数理解析研究所講究録
第2038巻 2017年 42-55

Navier wall law for nonstationary viscous incompressible flows

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1 Introduction

This note is a summary of the paper [7]. In fluid mechanics, it is a basic subject to understand the mathematical structure of flows near a solid wall with a rough surface. In the following we consider the initial-boundary value problem of the Navier-Stokes system in the two-dimensional rough boundary domain $\Omega^{\varepsilon}=\{(x_1,x_2)\in\mathbb{R}^2\mid \varepsilon\omega(\frac{x_1}{\varepsilon})<x_2<\infty\}$.

$$\begin{cases} \partial_t u^{\varepsilon}-\Delta u^{\varepsilon}+u^{\varepsilon}\cdot\nabla u^{\varepsilon}+\nabla p^{\varepsilon}=0\,, & t>0\,,\quad x\in\Omega^{\varepsilon}\,,\\ \nabla\cdot u^{\varepsilon}=0\,, & t\geq 0\,,\quad x\in\Omega^{\varepsilon}\,,\\ u^{\varepsilon}(x_1,x_2)\text{ is }2\pi\text{-periodic in }x_1\,, & t\geq 0\,,\\ u^{\varepsilon}|_{t=0}=u_0\,, & x\in\Omega^{\varepsilon}\,. \end{cases} \tag{NS$^{\varepsilon}$}$$

The Dirichlet (no-slip) boundary condition is imposed on the rough boundary $\partial\Omega^{\varepsilon}$.

$$u^{\varepsilon}=0 \qquad \text{on}\ \ \partial\Omega^{\varepsilon}\,. \tag{Di$^{\varepsilon}$}$$

The unknown functions $u^{\varepsilon}=u^{\varepsilon}(t,x)=(u_1^{\varepsilon}(t,x),u_2^{\varepsilon}(t,x))^{\top}$ and $p^{\varepsilon}=p^{\varepsilon}(t,x)$ are respectively the velocity field and the pressure field of the fluid. The initial data u_0 is assumed to be given by the zero-extension of some velocity field a on the half-plane $\mathbb{R}^2_+=\{x\in\mathbb{R}^2\mid x_2>0\}$. The boundary function $\omega:\mathbb{R}\to(-1,-\frac{1}{2})$ is assumed to be smooth and 2π-periodic. The parameter $\varepsilon=\frac{1}{N}$, $N\in\mathbb{N}$, characterizes the amplitude and the pulse width (namely the "roughness") of the rough boundary $\partial\Omega^{\varepsilon}$.

A typical approach to describe the averaged effect from such an irregular boundary on the fluid flow is to replace the actual rough boundary by an artificially flat one, but instead, the new boundary condition on this flat boundary is imposed so as to reflect the effect of the roughness of the original boundary. In our setting this process corresponds to consider the Navier-Stokes system in the half-plane $\mathbb{R}^2_+$.

$$\begin{cases} \partial_t u-\Delta u+u\cdot\nabla u+\nabla p=0\,, & t>0\,,\quad x\in\mathbb{R}^2_+\,,\\ \nabla\cdot u=0\,, & t\geq 0\,,\quad x\in\mathbb{R}^2_+\,,\\ u(x_1,x_2)\text{ is }2\pi\text{-periodic in }x_1\,, & t\geq 0\,,\\ u|_{t=0}=a\,, & x\in\mathbb{R}^2_+\,. \end{cases} \tag{NS^{0}}$$

with a new condition on the line $\partial\mathbb{R}^2_+$ which reflects the averaged effect of the rough boundary. An immediate example of the condition is the no-slip boundary condition:

$$u = 0 \quad \text{on } \partial\mathbb{R}^2_+\,, \tag{Di^0}$$

although it does not take the behavior of the flow near the rough boundary into account. The new boundary conditions derived through the above process are called the wall laws, and there is a lot of literature on the formal derivation in various settings. However, the derivation of wall laws often relies on formal computations and it is therefore important to justify the wall laws with a mathematical rigor. For the formal derivations of wall laws and its numerical validations, we refer to Achdou, Pironneau, and Valentin [1].

So far the justification of wall laws is discussed mathematically mainly for the stationary viscous incompressible flows subject to the no-slip boundary condition on the rough boundary. In the pioneering work of Jäger and Mikelić [8], the mathematical justification is given when the two-dimensional stationary channel flows are close to the small Poiseuille flow u^0. This result is extended for random rough boundaries and almost periodic boundaries by Basson and Gérard-Varet [2] and by Gérard-Varet and Masmoudi [6], respectively; see Dalibard and Gérard-Varet [4] for further generalization. In the papers mentioned above, the derivation of the wall law relies on the next formal expansion

$$u^\varepsilon(x) \sim u^0(x) + \partial_2 u^0_1(x_1, 0)\, v_{\rm bl}\Big(\frac{x}{\varepsilon}\Big)\,, \tag{$*$}$$

where $v_{\rm bl}$ is a boundary layer describing the influence from the roughness. The effective wall law in this approach is shown to be the Navier-slip condition (Navier wall law):

$$u_1 = \varepsilon\alpha\partial_2 u_1\,, \qquad u_2 = 0 \quad \text{on } \partial\mathbb{R}^2_+\,, \tag{Na$^\varepsilon$}$$

where the constant α depends only on the boundary function ω. In the periodic boundary case, the Navier wall law for stationary flows is justified in the following sense ([2, 6]); let u^N_ε be the stationary solution of the Navier-Stokes system with the condition (Na$^\varepsilon$). Then it is shown that u^N_ε is an $O(\varepsilon^{\frac{3}{2}})$ approximation of u^ε in the L^2 space.

In the justification of the Navier wall law [8, 2, 6, 4], the structure of the Poiseuille flow is essentially used. In view of applications, it is important to verify the Navier wall law also for the initial boundary value problem in order to show the generality of the method of the wall law. Nevertheless, in the nonstationary case, one naturally needs the high regularity of the ε-zero limit flow to make the formal expansion ($*$) rigorous. In our case the ε-zero limit flow u^0 is characterized as the solution to the Navier-Stokes system with the no-slip boundary condition (NS^0)-(Di^0). However, even if we take a smooth and compactly supported initial data, the solution u^0 of (NS^0)-(Di^0) is not of C^1-class including the initial time $t = 0$ as a space-time function. This regularity loss of the limit flow u^0, arising from the compatibility boundary condition on initial data, provides a central difficulty in the mathematical justification of the Navier wall law. Although recently Mikelić, Nečasová, and Neuss-Radu [10] discusses the Navier wall law for the nonstationary flow with an external force, its argument is based on the assumptions that the initial data is zero, and that the external force is smooth and identically zero near $t = 0$. Thus in [10] the regularity problem of the ε-zero limit flow is essentially avoided by these special assumptions, and it is still in question what condition, particularly for the initial data, is actually enough in order to verify the Navier wall law for (NS$^\varepsilon$)-(Di$^\varepsilon$).

