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Green-Naghdi and related models for shallow water waves

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Abstract. We develop a systematic procedure for extending the two-dimensional Green-Naghdi (GN) model to include higher-order dispersive effects, and present various model equations for both flat and uneven bottom topographies. We derive the linear dispersion relation for the extended GN models to explore the well-posedness of the linearized problem. Last, we show that these models permit the same Hamiltonian formulation as that of the original GN model.

1. Introduction

The Green-Naghdi (GN) model equation describes the one-dimensional (1D) propagation of the fully nonlinear and weakly dispersive surface gravity waves on fluid of finite depth. A large number of works have been devoted to the studies of the GN equation from both analytical and numerical points of view. See, for example, a review article [1] for the basic properties of the GN equation, and [2] for the recent advances, as well as monographs [3, 4] which detail the derivation and mathematical properties of the GN and other water wave equations.

The GN equation is a system of equations for the total depth of fluid h and the depth-averaged (or mean) horizontal velocity $\bar{u}$. It reads in an appropriate dimensionless form as

$$h_t+\epsilon(h\bar{u})_x=0, \quad \bar{u}_t+\epsilon\bar{u}\bar{u}_x+\eta_x=\frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt}+\epsilon\bar{u}\bar{u}_{xx}-\epsilon\bar{u}_x^2)\right\}_x, \quad (h=1+\epsilon\eta), \tag{1.1}$$

where η is the profile of the free surface and ϵ and δ are the nonlinearity and dispersion parameters, respectively. Unlike the classical Boussinesq system, the GN equation exhibits an exact solitary wave solution

$$h=1+(c^2-1)\,\mathrm{sech}^2\frac{\sqrt{3(c^2-1)}}{2c\delta}\xi, \quad \xi=x-ct, \quad (c>1). \tag{1.2}$$

There are several extensions of the 1D GN model. Recently, we have derived the extended GN models that take into account the arbitrary higher-order dispersive effects, and showed that they have the same Hamiltonian structure as that of the original GN model [5]. In this paper, we report on some results associated with the 2D extension of the GN model. Specifically, we generalize the 1D extended GN system mentioned above to the 2D system by making use of a novel asymptotic analysis, and show that it has the same Hamiltonian structure as that of the original 2D GN system. This 2D extension includes some new results. Among them, a highlight is an analysis of the linear dispersion relation for the extended GN equations which makes it possible to explore the well-posedness of the linearized model equations. The detail of the present work is found in a recent paper [6], and hence we shall summarize the main results.

The governing equation of the water wave problem is given in terms of the dimensionless variables by

$$\delta^2\nabla^2\phi + \phi_{zz} = 0, \quad -1+\beta b < z < \epsilon\eta, \tag{1.3}$$

$$\eta_t + \epsilon\nabla\phi\cdot\nabla\eta = \frac{1}{\delta^2}\phi_z, \quad z = \epsilon\eta, \tag{1.4}$$

$$\phi_t + \frac{\epsilon}{2\delta^2}\left\{\delta^2(\nabla\phi)^2 + \phi_z^2\right\} + \eta = 0, \quad z = \epsilon\eta, \tag{1.5}$$

$$\beta\delta^2\nabla b\cdot\nabla\phi = \phi_z, \quad z = -1+\beta b, \tag{1.6}$$

subjected to the boundary conditions

$$\lim_{|\boldsymbol{x}|\to\infty}\nabla\phi(\boldsymbol{x},z,t) = \boldsymbol{0}, \quad \lim_{|\boldsymbol{x}|\to\infty}\phi_z(\boldsymbol{x},z,t) = 0, \quad -1+\beta b < z < \epsilon\eta, \quad \lim_{|\boldsymbol{x}|\to\infty}\eta(\boldsymbol{x},t) = 0. \tag{1.7}$$

Here, $\phi = \phi(\boldsymbol{x},z,t)$ is the velocity potential with $\boldsymbol{x} = (x,y)$ being a vector in the horizontal plane and z the vertical coordinate pointing upwards, $\nabla = (\partial/\partial x, \partial/\partial y)$ is the 2D gradient operator, $\eta = \eta(\boldsymbol{x},t)$ is the profile of the free surface, $b = b(\boldsymbol{x})$ specifies the bottom topography, the parameter β measures the variation of the bottom topography, and the subscripts z and t appended to ϕ and η denote partial differentiations.

The dimensional quantities, with tildes, are related to the corresponding dimensionless ones by the relations $\tilde{\boldsymbol{x}} = l\boldsymbol{x}, \tilde{z} = h_0 z, \tilde{t} = (l/c_0)t, \tilde{\eta} = a\eta$, $\tilde{\phi} = (gla/c_0)\phi$ and $\tilde{b} = b_0 b$, where l, h_0, a, and b_0 denote a characteristic wavelength, water depth, wave amplitude and bottom profile, respectively. g is the acceleration due to the gravity, and $c_0 = \sqrt{gh_0}$ is the long wave phase velocity. There arise the following three independent dimensionless parameters from the above scalings of the variables:

$$\epsilon = \frac{a}{h_0}, \quad \delta = \frac{h_0}{l}, \quad \beta = \frac{b_0}{h_0}. \tag{1.8}$$

The nonlinearity parameter ϵ characterizes the magnitude of nonlinearity whereas the dispersion parameter δ characterizes the dispersion or shallowness, and the parameter β measures the variation of the bottom topography.

