,S_{r_0})$ is the loxodromic counting function and the parabolic counting function respectively.

Theorem 4. Let $M_i \to M$ be a degenerating sequence of hyperbolic three-manifolds of finite volume with limit manifold M. Let L_{γ_i} be a short geodesic of M_i of which the length $l_{\gamma_i} \to 0$ as $i \to \infty$ and result new cusp ζ on M. Then we have

$$\lim_{i\to\infty}\left(\frac{|\mathcal{P}|}{2\pi l_{\gamma_i}}\right)^{\frac{s}{2}} E_{\mathrm{lox},M_i,L_{\gamma_i}}(P,s) = E_{\mathrm{par},M,\zeta}(P,s),$$

where $|\mathcal{P}|$ denotes the Euclidean area determined by boundary torus of cusp ζ.

The outline of the proof of Theorem 4 is as follows. First, by estimating with counting function, we can show that for any $\varepsilon > 0$ there exists sufficiently large T_0 such that

$$\begin{aligned}&\left|2^{-s}e^{sg(r_0,l_i)}\int_{T_0+g(r_0,l_i)}^{\infty}\cosh(u)^{-s}dN_{\mathrm{lox},M_i,\gamma}(u;P,L_\gamma)\right|\\ &\qquad\le \frac{2^{-1}e^{(2-s)T_0+(s+1)g(r_0,l_i)+2r}(\log N(\gamma_0)+2r)}{\sinh(2r)-2r}\cdot\frac{s}{s-2}<\varepsilon. \qquad (12)\end{aligned}$$

Next, from Lemma 3, we have

$$\begin{aligned}\lim_{i\to\infty} 2^{-s}r_0^{s}e^{sg(r_0,l_{\gamma_i})}&\int_0^{T_0+g(r_0,l_{\gamma_i})}(\cosh(u))^{-s}dN_{\mathrm{lox},M_i,\gamma_i}(u;P,L_{\gamma_i})\\ &= r_0^{s}\int_0^{T_0}e^{-su}dN_{\mathrm{par},M,\zeta}(u;P,S_{r_0}). \qquad (13)\end{aligned}$$

Furthermore, we can evaluate $g(r_0,l_{\gamma_i})$ and then

$$\begin{aligned}2^{-s}r_0^{s}e^{sg(r_0,l_{\gamma_i})} &= 2^{-s}r_0^{s}\left(\sqrt{\frac{|\mathcal{P}|}{2\pi l_{\gamma_i}r_0^2}}+\sqrt{1+\frac{|\mathcal{P}|}{2\pi l_{\gamma_i}r_0^2}}\right)^{s}\\ &\to \left(\frac{|\mathcal{P}|}{2\pi l_{\gamma_i}}\right)^{\frac{s}{2}}, \qquad (14)\end{aligned}$$

as $l_{\gamma_i} \to 0$, i.e. $i \to \infty$. From (12)–(14), we obtain the assertion of Theorem 4.

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