The paper [7] is aimed to obtain a sufficient condition on the initial data, which should be reasonable and checkable, for the justification of the Navier wall law to (NS$^\varepsilon$)-(Di$^\varepsilon$). Before introducing the main theorem of [7], let us introduce some notations. Set $\Omega_p^0 = (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R}_+$ and $\Omega_p^\varepsilon = \{(x_1, x_2) \in (\mathbb{R}/2\pi\mathbb{Z}) \times \mathbb{R} \mid \varepsilon\omega(\frac{x_1}{\varepsilon}) < x_2 < \infty\}$. We denote by $L^2(\Omega_p^0)$ and $H^1(\Omega_p^0)$ the function spaces defined as follows.

$$\begin{aligned} L^2(\Omega_p^0) &= \Big\{v \in L^1_{loc}(\mathbb{R}^2_+) \mid v(x_1, x_2) \text{ is } 2\pi\text{-periodic in } x_1\,, \\ &\qquad \|v\|_{L^2(\Omega_p^0)} = \Big(\int_0^{2\pi}\int_0^\infty |v|^2\, \mathrm{d}x_2\, \mathrm{d}x_1\Big)^{\frac12} < \infty\Big\}\,, \\ H^1(\Omega_p^0) &= \Big\{v \in L^2(\Omega_p^0) \mid \|v\|_{H^1(\Omega_p^0)} = \Big(\int_0^{2\pi}\int_0^\infty \big(|v|^2 + |\nabla v|^2\big)\, \mathrm{d}x_2\, \mathrm{d}x_1\Big)^{\frac12} < \infty\Big\}\,, \\ H^1_0(\Omega_p^0) &= \{v \in H^1(\Omega_p^0);\ \gamma v = 0\}\,, \end{aligned}$$

where γ is the trace operator to the boundary $\partial\mathbb{R}^2_+$. By $L^2_\sigma(\Omega_p^0)$ and $H^1_{0,\sigma}(\Omega_p^0)$ we denote the completion of the $C^\infty_{0,\sigma}(\Omega_p^0)$ in the norm $\|\cdot\|_{L^2(\Omega_p^0)}$ and $\|\cdot\|_{H^1(\Omega_p^0)}$ respectively, where

$$\begin{aligned} C^\infty_{0,\sigma}(\Omega_p^0) = \Big\{v \in C^\infty(\mathbb{R}^2_+)^2 \mid\ & v(x_1, x_2) \text{ is } 2\pi\text{-periodic in } x_1\,, \quad \nabla\cdot v = 0 \text{ in } \mathbb{R}^2_+\,, \\ & v = 0 \text{ in a neighborhood of } \partial\mathbb{R}^2_+\,, \\ & v = 0 \text{ in } x_2 > R \text{ for some } R > 0\Big\}. \end{aligned}$$

The inner product in $L^2(\Omega_p^0)$ is denoted by $\langle u, v\rangle_{\Omega_p^0} = \int_0^{2\pi}\int_0^\infty u\cdot v\, \mathrm{d}x_2\, \mathrm{d}x_1$. We analogically define the function spaces $L^2(\Omega_p^\varepsilon)$, $H^1(\Omega_p^\varepsilon)$, $H^1_0(\Omega_p^\varepsilon)$, $C^\infty_{0,\sigma}(\Omega_p^\varepsilon)$, $L^2_\sigma(\Omega_p^\varepsilon)$, and $H^1_{0,\sigma}(\Omega_p^\varepsilon)$, and the inner product $\langle u, v\rangle_{\Omega_p^\varepsilon}$ in $L^2_\sigma(\Omega_p^\varepsilon)$. Moreover, we denote by $BC^1(\mathbb{R}^2_+)$ the space of bounded continuous functions in $\mathbb{R}^2_+$ having bounded continuous derivatives. In addition, for a function $\varphi : \mathbb{R}^2_+ \to \mathbb{R}$, we denote its zero extension to the domain Ω^ε by $e\varphi$.

Since the problem is two-dimensional, the unique and global solvability of (NS$^\varepsilon$)-(Di$^\varepsilon$) in the L^2 framework is well known; cf. Sohr [12]. For a given data $a \in L^2_\sigma(\Omega_p^0)$ let u^ε be the weak solution of (NS$^\varepsilon$)-(Di$^\varepsilon$) with the initial data $u_0 = ea$, and let u^0 and u^N_ε respectively be the weak solutions of (NS^0)-(Di^0) and (NS^0)-(Na$^\varepsilon$) with the same initial data a. For the regularity of u^ε, u^0, and u^N_ε with $a \in H^1_{0,\sigma}(\Omega_p^0)$, see Propositions 3, 4, and 5 below. The main result of the paper [7] is stated as follows.

Theorem 1. *If $a \in H^1_{0,\sigma}(\Omega_p^0) \cap BC^1(\mathbb{R}^2_+)^2$, then there exists a positive number T independent of $\varepsilon \in (0, e^{-1}]$ such that*

$$\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega_p^0)} \le C_T \varepsilon^{\frac32} |\log\varepsilon|^{\frac12}\,, \qquad 0 \le t \le T\,, \tag{1}$$

where the constant C_T is independent of t and ε, and depends on a and T.

Remark 2. In the setting of [10] we see that the order $O(\varepsilon^{\frac32}|\log\varepsilon|^{\frac12})$ can be improved to $O(\varepsilon^{\frac32})$. However, we need the special assumptions in [10], as is explained above.

We note that the assumption of Theorem 1 is easily checked for a given initial data. Moreover, our proof indicates that the condition $a \in BC^1(\mathbb{R}^2_+)^2$ is optimal to obtain the convergence rate $\varepsilon^{\frac32}$ (with a logarithmic correction) in the topology of $L^\infty(0, T; L^2(\Omega_p^0)^2)$.

The proof of Theorem 1 is carried out with the same spirit as in [10]. Indeed, based on the boundary layer analysis we construct a flow $u^\varepsilon_{\rm app}$ which approximates both u^ε and the Navier-slip solution u^N_ε. The key point is to introduce the boundary layer corrector of the form $\varepsilon v_{\rm bl}(\frac{x}{\varepsilon})$ in the approximation $u^\varepsilon_{\rm app}$, where $v_{\rm bl}$ is the solution to the boundary layer system (BL) in Section 2.2, which is analyzed in [6] in details.

As is already mentioned, the regularity of the ε-zero limit flow u^0, which is the solution to the system (NS^0)-(Di^0), should be investigated carefully. In fact, for the validity of the Navier wall law, we reveal that the next is a sufficient estimate:

$$\sum_{j=0}^{3} t^{\frac{j}{2}} \|\partial_1^j u^0(t)\|_{BC^1(\mathbb{R}^2_+)} + \sum_{k,l=0,1} t^{\frac{k+l+1}{2}} \|\partial_1^k \nabla^l \partial_t u^0(t)\|_{L^\infty(\mathbb{R}^2_+)} \le C_T\,, \quad 0 < t \le T\,, \tag{2}$$

which is naturally expected for the initial data of C^1 class. However, contrary to the proof of [10], the estimate (1) does not follow from a simple use of the Gronwall inequality on $[0,T]$ due to the singularity in the derivatives of the ε-zero limit flow u^0 near $t = 0$. To overcome this difficulty, we divide the time interval as $[0,T] = [0,\tau_\gamma] \cup [\tau_\gamma, T]$ with $\tau_\gamma = \varepsilon^2 |\log\varepsilon|^\gamma$, $\gamma \ge 0$, and derive the estimates of the difference $\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)}$ on each interval, which have different dependences on γ and ε:

$$\sup_{t\in[0,\tau_\gamma]} \|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)} \le C\varepsilon^{\frac{3}{2}} |\log\varepsilon|^{\frac{\gamma}{4}+\frac{1}{4}} e^{C|\log\varepsilon|^{\frac{\gamma}{2}-\frac{1}{2}}}\,,$$

$$\sup_{t\in[\tau_\gamma,T]} \|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)} \le C\varepsilon^{\frac{3}{2}} (|\log\varepsilon|^{\frac{\gamma}{4}+\frac{1}{4}} + |\log\varepsilon|^{1-\frac{\gamma}{2}}) e^{C|\log\varepsilon|^{\frac{\gamma}{2}-\frac{1}{2}}}\,.$$

Then the approximation (2) follows from finding the power γ to optimize the orders of ε in these two bounds, which is obviously $\gamma = 1$.