2. Derivation of the extended Green-Naghdi equations

(a) Extended Green-Naghdi system

Let $\bar{\boldsymbol{u}}$ be the depth averaged horizontal velocity defined by

$$\bar{\boldsymbol{u}} = \frac{1}{h}\int_{-1+\beta b}^{\epsilon\eta} \nabla\phi(\boldsymbol{x}, z, t)\mathrm{d}z, \quad h = 1 + \epsilon\eta - \beta b, \tag{2.1a}$$

with its components

$$\bar{u} = \frac{1}{h}\int_{-1+\beta b}^{\epsilon\eta} \phi_x(\boldsymbol{x}, z, t)\mathrm{d}z, \quad \bar{v} = \frac{1}{h}\int_{-1+\beta b}^{\epsilon\eta} \phi_y(\boldsymbol{x}, z, t)\mathrm{d}z. \tag{2.1b}$$

The horizontal component $\boldsymbol{u} = (u, v)$ and verical component w of the surface velocity are given respectively by

$$\boldsymbol{u}(\boldsymbol{x}, t) = \nabla\phi(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}, \tag{2.2a}$$

with its components

$$u(\boldsymbol{x}, t) = \phi_x(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}, \quad v(\boldsymbol{x}, t) = \phi_y(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}, \tag{2.2b}$$

and

$$w(\boldsymbol{x}, t) = \phi_z(\boldsymbol{x}, z, t)|_{z=\epsilon\eta}. \tag{2.3}$$

Let us now derive the equations for h and $\boldsymbol{u}$. First, we multiply (2.1a) by h and then apply the divergence operator to the resultant expression. This leads, after using (1.3) and (1.6), to

$$w = \delta^2\{-\nabla\cdot(h\bar{\boldsymbol{u}}) + \epsilon\boldsymbol{u}\cdot\nabla\eta\}. \tag{2.4}$$

Insersion of w from (2.4) into (1.4) now yields the evolution equation for $h = h(\boldsymbol{x}, t)$:

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0. \tag{2.5}$$

It is important that (2.5) is an *exact* equation without any approximation.

To obtain the equation of $\boldsymbol{u}$, we use the relation which follows from the definition of $\boldsymbol{u}$

$$\nabla\left(\phi_t|_{z=\epsilon\eta}\right) = \boldsymbol{u}_t + \epsilon w_t\nabla\eta - \epsilon\eta_t\nabla w. \tag{2.6}$$

Applying the gradient operator to (1.5) and using (2.6) as well as (2.4) and (2.5), we arrive at the evolution equation for $\boldsymbol{u}$:

$$\boldsymbol{u}_t + \epsilon w_t\nabla\eta + \frac{\epsilon}{2}\nabla\boldsymbol{u}^2 + \epsilon^2(\boldsymbol{u}\cdot\nabla\eta)\nabla w + \nabla\eta = \boldsymbol{0}. \tag{2.7}$$

Now, we introduce the new quantity $\boldsymbol{V}$ by

$$\boldsymbol{V} = \boldsymbol{u} + \epsilon w \nabla \eta. \tag{2.8}$$

To interpret the physical meaning of $\boldsymbol{V}$, we introduce the velocity potential evaluated at the free surface

$$\psi(\boldsymbol{x}, t) = \phi(\boldsymbol{x}, \epsilon\eta, t). \tag{2.9}$$

In view of the definition (2.2) and (2.3) of the surface velocity, the gradient of ψ is found to be

$$\nabla\psi = (\nabla\phi + \epsilon\phi_z \nabla\eta)|_{z=\epsilon\eta} = \boldsymbol{u} + \epsilon w \nabla\eta. \tag{2.10}$$

It immediately follows from (2.8) and (2.10) that

$$\boldsymbol{V} = \nabla\psi, \tag{2.11}$$

implying that $\boldsymbol{V}$ is equal to the 2D gradient of the velocity potential evaluated at the free surface, and it lies in the (x, y) plane. We can rewrite equation (2.7) in terms of $\boldsymbol{V}$, giving

$$\boldsymbol{V}_t + \epsilon\nabla\left(\boldsymbol{u}\cdot\boldsymbol{V} - \frac{1}{2}\boldsymbol{u}^2 - \frac{1}{2\delta^2}w^2 + \frac{\eta}{\epsilon}\right) = \boldsymbol{0}. \tag{2.12}$$

Equation (2.12) represents an exact conservation law for the vector $\boldsymbol{V}$. The system of equations (2.5) and (2.7) (or (2.12)) is a consequence deduced from the basic Euler system (1.3)-(1.6). The extended GN equations are obtained if one can express the variables $\boldsymbol{u}, w$ in equation (2.7) in terms of h and $\bar{\boldsymbol{u}}$. As will be shown below, this is always possible.

Then, we establish

Proposition 1. *The evolution equation for $\bar{\boldsymbol{u}}$ can be recast in the form*

$$\bar{\boldsymbol{u}}_t = \sum_{m=0}^{\infty} \delta^{2m}\boldsymbol{K}_m, \tag{2.13}$$

where $\boldsymbol{K}_m \in \mathbb{R}^2$ are vector functions of h and $\nabla\cdot\bar{\boldsymbol{u}}, \nabla\cdot\bar{\boldsymbol{u}}_t$ as well as the spatial derivatives of these variables.

The evolution equation (2.13) for $\bar{\boldsymbol{u}}$ is an infinite-order Boussinesq-type equation which, coupled with equation (2.5), constitutes the extended GN system. If one truncates the right-hand side of equation (2.13) at order δ^{2n}, then equation (2.13) yields the evolution equation for $\bar{\boldsymbol{u}}$ which is accurate to δ^{2n}

$$\bar{\boldsymbol{u}}_t = \sum_{m=0}^{n} \delta^{2m}\boldsymbol{K}_m. \tag{2.14}$$

We call the system of equations (2.5) and (2.14) for h and $\bar{\mathbf{u}}$ the δ^{2n} model. For the special case $n = 1$, it reduces to the original 2D GN model.