The important point is that, for the existence of the limit flow u^0 satisfying the estimate (2), there is no additional requirements for the compatibility boundary condition on initial data except for $a = 0$ on $\partial\mathbb{R}^2_+$. The key tool for the proof of the estimate (2) is the derivative estimates of the Stokes semigroup $\{e^{-t\mathbb{A}}\}_{t\ge0}$ in the L^∞ space and we apply the results of Solonnikov [13], Desch, Heiber, and Prüss [5], and Bae and Jin [3]. Moreover, under the condition $f \in BC^1(\mathbb{R}^2_+)^2$, $\nabla\cdot f = 0$ in $\mathbb{R}^2_+$, and $f = 0$ on $\partial\mathbb{R}^2_+$, we will show the next homogeneous estimate; $\|\nabla e^{-t\mathbb{A}} f\|_{L^\infty(\mathbb{R}^2_+)} + t^{\frac{1}{2}} \|\partial_t e^{-t\mathbb{A}} f\|_{L^\infty(\mathbb{R}^2_+)} \le C\|\nabla f\|_{L^\infty(\mathbb{R}^2_+)}$, which seems to have its own interest and is not found in the literature. We note that the L^∞ theory as above is a robust tool in verifying the Navier wall law systematically for the nonstationary problem within the natural compatibility condition.

2 Preliminaries

In this section we summarize the results which are needed in Section 3. In Section 2.1 we recall the results of L^2 regularity theory for the Navier-Stokes system in two dimensions. In Section 2.2 we give some remarks on the slip length of the Navier slip condition. In Section 2.3 we prove the estimate (2) by the L^∞ theory in the half-space.

2.1 L^2 regularity theory

In this section we collect the results for the unique solvability of the two-dimensional Navier-Stokes systems (NS$^\varepsilon$)-(Di$^\varepsilon$), (NS^0)-(Di^0), and (NS^0)-(Na$^\varepsilon$).

Proposition 3. *Let the initial data* $u_0 \in H^1_{0,\sigma}(\Omega^\varepsilon_p)$. *Then there exists a unique weak solution* $(u^\varepsilon, p^\varepsilon)$ *of* (NS$^\varepsilon$)-(Di$^\varepsilon$) *satisfying*

$$u^\varepsilon \in L^\infty(0,\infty\,;\, H^1_{0,\sigma}(\Omega^\varepsilon_p))\,, \qquad \partial_t u^\varepsilon\,, \nabla^2 u^\varepsilon\,, \nabla p^\varepsilon \in L^2(0,\infty\,;\, L^2(\Omega^\varepsilon_p))\,.$$

Proposition 4. *Let the initial data* $a \in H^1_{0,\sigma}(\Omega^0_p)$. *Then there exists a unique weak solution* (u^0, p^0) *of* (NS^0)-(Di^0) *satisfying*

$$u^0 \in L^\infty(0,\infty\,;\, H^1_{0,\sigma}(\Omega^0_p))\,, \qquad \partial_t u^0\,, \nabla^2 u^0\,, \nabla p^0 \in L^2(0,\infty\,;\, L^2(\Omega^0_p))\,.$$

Proposition 5. *Let the initial data* $a \in L^2_\sigma(\Omega^0_p) \cap H^1(\Omega^0_p)$. *Then there exists a unique weak solution* $(u^N_\varepsilon, p^N_\varepsilon)$ *of* (NS^0)-(Na$^\varepsilon$) *satisfying*

$$\nabla u^N_\varepsilon \in L^\infty(0,\infty\,;\, L^2(\Omega^0_p)), \qquad \partial_t u^N_\varepsilon\,, \nabla^2 u^N_\varepsilon\,, \nabla p^N_\varepsilon \in L^2(0,\infty\,;\, L^2(\Omega^0_p))\,.$$

We note that the assumption in Theorem 1 implies $u_0 = ea \in H^1_{0,\sigma}(\Omega^\varepsilon_p)$. For the proofs of Propositions 3 and 4, we refer to [12]. For the unique solvability of (NS^0)-(Na$^\varepsilon$), we refer to Saal [11] when the domain is the half space. Since we are working in the space of periodic L^2-functions, Proposition 5 can be proved in the same manner as [11].

2.2 Slip length of the Navier wall law

We recall the results for the constant α in Navier-slip boundary condition (Na$^\varepsilon$). In [6] it is defined as the (uniform) limit of the boundary layer corrector $v_{\rm bl}$:

$$\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \lim_{y_2\to\infty} v_{\rm bl}(y_1, y_2)\,. \tag{3}$$

Here $v_{\rm bl}$ is the solution to the boundary layer system (BL) below

$$\begin{cases} -\Delta v_{\rm bl} + \nabla q_{\rm bl} = 0\,, \qquad \nabla\cdot v_{\rm bl} = 0\,, \quad y_2 > \omega(y_1)\,, \\ v_{\rm bl}(\omega(y_1), y_2) = (-\omega(y_1), 0)\,, \end{cases} \tag{BL}$$

in the class $\int_0^{2\pi}\int_{\omega(y_1)}^\infty |\nabla v_{\rm bl}(y_1,y_2)|^2\,{\rm d}y_2\,{\rm d}y_1 < \infty$. See [6] for the unique existence of the boundary layer. As is pointed out in [4], one can derive the upper and lower bounds of α. The positivity of α in the next lemma plays a fundamental role in our argument.

Lemma 6. *Let the boundary function* $\omega : \mathbb{R} \to (-1, 0)$ *be smooth and* 2π*-periodic. Then the constant* α *in* (3) *satisfies*

$$0 < \alpha < 1\,.$$

Proof. We refer to Theorem 3.2 in [1] for the proof. □

2.3 Navier-Stokes system for non-decaying data in $\mathbb{R}^n_+$

The purpose of this section is to construct the mild solutions of the systems (NS^0)-(Di^0), which have a sufficient regularity for the justification of the Navier wall law to (NS$^\varepsilon$)-(Di$^\varepsilon$).

We consider the Navier-Stokes system in the n-dimensional half space $\mathbb{R}^n_+$, $n \geq 2$, subject to the no-slip boundary condition:

$$\begin{cases} \partial_t u^0 - \Delta u^0 + u^0 \cdot \nabla u^0 + \nabla p^0 = 0\,, & t > 0\,, \quad x \in \mathbb{R}^n_+\,, \\ \nabla \cdot u^0 = 0\,, & t \geq 0\,, \quad x \in \mathbb{R}^n_+\,, \\ u^0|_{\partial \mathbb{R}^n_+} = 0, & t > 0\,, \\ u^0|_{t=0} = a\,, & x \in \mathbb{R}^n_+\,. \end{cases} \tag{NS}$$

Throughout this section we use the standard notations for n-dimensional differential operators. In the analysis of (NS) the following notations are adopted: for an n-dimensional vector $u = (u_1, ..., u_n)^\top$ in $\mathbb{R}^n_+$, u' denotes its tangential part $u' = (u_1, ..., u_{n-1})^\top$. The tangential derivative $\nabla' = (\partial_1, ..., \partial_{n-1})$ is used in addition. Moreover, we denote by $BC^1(\mathbb{R}^n_+)$ the space of bounded continuous functions in $\mathbb{R}^n_+$ having bounded continuous derivatives. We write $\|\cdot\|_\infty$ and $\|\cdot\|_{BC^1}$ instead of $\|\cdot\|_{L^\infty(\mathbb{R}^n_+)}$ and $\|\cdot\|_{BC^1(\mathbb{R}^n_+)}$ for simplicity.

Firstly we recall the basic L^∞ estimates of the Stokes semigroup for bounded functions in $\mathbb{R}^n_+$. Let $\{e^{-t\mathbb{A}}\}_{t\geq 0}$ denote the Stokes semigroup in $\mathbb{R}^n_+$, and let $\mathbb{P}$ denote the Helmholtz projection. We refer to [5] and [13] for these definitions in the L^∞ setting. Set

$$L^\infty_\sigma(\mathbb{R}^n_+) = \Big\{ f \in L^\infty(\mathbb{R}^n_+)^n \mid \int_{\mathbb{R}^n_+} f \cdot \nabla \varphi \, \mathrm{d}x = 0 \quad \text{for all} \ \varphi \in \widehat{W}^{1,1}(\mathbb{R}^n_+) \Big\}\,,$$

where $\widehat{W}^{1,1}(\mathbb{R}^n_+)$ is the usual homogeneous Sobolev space.