(b) Expressions of the velocities $\boldsymbol{u}, w$ and $\boldsymbol{V}$ in terms of h and $\bar{\boldsymbol{u}}$

(i) Flat bottom topography

First, we solve the Laplace equation (1.3) in the case of a flat bottom topography, and express the surface velocity $\boldsymbol{u}, w$ and the velocity $\boldsymbol{V}$ in terms of the variables h and $\bar{\boldsymbol{u}}$. This enables us to obtain a closed system of equations for the latter variables, i.e., the extended GN system. Under the assumption $\delta^2 \ll 1$ which is relevant to the shallow water models, the solution of equation (1.3) subjected to the boundary condition (1.6) with $b = 0$ can be written explicitly in the form of an infinite series

$$\phi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty} \frac{(-1)^n \delta^{2n}}{(2n)!} (z+1)^{2n} \nabla^{2n} f, \tag{2.15}$$

where $f = f(\boldsymbol{x}, t)$ is the velocity potential at the fluid bottom. We substitute this expression into $(2.1a)$ and perform the integration with respect to z to obtain

$$\bar{\boldsymbol{u}} = \nabla f + \sum_{n=1}^{\infty} \frac{(-1)^n \delta^{2n} h^{2n}}{(2n+1)!} \nabla\nabla^{2n} f, \quad h = 1 + \epsilon\eta. \tag{2.16}$$

Using the formula $\nabla^2 f = \nabla \cdot (\nabla f)$, we can rewrite (2.16) in an alternative form

$$\nabla f = \bar{\boldsymbol{u}} - \sum_{n=1}^{\infty} \frac{(-1)^n \delta^{2n} h^{2n}}{(2n+1)!} \nabla\nabla^{2(n-1)}(\nabla \cdot \nabla f). \tag{2.17}$$

To derive the expansion of ∇f in terms of h and $\bar{\boldsymbol{u}}$, we look for the solution in the form of an infinite series in δ^2

$$\nabla f = \bar{\boldsymbol{u}} + \sum_{n=1}^{\infty} (-1)^n \delta^{2n} \boldsymbol{F}_n, \tag{2.18}$$

where $\boldsymbol{F}_n \in \mathbb{R}^2$ are unknown vector functions to be determined below. Substituting this expression into (2.17) and comparing the coefficients of δ^{2n} $(n = 1, 2, \ldots)$ on both sides, we obtain

$$\boldsymbol{F}_1 = -\frac{h^2}{6} \nabla(\nabla \cdot \bar{\boldsymbol{u}}), \tag{2.19a}$$

$$\boldsymbol{F}_n = -\frac{h^{2n}}{(2n+1)!} \nabla\nabla^{2(n-1)}(\nabla \cdot \bar{\boldsymbol{u}}) - \sum_{r=1}^{n-1} \frac{h^{2r}}{(2r+1)!} \nabla\nabla^{2(r-1)}(\nabla \cdot \boldsymbol{F}_{n-r}), \quad (n \geqslant 2). \tag{2.19b}$$

The recursion relation (2.19b) for $\boldsymbol{F}_n$ can be solved successively with the initial condition (2.19a), the first two of which read

$$\boldsymbol{F}_2 = -\frac{h^4}{120}\nabla\nabla^2(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{h^2}{36}\nabla\nabla\cdot\{h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}, \tag{2.20a}$$

$$\boldsymbol{F}_3 = -\frac{h^6}{5040}\nabla\nabla^4(\nabla\cdot\bar{\boldsymbol{u}}) - \frac{h^2}{6}\nabla(\nabla\cdot\boldsymbol{F}_2) - \frac{h^4}{120}\nabla\nabla^2(\nabla\cdot\boldsymbol{F}_1). \tag{2.20b}$$

The series expansions of $\boldsymbol{u}, w$ and $\boldsymbol{V}$ can be derived simply by substituting (2.18) with $\boldsymbol{F}_n$ from (2.19) and (2.20) into (2.2), (2.3) and (2.11), respectively. We write them up to order δ^4 for later use:

$$\boldsymbol{u} = \bar{\boldsymbol{u}} - \frac{\delta^2}{3}h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \delta^4\left[-\frac{1}{18}h^2\nabla\nabla\cdot\{h^2\nabla(\nabla\cdot\bar{\boldsymbol{u}})\} + \frac{1}{30}h^4\nabla\nabla^2(\nabla\cdot\bar{\boldsymbol{u}})\right] + O(\delta^6), \tag{2.21}$$

$$w = -\delta^2 h\nabla\cdot\bar{\boldsymbol{u}} - \frac{\delta^4}{3}h^2\nabla h\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + O(\delta^6), \tag{2.22}$$

$$\boldsymbol{V} = \bar{\boldsymbol{u}} - \frac{\delta^2}{3h}\nabla(h^3\nabla\cdot\bar{\boldsymbol{u}}) - \frac{\delta^4}{45h}\nabla[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}] + O(\delta^6). \tag{2.23}$$

(ii) Uneven bottom topography

The effect of an uneven bottom topography on the propagation characteristics of water waves is prominent in the coastal zone. Here, we provide the formulas of $\boldsymbol{u}, w$ and $\boldsymbol{V}$ in terms of $h, \bar{\boldsymbol{u}}$ and b. In this case, the solution of the Laplace equation (1.3) subjected to the boundary condition (1.7) can be written in the form

$$\phi(\boldsymbol{x}, z, t) = \sum_{n=0}^{\infty}(z + 1 - \beta b)^n\phi_n(\boldsymbol{x}, t), \tag{2.24}$$

where the orders of unknown functions ϕ_n are to be determined. Substituting (2.24) into equation (1.3), we obtain the recursion relation that determines ϕ_n. For small δ^2, the first three of ϕ_n are found to be

$$\phi_1 = \beta\delta^2(\nabla b\cdot\nabla\phi_0)\{1 - \beta^2\delta^2(\nabla b)^2\} + O(\delta^6), \tag{2.25a}$$

$$\phi_2 = -\frac{\delta^2}{2}\nabla^2\phi_0 + \beta^2\delta^4\left\{\frac{1}{2}(\nabla b)^2\nabla^2\phi_0 + \nabla b\cdot\nabla(\nabla b\cdot\nabla\phi_0) + \frac{1}{2}\nabla^2 b(\nabla b\cdot\nabla\phi_0)\right\} + O(\delta^6), \tag{2.25b}$$

$$\phi_3 = -\frac{\beta\delta^4}{6}\{\nabla^2(\nabla b\cdot\nabla\phi_0) + 2\nabla b\cdot\nabla(\nabla^2\phi_0) + \nabla^2 b\nabla^2\phi_0\} + O(\delta^6). \tag{2.25c}$$

The depth-averaged horizontal velocity $\bar{\boldsymbol{u}}$ and the surface velocity $(\boldsymbol{u}, w)$ are expressed in terms of ϕ_n by introducing (2.24) into (2.1)-(2.3). Explicitly, they read

$$\bar{\boldsymbol{u}} = \sum_{n=0}^{\infty}\frac{h^n}{n+1}\nabla\phi_n - \beta\nabla b\sum_{n=1}^{\infty}h^{n-1}\phi_n, \quad h = 1 + \epsilon\eta - \beta b, \tag{2.26}$$

$$\boldsymbol{u} = \sum_{n=0}^{\infty} h^n \nabla\phi_n - \beta\nabla b \sum_{n=1}^{\infty} nh^{n-1}\phi_n, \tag{2.27}$$

$$w = \sum_{n=1}^{\infty} nh^{n-1}\phi_n. \tag{2.28}$$

Inverting (2.26) and using (2.25), we can express $\nabla\phi_0$ in terms of $\bar{\boldsymbol{u}}, h$ and b. The approximate expression which retains the terms of order δ^2 is given by

$$\nabla\phi_0 = \bar{\boldsymbol{u}} + \delta^2\left[\frac{h^2}{6}\nabla(\nabla\cdot\bar{\boldsymbol{u}}) - \frac{\beta}{2}\{h\nabla(\nabla b\cdot\bar{\boldsymbol{u}}) + (h\nabla\cdot\bar{\boldsymbol{u}})\nabla b\} + \beta^2(\nabla b\cdot\bar{\boldsymbol{u}})\nabla b\right] + O(\delta^4). \tag{2.29}$$

Substitution of (2.25) with (2.29) into (2.27) and (2.28) yields the approximate expressions of $\boldsymbol{u}$ and w

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \delta^2\left[-\frac{h^2}{3}\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{\beta}{2}\{h\nabla(\nabla b\cdot\bar{\boldsymbol{u}}) + (h\nabla\cdot\bar{\boldsymbol{u}})\nabla b\}\right] + O(\delta^4), \tag{2.30}$$

$$w = \delta^2(-h\nabla\cdot\bar{\boldsymbol{u}} + \beta\nabla b\cdot\bar{\boldsymbol{u}}) + O(\delta^4). \tag{2.31}$$

Last, by making use of (2.30) and (2.31), $\boldsymbol{V}$ from (2.8) is shown to have an approximate expression

$$\boldsymbol{V} = \bar{\boldsymbol{u}} + \frac{\delta^2}{h}\left[-\frac{1}{3}\nabla(h^3\nabla\cdot\bar{\boldsymbol{u}}) + \frac{\beta}{2}\{\nabla(h^2\nabla b\cdot\bar{\boldsymbol{u}}) - h^2\nabla b(\nabla\cdot\bar{\boldsymbol{u}})\} + \beta^2 h\nabla b(\nabla b\cdot\bar{\boldsymbol{u}})\right] + O(\delta^4). \tag{2.32}$$

(c) Linear dispersion relation for the extended GN system

Here, we show that the exact linear dispersion relation for the current water wave problem can be derived from the extended GN system, and discuss its structure. We consider the flat bottom case for simplicity. Linearization of equations (2.5) and (2.7) about the uniform state $h = 1$ and $\bar{\boldsymbol{u}} = \boldsymbol{0}$ yields the system of linear equations for η and $\bar{\boldsymbol{u}}$

$$\eta_t + \nabla\cdot\bar{\boldsymbol{u}} = 0, \quad \boldsymbol{u}_t + \nabla\eta = \boldsymbol{0}. \tag{2.33}$$

We eliminate the variable η from the system of equations (2.33) and obtain the linear wave equation for $\bar{\boldsymbol{u}}$

$$\boldsymbol{u}_{tt} - \nabla(\nabla\cdot\bar{\boldsymbol{u}}) = \boldsymbol{0}. \tag{2.34}$$

Recall that the variable $\boldsymbol{u}$ is a linear function of $\bar{\boldsymbol{u}}$ and its spatial derivatives. It follows from (2.2a) and (2.15) with $h = 1$ that

$$\boldsymbol{u} = \nabla f + \sum_{n=1}^{\infty} \frac{(-1)^n\delta^{2n}}{(2n)!}\nabla\nabla^{2(n-1)}(\nabla\cdot\nabla f), \tag{2.35}$$

where ∇f is given by (2.18) with F_n being given by

$$\boldsymbol{F}_n = \alpha_n \nabla\nabla^{2(n-1)}(\nabla\cdot\bar{\boldsymbol{u}}), \quad n \geqslant 1. \tag{2.36}$$

Here, the coefficients α_n are determined by the recuirsion relation

$$\alpha_1 = -\frac{1}{6}, \quad \alpha_n = -\frac{1}{(2n+1)!} - \sum_{r=1}^{n-1} \frac{\alpha_{n-r}}{(2r+1)!}, \quad n \geqslant 2. \tag{2.37}$$