Proposition 7. *For $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ we have*

$$\|e^{-t\mathbb{A}} f\|_\infty + t^{\frac{1}{2}} \|\nabla e^{-t\mathbb{A}} f\|_\infty + t \|\partial_t e^{-t\mathbb{A}} f\|_\infty \leq C \|f\|_\infty\,, \qquad t > 0\,. \tag{4}$$

For $F \in BC^1(\mathbb{R}^n_+)^{n\times n}$ satisfying $F = 0$ on $\partial\mathbb{R}^n_+$, we have

$$\|e^{-t\mathbb{A}} \mathbb{P} \nabla \cdot F\|_\infty \leq C t^{-\frac{1}{2}} \|F\|_\infty\,, \qquad t > 0\,, \tag{5}$$

$$\|\nabla e^{-t\mathbb{A}} \mathbb{P} \nabla \cdot F\|_\infty \leq C t^{-\frac{1}{2}} \|\nabla F\|_\infty\,, \qquad t > 0\,. \tag{6}$$

Here all the constants C above are independent of t.

Proof. For the estimate (4) we refer to [5]. The estimates (5) and (6) are proved in [13] and [3], respectively. □

The homogeneous L^∞ estimates of the Stoke semigroup are provided in the next.

Proposition 8. *For $f \in BC^1(\mathbb{R}^n_+)^n$ satisfying $\nabla \cdot f = 0$ in $\mathbb{R}^n_+$ and $f = 0$ on $\partial\mathbb{R}^n_+$, we have*

$$\|\nabla e^{-t\mathbb{A}} f\|_\infty + t^{\frac{1}{2}} \|\partial_t e^{-t\mathbb{A}} f\|_\infty \leq C \|\nabla f\|_\infty\,, \qquad t > 0\,, \tag{7}$$

where the constant C is independent of t.

Proof. By the definition of the Stokes semigroup, $w(t) = e^{-t\mathbb{A}} f$ solves the Stokes system:

$$\begin{cases} \partial_t w - \Delta w + \nabla r = 0\,, & t > 0\,, \quad x \in \mathbb{R}^n_+\,, \\ \nabla \cdot w = 0\,, & t \geq 0\,, \quad x \in \mathbb{R}^n_+\,, \\ w|_{\partial \mathbb{R}^n_+} = 0\,, & t > 0\,, \\ w|_{t=0} = f\,, & x \in \mathbb{R}^n_+\,, \end{cases} \tag{S}$$

(see [5] for the convergence $\lim_{t\downarrow 0}\|e^{-t\mathbb{A}}f-f\|_\infty=0$ with $f\in BC^1(\mathbb{R}^n_+)^n$). Since $w(t)$ can be interpreted as the solution of the inhomogeneous heat equations, $w(t)$ satisfies

$$w(t) \,=\, e^{t\Delta_D}f-\int_0^t e^{(t-s)\Delta_D}\nabla r(s)\,\mathrm{d}s\,. \tag{8}$$

Here and in the following, we denote by $e^{t\Delta_D}$ (resp. $e^{t\Delta_N}$) the solution operator of the heat equation in $\mathbb{R}^n_+$ with the zero Dirichlet (resp. zero Neumann) boundary condition. To estimate the right-hand side of (8) we express the pressure r in terms of f. Taking the divergence of $(\mathrm{S})_1$, and using the condition $\partial_n w_n=-\nabla'\cdot w'$ in $\mathbb{R}^n_+$, we see that r satisfies the Laplace equation with the inhomogeneous Neumann boundary condition:

$$\begin{cases} \Delta r \,=\, 0\,, & x\in\mathbb{R}^n_+\,,\\ \partial_n r|_{\partial\mathbb{R}^n_+} \,=\, -\nabla'\cdot\gamma\partial_n w'\,, \end{cases}$$

where γ is the trace operator to the boundary $\partial\mathbb{R}^n_+$. Then $\nabla r(t)$ is given by

$$\begin{aligned}\nabla r(t) \,&=\, \nabla(-\Delta')^{-\frac12}e^{-x_n(-\Delta')^{\frac12}}\nabla'\cdot\gamma\partial_n w'(t)\\ &=\, e^{-x_n(-\Delta')^{\frac12}}\begin{pmatrix} S_0\nabla'\cdot\gamma\partial_n w'(t)\\ -\nabla'\cdot\gamma\partial_n w'(t)\end{pmatrix}\,,\end{aligned} \tag{9}$$

where $(-\Delta')^{\frac12}$ is the half Laplacian in $\mathbb{R}^{n-1}$, $\{e^{-x_n(-\Delta')^{\frac12}}\}_{x_n\ge 0}$ is the Poisson semigroup, and S_0 denotes the Riesz operator in $\mathbb{R}^{n-1}$ defined as $S_0=\nabla'(-\Delta')^{-\frac12}$. Then by the Ukai formula (see [14] for the details) we have

$$\gamma\partial_n w'(t) \,=\, \gamma\partial_n e^{t\Delta_D}\big(f'+S_0 f_n\big)\,. \tag{10}$$

Inserting (9) and (10) to (8) we find

$$e^{-t\mathbb{A}}f \,=\, w(t) \,=\, e^{t\Delta_D}f-\int_0^t e^{(t-s)\Delta_D}e^{-x_n(-\Delta')^{\frac12}}\gamma\begin{pmatrix} S_0\nabla'\cdot\partial_n e^{s\Delta_D}\big(f'+S_0f_n\big)\\ -\nabla'\cdot\partial_n e^{s\Delta_D}\big(f'+S_0f_n\big)\end{pmatrix}\mathrm{d}s\,.$$

We only prove the estimate $\|\nabla e^{-t\mathbb{A}}f\|_\infty\le C\|\nabla f\|_\infty$. The estimate $t^{\frac12}\|\partial_t e^{-t\mathbb{A}}f\|_\infty\le C\|\nabla f\|_\infty$ is proved in the same way. The relation $\partial_n e^{s\Delta_D}g=e^{s\Delta_N}\partial_n g$ for $g\in BC^1(\mathbb{R}^n_+)$ with $g=0$ on $\partial\mathbb{R}^n_+$, and the maximum principle $\|e^{-x_n(-\Delta')^{\frac12}}\gamma g\|_\infty\le\|g\|_\infty$ yield

$$\begin{aligned}\|\nabla e^{-t\mathbb{A}}f\|_\infty &\le C\|\nabla f\|_\infty\\ &\quad+C\int_0^t(t-s)^{-\frac12}\big(\|S_0\nabla'\cdot e^{s\Delta_N}\partial_n f'\|_\infty+\|S_0\nabla'\cdot S_0e^{s\Delta_N}\partial_n f_n\|_\infty\big)\,\mathrm{d}s\\ &\quad+C\int_0^t(t-s)^{-\frac12}\big(\|\nabla'\cdot e^{s\Delta_N}\partial_n f'\|_\infty+\|\nabla'\cdot S_0e^{s\Delta_N}\partial_n f_n\|_\infty\big)\,\mathrm{d}s\,.\end{aligned}$$

Then the claim follows from the next estimate

$$\begin{aligned}&\|S_0\nabla'\cdot e^{s\Delta_N}\partial_n f'\|_\infty+\|S_0\nabla'\cdot S_0e^{s\Delta_N}\partial_n f_n\|_\infty\\ &+\|\nabla'\cdot e^{s\Delta_N}\partial_n f'\|_\infty+\|\nabla'\cdot S_0e^{s\Delta_N}\partial_n f_n\|_\infty\le Cs^{-\frac12}\|f\|_\infty\,,\qquad s>0\,.\end{aligned}$$

We can show this estimate by the derivative estimates of the Gauss kernel. The details are omitted; see Proposition 3.2 and Appendix C in [7]. This completes the proof. □

Remark 9. The semigroup property of $\{e^{-t\mathbb{A}}\}_{t\geq 0}$ yields the next estimates. For $f \in L^\infty_\sigma(\mathbb{R}^n_+)$ we have

$$\sum_{j=1}^{2} t^{\frac{j+1}{2}} \|(\nabla')^j \nabla e^{-t\mathbb{A}} f\|_\infty + t^{\frac{3}{2}} \|\nabla' \partial_t e^{-t\mathbb{A}} f\|_\infty \leq C\|f\|_\infty\,, \qquad t > 0\,.$$

For $f \in BC^1(\mathbb{R}^n_+)^n$ satisfying $\nabla \cdot f = 0$ in $\mathbb{R}^n_+$ and $f = 0$ on $\partial\mathbb{R}^n_+$, we have

$$\sum_{j=1}^{3} t^{\frac{j}{2}} \|(\nabla')^j \nabla e^{-t\mathbb{A}} f\|_\infty + \sum_{k=0,1} t^{\frac{k+2}{2}} \|(\nabla')^k \nabla \partial_t e^{-t\mathbb{A}} f\|_\infty \leq C\|\nabla f\|_\infty\,, \qquad t > 0\,.$$

For $F \in BC^1(\mathbb{R}^n_+)^{n\times n}$ satisfying $F = 0$ on $\partial\mathbb{R}^n_+$, we have

$$\begin{aligned}&\sum_{j=1}^{3} t^{\frac{j+1}{2}} \|(\nabla')^j \nabla e^{-t\mathbb{A}} \mathbb{P}\nabla \cdot F\|_\infty \\ &+ \sum_{k,l=0,1} t^{\frac{k+l+2}{2}} \|(\nabla')^k \nabla^l \partial_t e^{-t\mathbb{A}} \mathbb{P}\nabla \cdot F\|_\infty \leq C\|\nabla F\|_\infty\,, \qquad t > 0\,.\end{aligned}$$

Here all the constants C above are independent of t.

Finally we prove the existence of the mild solution to (NS) which has a sufficient regularity for the proof of Theorem 1. The time-local existence for the mild solution of (NS) in the L^∞ space is already proved by [13], Maremonti [9], and [3]. But we revisit this problem here in order to study the derivative estimates of solutions near $t = 0$, under the assumption for the initial data as in Theorem 1. We also note that, under the compatibility condition of Theorem 1, the L^2 solution of (NS^0)-(Di^0) in Proposition 4 satisfies the estimates stated in Theorems 10 with $n = 2$.

Theorem 10. *Let the initial data a satisfy $a \in BC^1(\mathbb{R}^n_+)^n$, $\nabla \cdot a = 0$ in $\mathbb{R}^n_+$, and $a = 0$ on $\partial\mathbb{R}^n_+$. Then there exists a unique mild solution (u^0, p^0) of* (NS) *satisfying the following property; there exists a positive number $T_1 < 1$ such that*

$$\sum_{j=0}^{3} t^{\frac{j}{2}} \|(\nabla')^j u^0(t)\|_{BC^1} + \sum_{k,l=0,1} t^{\frac{k+l+1}{2}} \|(\nabla')^k \nabla^l \partial_t u^0(t)\|_\infty + t^{\frac{1}{2}} \|\nabla p^0(t)\|_\infty \leq C\,, \quad 0 < t < T_1\,,$$

where the constant C depends only on $\|a\|_{BC^1}$.

Proof. The proof is a simple application of the fixed point theorem, using the linear estimates in Remark 9. We omit the details here; see Theorem 3.6 in [7]. □

3 Outlined proof of Theorem 1

In this section we give an outlined proof of Theorem 1. As is explained in the introduction, we divide a time interval as $[0, T] = [0, \varepsilon^2 |\log \varepsilon|] \cup [\varepsilon^2 |\log \varepsilon|, T]$, and derive the estimates of the difference $\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)}$ on each interval. In Section 3.1 we show the short time estimate of the difference. In Section 3.2 the finite time estimate is established.

3.1 Navier wall law near the initial time

The next proposition corresponds to Theorem 1 near the initial time $t \in [0, \varepsilon^2|\log\varepsilon|]$.

Proposition 11. *Let the initial data a satisfy the assumption in Theorem 1. Then we have*

$$\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)} \le C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}\,, \qquad 0 \le t \le \varepsilon^2|\log\varepsilon|\,,$$

where the constant C is independent of t and ε.

Proof. Let u^0 be the solution of (NS^0)-(Di^0) in Proposition 4 satisfying the estimate in Theorem 10. We deduce the estimate of $\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)}$ from the triangle inequality

$$\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega^0_p)} \le \|u^\varepsilon(t) - eu^0(t)\|_{L^2(\Omega^\varepsilon_p)} + \|u^N_\varepsilon(t) - u^0(t)\|_{L^2(\Omega^0_p)}\,, \tag{11}$$

where and in the following (eu^0, ep^0) denotes the zero extension of (u^0, p^0).

Firstly we estimate the term $\|u^\varepsilon(t) - eu^0(t)\|_{L^2(\Omega^\varepsilon_p)}$. From the integration by parts we see that $w^\varepsilon(t) = u^\varepsilon(t) - eu^0(t)$ satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|w^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon_p)} &= -2\langle w^\varepsilon\cdot\nabla w^\varepsilon + w^\varepsilon\cdot\nabla\tilde{u}^0 + \tilde{u}^0\cdot\nabla w^\varepsilon, w^\varepsilon\rangle_{\Omega^\varepsilon_p} - 2\|\nabla w^\varepsilon\|^2_{L^2(\Omega^\varepsilon_p)}\\ &\quad - 2\int_0^{2\pi}\partial_2 u^0(x_1,0)\cdot w^\varepsilon(x_1,0)\,\mathrm{d}x_1 - 2\int_0^{2\pi} p^0(x_1,0)w^\varepsilon_2(x_1,0)_2\,\mathrm{d}x_1.\end{aligned}$$

Noticing $\langle w^\varepsilon\cdot\nabla w^\varepsilon, w^\varepsilon\rangle_{\Omega^\varepsilon_p} = 0$ and $\langle\tilde{u}^0\cdot\nabla w^\varepsilon, w^\varepsilon\rangle_{\Omega^\varepsilon_p} = 0$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|w^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon_p)} + 2\|\nabla w^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon_p)}\\ &= -2\langle w^\varepsilon\cdot\nabla\tilde{u}^0, w^\varepsilon\rangle_{\Omega^\varepsilon_p}\\ &\quad - 2\int_0^{2\pi}\partial_2 u^0(x_1,0)\cdot w^\varepsilon(x_1,0)\,\mathrm{d}x_1 - 2\int_0^{2\pi} p^0(x_1,0)w^\varepsilon_2(x_1,0)\,\mathrm{d}x_1.\end{aligned}$$

Then by a direct computation and the the Gronwall inequality we have

$$\begin{aligned}&\|w^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon_p)} + \int_0^t\|\nabla w^\varepsilon(s)\|^2_{L^2(\Omega^\varepsilon_p)}\,\mathrm{d}s\\ &\le Ce^{C\int_0^t\left(\|\nabla u^0(s)\|_{L^\infty(\mathbb{R}^2_+)} + \varepsilon^{-1}|\log\varepsilon|^{-\frac{1}{2}}\|\nabla p^0(s)\|_{L^\infty(\mathbb{R}^2_+)}\right)\mathrm{d}s}\\ &\quad\times\left(\varepsilon\int_0^t\|\nabla u^0(s)\|^2_{L^\infty(\mathbb{R}^2_+)}\,\mathrm{d}s + \varepsilon^2|\log\varepsilon|^{\frac{1}{2}}\int_0^t\|\nabla p^0(s)\|_{L^\infty(\mathbb{R}^2_+)}\,\mathrm{d}s\right), \quad t \ge 0\,,\end{aligned}$$

where we have used the condition $w^\varepsilon(0) = 0$, and the constant C is independent of t and ε. Then the estimates of (u^0, p^0) in Theorem 10 with $n = 2$ yield the short-time estimate

$$\|u^\varepsilon(t) - eu^0(t)\|_{L^2(\Omega^\varepsilon_p)} \le C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}\,, \quad 0 \le t \le \varepsilon^2|\log\varepsilon|\,. \tag{12}$$