Inserting (2.18) with (2.36) into (2.35), we obtain the expression of $\boldsymbol{u}$ in terms of $\bar{\boldsymbol{u}}$

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \sum_{n=1}^{\infty}(-1)^n \delta^{2n}\left\{\frac{1}{(2n)!} + \sum_{r=0}^{n-1}\frac{\alpha_{n-r}}{(2r)!}\right\}\nabla\nabla^{2(n-1)}(\nabla\cdot\bar{\boldsymbol{u}}). \tag{2.38}$$

The linear dispersion relation for the extended GN system can be derived from (2.34) and (2.38). It reads

$$\omega^2 = \frac{k^2}{D(k\delta)}, \quad (k = |\boldsymbol{k}|), \quad D(k\delta) = 1 + \sum_{n=1}^{\infty}(k\delta)^{2n}\left\{\frac{1}{(2n)!} + \sum_{r=0}^{n-1}\frac{\alpha_{n-r}}{(2r)!}\right\}. \tag{2.39}$$

The explicit form of D follows by using (2.37), giving $D(k\delta) = k\delta \coth k\delta$. Thus, we obtain

$$\omega^2 = \frac{k}{\delta}\tanh k\delta, \tag{2.40}$$

which coincides perfectly with that derived from the linearized system of equations for the current water wave problem (1.3)-(1.7).

In accordance with the above result, we provide the following proposition.

Proposition 2. *The linear dispersion relation of the δ^{2n} model (i.e., the system of equations (2.5) and (2.14)) is given by*

$$\omega^2 = \frac{k^2}{D_{2n}(k\delta)}, \tag{2.41a}$$

with

$$D_{2n}(\kappa) = 1 + \sum_{r=1}^{n}\frac{(-1)^{r-1}2^{2r}}{(2r)!}B_r\kappa^{2r}, \quad B_r = \frac{2(2r)!}{(2\pi)^{2r}}\sum_{j=1}^{\infty}\frac{1}{j^{2r}}, \quad r \geqslant 1, \tag{2.41b}$$

where B_r are Bernoulli's numbers.

Using the inequality for the Bernoulli numbers, we can show that D_{2n} with odd n are positive for all $k\delta$ whereas D_{2n} with even n exhibit single positive zero. It turns out that the δ^{2n} models with even n exhibit an unphysical dispersion characteristic which leads to the ill-posedness result for the linearized systems of equations. In accordance with these observations, the δ^{2n} models with odd n may be more tractable as the practical model equations than the δ^{2n} models with even n. This issue deserves further research.

3. Approximate model equations

(a) The δ^4 model

For the purpose of deriving the δ^4 model with a flat bottom topography, we only need the evolution equation for $\bar{\boldsymbol{u}}$ since the equation for h is already at hand, as indicated by equation (2.5). Actually, substituting (2.21)-(2.23) into (2.12) and retaining terms up to order δ^4, we finally arrive at the evolution equation for $\bar{\boldsymbol{u}}$:

$$\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \delta^2 R_1 + \delta^4 R_2 + O(\delta^6), \tag{3.1a}$$

with

$$R_1 = \frac{1}{3h}\nabla\Big[h^3\{\nabla\cdot\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)(\nabla\cdot\bar{\boldsymbol{u}}) - \epsilon(\nabla\cdot\bar{\boldsymbol{u}})^2\}\Big], \tag{3.1b}$$

$$R_2 = \frac{1}{45h}\nabla\Big[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t) + \epsilon h^5(\nabla^2(\nabla\cdot\bar{\boldsymbol{u}}))\bar{\boldsymbol{u}} - 5\epsilon h^5(\nabla\cdot\bar{\boldsymbol{u}})\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \epsilon\nabla h^5\times(\bar{\boldsymbol{u}}\times\nabla(\nabla\cdot\bar{\boldsymbol{u}}))\}$$
$$-2\epsilon h^5\{\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}^2\Big] - \frac{\epsilon}{45h}\Big[\nabla\cdot\{h^5\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}\nabla(\nabla\cdot\bar{\boldsymbol{u}}) + \frac{h^5}{2}\nabla\{\nabla(\nabla\cdot\bar{\boldsymbol{u}})\}^2\Big]. \tag{3.1c}$$

The linear dispersion relation for the δ^4 model is given by

$$\omega^2 = \frac{k^2}{1 + \frac{1}{3}(k\delta)^2 - \frac{1}{45}(k\delta)^4}, \quad k = |\boldsymbol{k}|. \tag{3.2}$$

The property of (3.2) has been discussed in the 1D case. See Matsuno [5]. Note that ω from (3.2) exhibits a singularity at $k\delta \simeq 4.19$.

Various reductions are possible for the δ^4 model. Indeed, if we neglect the δ^4 terms in equation (3.1), then it reduces to the 2D GN system when coupled with equation (2.5)

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \tag{3.3a}$$

$$\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\delta^2}{3h}\nabla\left[h^3\{\nabla\cdot\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)(\nabla\cdot\bar{\boldsymbol{u}}) - \epsilon(\nabla\cdot\bar{\boldsymbol{u}})^2\}\right], \tag{3.3b}$$

whereas the δ^4 model reduces to the classical 2D Boussinesq system

$$h_t + \epsilon\nabla\cdot(h\bar{\boldsymbol{u}}) = 0, \tag{3.4a}$$

$$\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\delta^2}{3}\nabla(\nabla\cdot\bar{\boldsymbol{u}}_t), \tag{3.4b}$$

after neglecting the $\epsilon\delta^2$ and higher-order terms. On the other hand, the 1D forms of equations (2.5) and (3.1) become

$$h_t + \epsilon(h\bar{u})_x = 0, \tag{3.5a}$$

$$\bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \epsilon\bar{u}_x^2)\right\}_x$$

$$+\frac{\delta^4}{45h}\left[\left\{h^5(\bar{u}_{xxt} + \epsilon\bar{u}\bar{u}_{xxx} - 5\epsilon\bar{u}_x\bar{u}_{xx})\right\}_x - 3\epsilon h^5\bar{u}_{xx}^2\right]_x + O(\delta^6), \tag{3.5b}$$

which are in agreement with equations (2.5) and (2.21) of Matsuno [5], respectively.

(b) The GN model with an uneven bottom topography

In accordance with the method developed in §2, let us derive the GN model which takes into account an uneven bottom topography. The evolution equation for $\bar{\boldsymbol{u}}$ follows by substituting (2.30)-(2.32) into equation (2.12) and retaining tems of order δ^2. We can write it compactly as

$$\left(1 + \frac{\delta^2}{h}\mathcal{L}(h,b)\right)\bar{\boldsymbol{u}}_t + \epsilon(\bar{\boldsymbol{u}}\cdot\nabla)\bar{\boldsymbol{u}} + \nabla\eta = \frac{\epsilon\delta^2}{3h}\nabla\left[h^3\{(\bar{\boldsymbol{u}}\cdot\nabla)\nabla\cdot\bar{\boldsymbol{u}} - (\nabla\cdot\bar{\boldsymbol{u}})^2\}\right] + \epsilon\delta^2 Q, \tag{3.6a}$$

with

$$Q = -\frac{\beta}{2h}\left[\nabla\{h^2\bar{\boldsymbol{u}}\cdot\nabla(\nabla b\cdot\bar{\boldsymbol{u}})\} - h^2\{\bar{\boldsymbol{u}}\cdot\nabla(\nabla\cdot\bar{\boldsymbol{u}}) - (\nabla\cdot\bar{\boldsymbol{u}})^2\}\nabla b\right] - \beta^2\{(\bar{\boldsymbol{u}}\cdot\nabla)^2 b\}\nabla b, \tag{3.6b}$$

where $\mathcal{L}(h,b)$ is a linear differential operator defined by

$$\mathcal{L}(h,b)\boldsymbol{f} = -\frac{1}{3}\nabla(h^3\nabla\cdot\boldsymbol{f}) + \frac{\beta}{2}\{\nabla(h^2\nabla b\cdot\boldsymbol{f}) - h^2\nabla b(\nabla\cdot\boldsymbol{f})\} + \beta^2 h\nabla b(\nabla b\cdot\boldsymbol{f}), \tag{3.6c}$$

for any vector function $\boldsymbol{f} \in \mathbb{R}^2$.

(c) Remark

The dispersion relation for the δ^4 model exhibits a singularity at $k\delta \simeq 4.19$, and this feature may limit the range of applicability of the model. The simplest extended GN model which avoids this undesirable behavior in higher wavenumber is the 1D δ^6 model with a flat bottom topography. The evolution equation for $\bar{u}$ which extends equation (3.5b) to order δ^6 can now be written in the form

$$\bar{u}_t + \epsilon\bar{u}\bar{u}_x + \eta_x = \frac{\delta^2}{3h}\left\{h^3(\bar{u}_{xt} + \epsilon\bar{u}\bar{u}_{xx} - \epsilon\bar{u}_x^2)\right\}_x$$

$$+\frac{\delta^4}{45h}\left[\left\{h^5(\bar{u}_{xxt} + \epsilon\bar{u}\bar{u}_{xxx} - 5\epsilon\bar{u}_x\bar{u}_{xx})\right\}_x - 3\epsilon h^5\bar{u}_{xx}^2\right]_x$$

$$+\frac{\delta^6}{945h}\Big[\{h^7(2\bar{u}_{xxxxt}+2\epsilon\bar{u}\bar{u}_{xxxxx}-14\epsilon\bar{u}_x\bar{u}_{xxxx}-30\epsilon\bar{u}_{xx}\bar{u}_{xxx})\}_x$$
$$+\{h^6h_x(14\bar{u}_{xxxt}+14\epsilon\bar{u}\bar{u}_{xxxx}-112\epsilon\bar{u}_x\bar{u}_{xxx}+42\epsilon\bar{u}_{xx}^2)\}_x$$
$$+\{h^5(hh_x)_x(7\bar{u}_{xxt}+7\epsilon\bar{u}\bar{u}_{xxx}-63\epsilon\bar{u}_x\bar{u}_{xx})\}_x+\epsilon\{10h^7\bar{u}_{xxx}^2-35h^5(hh_x)_x\bar{u}_{xx}^2\}\Big]_x. \quad (3.7)$$

The linear dispersion relation for the system of equations (3.5a) and (3.7) is then given by

$$\omega^2=\frac{k^2}{1+\frac{1}{3}(k\delta)^2-\frac{1}{45}(k\delta)^4+\frac{2}{945}(k\delta)^6}. \quad (3.8)$$

Obviously, the singularity does not occur in ω for arbitrary values of $k\delta$, as opposed to the δ^4 model. This ensures the well-posedness of the system of linearized equations for the model.