Next we estimate the term $\|u_\varepsilon^N(t)-u^0(t)\|_{L^2(\Omega_p^0)}$ in (11). From the integration by parts we see that $z^\varepsilon(t) = u_\varepsilon^N(t) - u^0(t)$ satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|z^\varepsilon(t)\|_{L^2(\Omega_p^0)}^2 + 2\|\nabla z^\varepsilon(t)\|_{L^2(\Omega_p^0)}^2 \\
&= -2\langle z^\varepsilon\cdot\nabla z^\varepsilon + z^\varepsilon\cdot\nabla u^0 + u^0\cdot\nabla z^\varepsilon, z^\varepsilon\rangle_{\Omega_p^0} - 2\int_0^{2\pi}\partial_2 z_1^\varepsilon(x_1,0)z_1^\varepsilon(x_1,0)\,\mathrm{d}x_1 \\
&= -2\langle z^\varepsilon\cdot\nabla u^0, z^\varepsilon\rangle_{\Omega_p^0} - 2\int_0^{2\pi}\partial_2 z_1^\varepsilon(x_1,0)z_1^\varepsilon(x_1,0)\,\mathrm{d}x_1\,.
\end{aligned}$$

From a direct computation and the the Gronwall inequality again, we obtain

$$\begin{aligned}
&\|z^\varepsilon(t)\|_{L^2(\Omega_p^0)}^2 + 2\int_0^t\|\nabla z^\varepsilon(s)\|_{L^2(\Omega_p^0)}^2\,\mathrm{d}s + \varepsilon\alpha\int_0^t\|\partial_2 u_{\varepsilon,1}^N(\cdot,0)(s)\|_{L^2(0,2\pi)}^2\,\mathrm{d}s \\
&\le Ce^{C\int_0^t\|\nabla u^0(s)\|_{L^\infty(\mathbb{R}_+^2)}\,\mathrm{d}s}\Big(\varepsilon\alpha\int_0^t\|\nabla u^0(s)\|_{L^\infty(\mathbb{R}_+^2)}^2\,\mathrm{d}s\Big) \\
&\le Ce^{Ct}\varepsilon\alpha t\,,\quad 0\le t\le T_1\,,
\end{aligned}$$

where we have used the condition $z^\varepsilon(0) = 0$, and the constant C does not depend on t and ε. Hence we obtain the short-time estimate

$$\|u_\varepsilon^N(t) - u^0(t)\|_{L^2(\Omega_p^0)} \le C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}\,,\quad 0\le t\le\varepsilon^2|\log\varepsilon|\,. \tag{13}$$

Inserting (12) and (13) to (11) we obtain the desired estimate. The proof is completed. □

3.2 Navier wall law in a finite time period

This section is devoted to the outlined proof of Theorem 1. In virtue of Proposition 11, it suffices to show the estimate (1) for $t\in[\varepsilon^2|\log\varepsilon|, T]$ with some finite $T>0$.

We follow the strategy in [10]. Using the boundary layer corrector, we construct a function $u_{\mathrm{app}}^\varepsilon$ which approximates both u^ε and u_ε^N in the domain Ω^ε. In the first step of the construction we define the next approximation function:

$$\tilde u_{\mathrm{app}}^\varepsilon(t,x) = \begin{cases}
\partial_2 u_1^0(t,x_1,0)\Big(\begin{pmatrix} x_2 \\ 0\end{pmatrix} + \varepsilon v_{\mathrm{bl}}(\frac{x}{\varepsilon})\Big) & \\
\quad +\varepsilon\alpha\varphi(\frac{x_2}{\varepsilon})\Big(-U^0(t,x_1,-x_2) + \begin{pmatrix}\partial_2 u_1^0(t,x_1,0)\\ 0\end{pmatrix}\Big)\,, & x_2\le 0\,,\\
u^0(t,x) + \varepsilon\alpha U^0(t,x) & \\
\quad +\varepsilon\partial_2 u_1^0(t,x_1,0)\Big(v_{\mathrm{bl}}(\frac{x}{\varepsilon}) - \begin{pmatrix}\alpha\\ 0\end{pmatrix}\Big)\chi(\frac{x_2}{4\varepsilon|\log\varepsilon|})\,, & x_2>0\,.
\end{cases}$$

Here v_{bl} is the boundary layer corrector introduced in Section 2.2. The new vector field $U^0(t,x) = (U_1^0(t,x), U_2^0(t,x))^\top$ is the (mild) solution to the perturbed Stokes system (PS):

$$\begin{cases}
\partial_t U^0 - \Delta U^0 + u^0\cdot\nabla U^0 + U^0\cdot\nabla u^0 + \nabla P^0 = 0\,, & t>0\,,\quad x\in\mathbb{R}_+^2\,,\\
\nabla\cdot U^0 = 0\,, & t\ge 0\,,\quad x\in\mathbb{R}_+^2\,,\\
U_1^0(x_1,0) = \partial_2 u_1^0(x_1,0)\,,\quad U_2^0(x_1.0) = 0\,, & t>0\,,\\
U^0|_{t=0} = 0\,, & x\in\mathbb{R}_+^2\,.
\end{cases} \tag{PS}$$

In fact, we have the next estimates for (U^0, P^0) (Theorem 3.9 and Lemma 2.6 in [7]); there exists a positive number T_2, which is smaller than T_1 in Theorem 10, such that

$$
\begin{aligned}
&t^{-\frac{1}{4}}\|U^0(t)\|_{L^2(\Omega_p^0)} + \|U^0(t)\|_{L^\infty(\mathbb{R}^2_+)} + \sum_{j=0}^{2} t^{\frac{j+1}{2}}\|\partial_1^j \nabla U^0(t)\|_{L^\infty(\mathbb{R}^2_+)} \\
&\quad + \sum_{k=0,1} t^{\frac{k}{2}+1}\|\partial_1^k \partial_t U^0(t)\|_{L^\infty(\mathbb{R}^2_+)} + t\|\nabla P^0(t)\|_{L^\infty(\mathbb{R}^2_+)} \le C\,, \qquad 0 < t \le T_2\,,
\end{aligned}
\tag{14}
$$

where the constant C depends only on $\|a\|_{H^1(\Omega_p^0)}$ and $\|a\|_{BC^1(\mathbb{R}^2_+)}$. The smooth cut-off functions φ and χ satisfy the next conditions; $\varphi(X) = 0$ if $X \le -\frac{1}{4}$ and $\varphi(X) = 1$ if $X \ge 0$, and $\chi(X) = 1$ if $X \le 2$ and $\chi(X) = 0$ if $X \ge 3$, respectively.

Next we summarize the properties of the approximation function $\tilde{u}^\varepsilon_{\rm app}$. By the choice of the cut-off function φ and χ we have

$$
\lim_{x_2 \uparrow 0} \tilde{u}^\varepsilon_{\rm app}(t,x) = \lim_{x_2 \downarrow 0} \tilde{u}^\varepsilon_{\rm app}(t,x)\,, \quad \lim_{x_2 \uparrow 0} \partial_2 \tilde{u}^\varepsilon_{\rm app}(t,x) = \lim_{x_2 \downarrow 0} \partial_2 \tilde{u}^\varepsilon_{\rm app}(t,x)\,, \qquad t > 0\,,
$$
$$
\tilde{u}^\varepsilon_{\rm app}(t)|_{\partial\Omega^\varepsilon} = 0\,, \qquad t > 0\,.
$$

However, $\tilde{u}^\varepsilon_{\rm app}$ is not divergence free in general. To recover this condition, let us introduce the Bogovskiĭ corrector. Set $D^\varepsilon = \{x \in \mathbb{R}^2 \mid 0 < x_1 < 2\pi,\ \ \varepsilon\omega(\frac{x_1}{\varepsilon}) < x_2 < 12\varepsilon|\log\varepsilon|\}$.