4. Hamiltonian structure

(a) Hamiltonian

In this section, we show that the 2D extended GN system derived in §2 can be formulated as a Hamiltonian form. First, recall that the basic Euler system of equations (1.3)-(1.6) conserves the total energy (or Hamiltonian) H which is the sum of the kinetic energy K and the potential energy U:

$$H=K+U=\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\left[\int_{-1+\beta b}^{\epsilon\eta}\left\{(\nabla\phi)^2+\frac{1}{\delta^2}\phi_z^2\right\}\mathrm{d}z\right]\mathrm{d}\boldsymbol{x}+\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\eta^2\mathrm{d}\boldsymbol{x}. \quad (4.1)$$

The integrand of K is then modified, after using (1.3) and (1.6), as well as the definitions (2.2), (2.3) and (2.9) in the form

$$K=\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}[h\bar{\boldsymbol{u}}\cdot\nabla\psi]\mathrm{d}\boldsymbol{x}. \quad (4.2)$$

Thus, the Hamiltonian can be rewritten in a simple form

$$H=\frac{\epsilon^2}{2}\int_{\mathbb{R}^2}\left[h\bar{\boldsymbol{u}}\cdot\nabla\psi+\frac{1}{\epsilon^2}(h-1+\beta b)^2\right]\mathrm{d}\boldsymbol{x}, \quad (4.3)$$

where we have replaced η by $(h-1+\beta b)/\epsilon$ in the expression of the potential energy. The quantity $\nabla\psi(=\boldsymbol{V})$ expressed in terms of h and $\bar{\boldsymbol{u}}$ is available in the form of a series expansion. See (2.32) for the expression of $\boldsymbol{V}$ up to order δ^2. Inserting this into (4.3), we obtain a series expansion of H in powers of δ^2

$$H=\epsilon^2\sum_{n=0}^{\infty}\delta^{2n}H_n, \quad (4.4a)$$

with the first two of H_n being given by

$$H_0 = \frac{1}{2}\int_{\mathbb{R}^2}\left[h\bar{\boldsymbol{u}}^2 + \frac{1}{\epsilon^2}(h-1+\beta b)^2\right]\mathrm{d}\boldsymbol{x}, \tag{4.4b}$$

$$H_1 = \frac{1}{6}\int_{\mathbb{R}^2}\left[h^3(\nabla\cdot\bar{\boldsymbol{u}})^2 - 3\beta h^2(\nabla b\cdot\bar{\boldsymbol{u}})\nabla\cdot\bar{\boldsymbol{u}} + 3\beta^2 h(\nabla b\cdot\bar{\boldsymbol{u}})^2\right]\mathrm{d}\boldsymbol{x}. \tag{4.4c}$$

(b) Momentum density

The momentum density $\boldsymbol{m}$ is the fundamental quantity in formulating the extended GN system as a Hamiltonian form. It is given by the following relation

$$\epsilon\boldsymbol{m} = \frac{\delta H}{\delta\bar{\boldsymbol{u}}}, \tag{4.5}$$

where the operator $\delta/\delta\bar{\boldsymbol{u}}$ is the variational derivative defined by

$$\frac{\partial}{\partial\epsilon}H(\bar{\boldsymbol{u}}+\epsilon\boldsymbol{w})\big|_{\epsilon=0} = \int_{\mathbb{R}^2}\frac{\delta H}{\delta\bar{\boldsymbol{u}}}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}, \tag{4.6}$$

for arbitrary vector function $\boldsymbol{w}\in\mathbb{R}^2$. As seen from (4.4) and its higher-order analog, the integrand of K is quadratic in $\bar{\boldsymbol{u}}$, and hence K obeys the scaling law

$$K(\epsilon\bar{\boldsymbol{u}}, h, b) = \epsilon^2 K(\bar{\boldsymbol{u}}, h, b). \tag{4.7}$$

Putting $\boldsymbol{w}=\bar{\boldsymbol{u}}$ in (4.6) and noting that the potential energy U is independent of $\bar{\boldsymbol{u}}$, we see that

$$\frac{\partial}{\partial\epsilon}K((1+\epsilon)\bar{\boldsymbol{u}}, h, b)\big|_{\epsilon=0} = \int_{\mathbb{R}^2}\frac{\delta K}{\delta\bar{\boldsymbol{u}}}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^2}\frac{\delta H}{\delta\bar{\boldsymbol{u}}}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}. \tag{4.8}$$

On the other hand, in view of (4.7), $\frac{\partial}{\partial\epsilon}K((1+\epsilon)\bar{\boldsymbol{u}}, h, b))\big|_{\epsilon=0} = 2K(\bar{\boldsymbol{u}}, h, b)$. Hence, (4.8) gives, after introducing $\boldsymbol{m}$ from (4.5), $K = \frac{\epsilon}{2}\int_{\mathbb{R}^2}\boldsymbol{m}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}$, so that H is expressed compactly as

$$H = \frac{1}{2}\int_{\mathbb{R}^2}[\epsilon\,\boldsymbol{m}\cdot\bar{\boldsymbol{u}} + (h-1+\beta b)^2]\,\mathrm{d}\boldsymbol{x}. \tag{4.9}$$

Comparing (4.3) and (4.9), we obtain the key relation which connects the variable $\nabla\psi$ with the momentum density $\boldsymbol{m}$:

$$\boldsymbol{m} = \epsilon h\nabla\psi. \tag{4.10}$$

Note that the kinetic energy obeys the scaling law $K(\epsilon\boldsymbol{m}, h, b) = \epsilon^2 K(\boldsymbol{m}, h, b)$, and hence $K = \frac{1}{2}\int_{\mathbb{R}^2}\delta H/\delta\boldsymbol{m}\cdot\boldsymbol{m}\,\mathrm{d}\boldsymbol{x}$. Comparing this expression with $K = \frac{\epsilon}{2}\int_{\mathbb{R}^2}\boldsymbol{m}\cdot\bar{\boldsymbol{u}}\,\mathrm{d}\boldsymbol{x}$, we obtain the dual relation to (4.5)

$$\epsilon\bar{\boldsymbol{u}} = \frac{\delta H}{\delta\boldsymbol{m}}. \tag{4.11}$$

(c) Evolution equation for the momentum density

To derive the evolution equation for the momentum density $\boldsymbol{m}$, we first note the following formula which provides the variational derivative of H with respect to h:

$$\frac{\delta H}{\delta h} = \epsilon^2 \left(\frac{1}{2}\boldsymbol{u}^2 + \frac{w^2}{2\delta^2} - \boldsymbol{u}\cdot\bar{\boldsymbol{u}} + hw\nabla\cdot\bar{\boldsymbol{u}} - \beta w \nabla b \cdot \bar{\boldsymbol{u}} \right) + h - 1 + \beta b. \tag{4.12}$$

By using (2.5), (2.12) and (4.12), we establish.