Lemma 12. *There exists $z^\varepsilon = z^\varepsilon(t,x) \in W^{1,p}_0(D^\varepsilon)$ $(1 < p < \infty)$ satisfying*

$$
\nabla \cdot z^\varepsilon(t) = \nabla \cdot \tilde{u}^\varepsilon_{\rm app}(t)\,, \qquad 0 < t \le T_2\,,
$$

and

$$
t^{\frac{1}{2}}\|z^\varepsilon(t)\|_{W^{1,p}(D^\varepsilon)} + t^{\frac{3}{2}}\|\partial_t z^\varepsilon(t)\|_{W^{1,p}(D^\varepsilon)} \le C_{T_2,p}\varepsilon^{1+\frac{1}{p}}, \qquad 0 < t \le T_2\,. \tag{15}
$$

Here the constant $C_{T_2,p}$ is independent of ε and t, and depends on T_2 and p.

Proof. See Appendix B in [7] for the proof. □

Let $Z^\varepsilon = Z^\varepsilon(t,x)$ denote the periodic extension of z^ε. Namely Z^ε is 2π-periodic in x_1 and $Z^\varepsilon(t,x) = z^\varepsilon(t,x)$ if $0 < x_1 < 2\pi$. Finally we set the divergence free approximation

$$
u^\varepsilon_{\rm app}(t) = \tilde{u}^\varepsilon_{\rm app}(t) - Z^\varepsilon(t)\,, \qquad 0 < t \le T_2\,.
$$

We note that $u^\varepsilon_{\rm app}$ satisfies the no-slip condition $u^\varepsilon_{\rm app} = 0$ on $\partial\Omega^\varepsilon$ in the trace sense.

Outlined proof of Theorem 1. As is stated in the beginning of this section, it remains to estimate the difference u^ε and u^N_ε for $t \in [\varepsilon^2|\log\varepsilon|, T]$ with $T = T_2$, where T_2 is the number in (14). To obtain this estimate, we start from the triangle inequality

$$
\|u^\varepsilon(t) - u^N_\varepsilon(t)\|_{L^2(\Omega_p^0)} \le \|u^\varepsilon(t) - u^\varepsilon_{\rm app}(t)\|_{L^2(\Omega_p^\varepsilon)} + \|u^N_\varepsilon(t) - u^\varepsilon_{\rm app}(t)\|_{L^2(\Omega_p^0)}\,. \tag{16}
$$

For the difference $\|u^N_\varepsilon(t) - u^\varepsilon_{\rm app}(t)\|_{L^2(\Omega_p^0)}$, the next bound is an immediate consequence from the construction of $u^\varepsilon_{\rm app}(t)$ (Lemma 2.9 in [7]):

$$
\|u^N_\varepsilon(t) - u^\varepsilon_{\rm app}(t)\|_{L^2(\Omega_p^0)} \le C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}\,, \qquad \varepsilon^2|\log\varepsilon| \le t \le T_2\,,
$$

where the constant C is independent of t and ε. Hence it suffices to estimate $\|u^\varepsilon(t) - u^\varepsilon_{\mathrm{app}}(t)\|_{L^2(\Omega_p^\varepsilon)}$ in the right-hand side of (16). Setting $W^\varepsilon(t) = u^\varepsilon(t) - u^\varepsilon_{\mathrm{app}}(t)$, from the integration by parts we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W^\varepsilon(t)\|^2_{L^2(\Omega_p^\varepsilon)} = \sum_{i=1}^{7} I_i(t)\,,$$

where each I_i is defined below.

$$\begin{aligned}
I_1 &= -2\|\nabla W^\varepsilon\|^2_{L^2(\Omega_p^\varepsilon)} + 2\big\langle -W^\varepsilon\cdot\nabla W^\varepsilon - u^\varepsilon_{\mathrm{app}}\cdot\nabla W^\varepsilon - W^\varepsilon\cdot\nabla u^\varepsilon_{\mathrm{app}}\\
&\quad + \frac{1}{\varepsilon}\partial_2 u_1^0(x_1,0)(\nabla q_{\mathrm{bl}})\big(\frac{x}{\varepsilon}\big)\chi\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big) - \partial_t z^\varepsilon, W^\varepsilon\big\rangle_{\Omega_p^\varepsilon} + 2\langle\nabla z^\varepsilon,\nabla W^\varepsilon\rangle_{\Omega_p^\varepsilon}\,,\\
I_2 &= 2\big\langle \partial_1^2\partial_2 u_1^0(x_1,0)\Big(\begin{pmatrix} x_2\\ 0\end{pmatrix} + \varepsilon v_{\mathrm{bl}}\big(\frac{x}{\varepsilon}\big)\Big) + 2\partial_1\partial_2 u_1^0(x_1,0)(\partial_1 v_{\mathrm{bl}})\big(\frac{x}{\varepsilon}\big)\\
&\quad + \varepsilon\alpha\varphi\big(\frac{x_2}{\varepsilon}\big)\Big(-\partial_1^2 U^0(x_1,-x_2) + \begin{pmatrix}\partial_1^2\partial_2 u_1^0(x_1,0)\\ 0\end{pmatrix}\Big)\\
&\quad + \frac{\alpha}{\varepsilon}\varphi''\big(\frac{x_2}{\varepsilon}\big)\Big(-U^0(x_1,-x_2) + \begin{pmatrix}\partial_2 u_1^0(x_1,0)\\ 0\end{pmatrix}\Big)\\
&\quad + 2\alpha\varphi'\big(\frac{x_2}{\varepsilon}\big)\partial_2 U^0(x_1,-x_2) - \varepsilon\alpha\varphi\big(\frac{x_2}{\varepsilon}\big)\partial_2^2 U^0(x_1,-x_2), W^\varepsilon\big\rangle_{\Omega_p^\varepsilon\setminus\Omega_p^0}\,,\\
I_3 &= 2\big(-\langle u^\varepsilon_{\mathrm{app}}\cdot\nabla u^\varepsilon_{\mathrm{app}}, W^\varepsilon\rangle_{\Omega_p^\varepsilon}\\
&\quad + \big\langle u^0\cdot\nabla u^0 + \varepsilon\alpha(u^0\cdot\nabla U^0 + U^0\cdot\nabla u^0)), W^\varepsilon\big\rangle_{\Omega_p^0}\big)\,,\\
I_4 &= 2\big\langle\big\{\varepsilon\partial_1^2\partial_2 u_1^0(x_1,0)\chi\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big) + \varepsilon(4\varepsilon|\log\varepsilon|)^{-2}\partial_2 u_1^0(x_1,0)\chi''\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big)\\
&\quad - \varepsilon\partial_t\partial_2 u_1^0(x_1,0)\chi\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big)\big\}\Big(v_{\mathrm{bl}}\big(\frac{x}{\varepsilon}\big) - \begin{pmatrix}\alpha\\ 0\end{pmatrix}\Big), W^\varepsilon\big\rangle_{\Omega_p^0}\,,\\
I_5 &= 2\big\langle 2\partial_1\partial_2 u_1^0(x_1,0)\chi\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big)(\partial_1 v_{\mathrm{bl}})\big(\frac{x}{\varepsilon}\big), W^\varepsilon\big\rangle_{\Omega_p^0}\,,\\
I_6 &= 2\Big\langle 2(4\varepsilon|\log\varepsilon|)^{-1}\partial_2 u_1^0(x_1,0)\chi'\big(\frac{x_2}{4\varepsilon|\log\varepsilon|}\big)(\partial_2 v_{\mathrm{bl}})\big(\frac{x}{\varepsilon}\big), W^\varepsilon\Big\rangle_{\Omega_p^0}\,,\\
I_7 &= 2\big\langle\nabla(p^0 + \varepsilon\alpha P^0), W^\varepsilon\big\rangle_{\Omega_p^0}\,.
\end{aligned}$$