Proposition 3. *The evolution equation for the momentum density* $\mathbf{m}$ *can be put into the form of local conservation law*

$$\left(\frac{\boldsymbol{m}}{h}\right)_t + \nabla\left(\frac{\epsilon\bar{\boldsymbol{u}}\cdot\boldsymbol{m}}{h} + \frac{\delta H}{\delta h}\right) = \mathbf{0}. \tag{4.13}$$

The velocity $\bar{\boldsymbol{u}}$ in equation (4.13) can be expressed in terms of h and $\boldsymbol{m}$. Thus, the resulting evolution equation, when coupled with equation (2.5) for h, constitutes a closed system of equations for h and $\boldsymbol{m}$. This system is equivalent to the extended GN system and will be used for establishing the Hamiltonian formulation of the latter system.

(d) Hamiltonian formulation

In this section, we demonstrate that the 2D extended GN system can be formulated as a Hamiltonian system. To this end, we introduce the noncanonical Lie-Poisson bracket between any pair of smooth functional F and G

$$\{F, G\} = -\int_{\mathbb{R}^2} \left[\sum_{i,j=1}^{2} \frac{\delta F}{\delta m_i}(m_j\partial_i + \partial_j m_i)\frac{\delta G}{\delta m_j} + h\frac{\delta F}{\delta \boldsymbol{m}}\cdot\nabla\frac{\delta G}{\delta h} + \frac{\delta F}{\delta h}\nabla\cdot\left(h\frac{\delta G}{\delta \boldsymbol{m}}\right)\right] \mathrm{d}\boldsymbol{x}, \tag{4.14}$$

where we have put $\boldsymbol{m} = (m_1, m_2)$ and $\partial_1 = \partial/\partial x, \partial_2 = \partial/\partial y$. Note that the partial derivatives ∂_i $(i = 1, 2)$ operate on all terms they multiply to the right. Then, our main result is given by the following theorem.

Theorem 1. *The 2D extended GN system (2.5) and (2.12) (or equivalently, (4.13)) can be written in the form of Hamilton's equations*

$$h_t = \{h, H\}, \tag{4.15a}$$

$$m_{i,t} = \{m_i, H\}, \quad (i = 1, 2). \tag{4.15b}$$

It follows from theorem 1 that the truncated system like the δ^{2n} model (2.5) and (2.14) has the same Hamiltonian structure as that of the original GN model [7].

5. Relation to Zakharov's Hamiltonian formulation

(a) Zakharov's formulation

Zakharov [8] showed that the water wave problem (1.3)-(1.7) permits a canonical Hamiltonian formulation in terms of the canonical variables η and ψ. If we change the variables from (η, ψ) to $(h, \nabla\psi)$, then the equations of motion for the variables h and $\nabla\psi$ are written in the form

$$h_t = -\frac{1}{\epsilon}\nabla\cdot\frac{\delta H}{\delta \nabla\psi}, \quad \nabla\psi_t = -\frac{1}{\epsilon}\nabla\frac{\delta H}{\delta h}, \tag{5.1}$$

where the Hamiltonian H is given by (4.1) rewritten in terms of the variables h and $\nabla\psi$. If we define the Poisson bracket between any pair of smooth functionals F and G by

$$\{F, G\} = -\frac{1}{\epsilon}\int_{\mathbb{R}^2}\left[\frac{\delta F}{\delta h}\left(\nabla\cdot\frac{\delta G}{\delta\nabla\psi}\right) - \left(\nabla\cdot\frac{\delta F}{\delta\nabla\psi}\right)\frac{\delta G}{\delta h}\right]\mathrm{d}\boldsymbol{x}, \tag{5.2}$$

then the system of equations (5.1) can be put into the form of Hamilton's equations

$$h_t = \{h, H\}, \quad \nabla\psi_t = \{\nabla\psi, H\}. \tag{5.3}$$

(b) Transformation of the Zakharov system to the extended GN system

The Zakharov system (5.3) can be transformed to the extended GN system (4.15) by performoming a sequence of dependent variable transformations $(\eta, \psi) \rightarrow (h, \nabla\psi) \rightarrow (h, \mathbf{m})$. In the process, the relation (4.10) plays the centrl role. Consequently, we establish the following theorem.

Theorem 2. *Zakharov's system of equations (5.3) is equivalent to the extended GN system (4.15).*

One can show that the noncanonical Lie-Poisson bracket (4.14) has the the skew-symmetry, and satisfies the Jacobi identity. In particular, the latter follows by applying a sequence of transformations mentioned above to the Jacobi identity for Zakharov's canonical Poisson bracket.

6. Concluding remarks

In this paper, we have developed a systematic procedure for extending the 2D GN model to include higher-order dispersive effects while preserving full nonlinearity of the original GN model, and presented various model equations for both flat and uneven bottom topographies. We have derived the linear dispersion relations for the δ^{2n} model, and examined its properties, revealing that the models with odd n have smooth dispersion relations without any singularities, whereas the models with even n exhibit single positive zero. We have also shown that the δ^{2n} models with $n \geq 2$ have the same Hamiltonian structure as that of the original GN model.

There are a number of interesting problems associated with the extended GN equations that are worthy of further study. In conclusion, we list some of them:

(i) The effect of higher-order dispersion on the wave characteristics in comparison with that predicted by other asymptotic models like Boussinesq equations.

(ii) The identification of physically relevant models among various extended GN equations.

(iii) Numerical computations of the initial value problems as well as solitary and periodic wave solutions.

(iv) The justification of the asymptotic models by means of the rigorous mathematical analysis.

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