In the above $\langle\cdot,\cdot\rangle_{\Omega_p^\varepsilon\setminus\Omega_p^0}$ is the inner product $\langle u,v\rangle_{\Omega_p^\varepsilon\setminus\Omega_p^0} = \int_0^{2\pi}\int_{\varepsilon\omega(x_1/\varepsilon)}^0 u\cdot v\,\mathrm{d}x_2\,\mathrm{d}x_1$, and in the following we use the notation $\|u\|_{L^2(\Omega_p^\varepsilon\setminus\Omega_p^0)} = \langle u,u\rangle^{1/2}_{\Omega_p^\varepsilon\setminus\Omega_p^0}$. We note that the pressure term p^ε in (NS$^\varepsilon$) is eliminated by the identity $\langle\nabla p^\varepsilon, W^\varepsilon\rangle_{\Omega_p^\varepsilon} = 0$, which follows from the $W^\varepsilon|_{\partial\Omega^\varepsilon} = 0$ and $\nabla\cdot W^\varepsilon = 0$ in Ω^ε. Then, applying the estimates of (u^0,p^0) and (U^0,P^0), by the direct calculation and the Gronwall inequality on $[\varepsilon^2|\log\varepsilon|, t]$ we obtain

$$\begin{aligned}
&\|W^\varepsilon(t)\|^2_{L^2(\Omega_p^\varepsilon)} + \int_{\varepsilon^2|\log\varepsilon|}^t \|\nabla W^\varepsilon(s)\|^2_{L^2(\Omega_p^\varepsilon)}\,\mathrm{d}s\\
&\le C\Bigg(\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|^2_{L^2(\Omega_p^\varepsilon)} + \int_{\varepsilon^2|\log\varepsilon|}^{T_2}\varepsilon^{\frac{5}{2}}|\log\varepsilon|s^{-\frac{3}{2}}\|W^\varepsilon(s)\|_{L^2(\Omega_p^\varepsilon)}\,\mathrm{d}s\\
&\quad + \int_{\varepsilon^2|\log\varepsilon|}^{T_2}(\varepsilon^3 + \varepsilon^3 s^{-1} + \varepsilon^5 s^{-2})\,\mathrm{d}s\Bigg), \quad \varepsilon^2|\log\varepsilon| \le t \le T_2\,,
\end{aligned}$$

where C is independent of both t and ε. By the Young inequality we observe

$$\sup_{\varepsilon^2|\log\varepsilon|\leq t\leq T_2} \|W^\varepsilon(t)\|^2_{L^2(\Omega^\varepsilon_p)} \leq C\big(\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|^2_{L^2(\Omega^\varepsilon_p)} + \varepsilon^3|\log\varepsilon|\big)\,.$$

The estimate of the term $\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|^2_{L^2(\Omega^\varepsilon_p)} = \|(u^\varepsilon - u^\varepsilon_{app})(\varepsilon^2|\log\varepsilon|)\|^2_{L^2(\Omega^\varepsilon_p)}$ is as follows. Using the zero extension eu^0 of u^0, which is already introduced in the proof of Proposition 11, we have the next bound of $\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|^2_{L^2(\Omega^\varepsilon_p)}$.

$$\begin{aligned}
\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p)} &\leq \|(u^\varepsilon - eu^0)(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p)} + \|(u^\varepsilon_{\mathrm{app}} - eu^0)(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p)} \\
&\leq \|(u^\varepsilon - eu^0)(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p)} + \|\tilde{u}^\varepsilon_{\mathrm{app}}(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p\setminus\Omega^0_p)} \\
&\quad + \|(\tilde{u}^\varepsilon_{\mathrm{app}} - u^0 - \varepsilon\alpha U^0)(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^0_p)} \\
&\quad + \|\varepsilon\alpha U^0(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^0_p)} + \|z^\varepsilon(\varepsilon^2|\log\varepsilon|)\|_{L^2(D^\varepsilon)}\,.
\end{aligned}$$

By the short time estimate (12) in Proposition 11, and Lemma 12 with the the Poincaré inequality, we have

$$\|W^\varepsilon(\varepsilon^2|\log\varepsilon|)\|_{L^2(\Omega^\varepsilon_p)} \leq C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}.$$

Finally we obtain

$$\|u^\varepsilon(t) - u^\varepsilon_{app}(t)\|_{L^2(\Omega^\varepsilon_p)} \leq C\varepsilon^{\frac{3}{2}}|\log\varepsilon|^{\frac{1}{2}}, \quad \varepsilon^2|\log\varepsilon| \leq t \leq T_2\,.$$

Now we have the estimate (1). This completes the proof of Theorem 1. □

References

[1] Y. Achdou, O. Pironneau, F. Valentin, Effective boundary conditions for laminar flows over periodic rough boundaries, J. Comput. Phys. 147 (1998) 187-218.

[2] A. Basson, D. Gérard-Varet, Wall laws for fluid flows at a boundary with random roughness, Comm. Pure Appl. Math. 61 (7) (July 2008) 941-987.

[3] H.-O. Bae, B. J. Jin, Existence of strong mild solution of the Navier-Stokes equations in the half space with nondecaying initial data, J. Korean Math. Soc. 49 (1) (2012) 113-138.

[4] A.-L. Dalibard, D. Gérard-Varet, Effective boundary condition at a rough surface starting from a slip condition, J. Differential Equations 251 (2011) 3297-3658.

[5] W. Desch, M. Hieber, J. Prüss, L^p-theory of the Stokes equation in a half space, J. Evolution Equations 1 (2001) 115-142.

[6] D. Gérard-Varet, N. Masmoudi, Relevance of the slip condition for fluid flows near an irregular boundary, Comm. Math. Phys. 295 (2010) 99-137.

[7] M. Higaki: Navier wall law for nonstationary viscous incompressible flows, J. Differential Equations 260 (10) (2016) 7358-7396.

[8] W. Jäger, A. Mikelić, On the roughness-induced effective boundary conditions for an incompressible viscous flow, J. Differential Equations 170 (1) (2001) 96-122.

[9] P. Maremonti, Stokes and Navier-Stokes problems in a half space: the existence and uniqueness of solutions a priori nonconvergent to a limit at infinity, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 362 (2008) 176-240; translation in J. Math. Sci. (N. Y.) 159 (4) (2009) 486-523.

[10] A. Mikelić, S. Nečasová, M. Neuss-Radu: Effective slip law for general viscous flows over an oscillating surface, Math. Models Methods Appl. Sci. 36 (15) (2013) 2086-2100.

[11] J. Saal: Existence and regularity of weak solutions for the Navier-Stokes equations with partial slip boundary conditions (English summary), In: RIMS Kôkyûroku Bessatsu, B1, Kyoto, 2007, pp. 331-342.

[12] H. Sohr: The Navier-Stokes Equations. An elementary Functional Analytic Approach, Birkhäuser-Verlag, Basel, 2001.

[13] V. A. Solonnikov, On nonstationary Stokes problem and Navier-Stokes problem in a half-space with initial data nondecreasing at infinity, Problemy Mathematich eskogo Analiza, 25 (2003) 189-210; translation in J. Math. Sci. (N. Y.) 114 (5) (2003) 1726-1740.

[14] S. Ukai, A solution formula for the Stokes equation in $\mathbb{R}^n_+$, Comm. Pure Appl. Math. 40 (1987) 611-621.