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# SMOOTHING DUE TO MIXING IN THE STATIONARY LINEARIZED BOLTZMANN EQUATION

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## 1. INTRODUCTION

We consider the stationary linearized Boltzmann equation

$$\zeta\cdot\nabla f(x,\zeta)=L(f), \tag{1.1}$$

where $\zeta\in\mathbb{R}^{3}$ and $x\in\Omega$, a $C^{1}$ bounded convex domain in $\mathbb{R}^{3}$. The linear collision operator $L$ here is corresponding to the Grad's cutoff hard potential gases, hard sphere model, or cutoff Maxwellian gases, which are indicated by $0\leq\gamma\leq 1$. Furthermore, we assume the cross section is a product of a function of the length of relative velocity and a function of the deflecting angle. Under this assumption, $L$ has the following know properties (See [1, 4, 6]). $L$ can be decomposed into a multiply operator and integral operator.

$$L(f)=-\nu(|\zeta|)f+K(f), \tag{1.2}$$

where $K(f)(x,\zeta)=\int_{\mathbb{R}^{3}}k(\zeta,\zeta_{*})f(x,\zeta_{*})d\zeta_{*}$ is symmetric. The precise expression of the collision frequency is

$$\nu(|\zeta|)=\beta_{0}\int_{\mathbb{R}^{3}}e^{-|\eta|^{2}}|\eta-\zeta|^{\gamma}d\eta, \tag{1.3}$$

where $\beta_{0}$ is a constant that comes from to angular part of the cross section. Let $0<\delta$. The collision frequency $\nu(|\zeta|)$ and the collision kernel $k(\zeta,\zeta_{*})$ statisfies

(1.4)

$$\nu_{0}(1+|\zeta|)^{\gamma}\leq\nu(|\zeta|)\leq\nu_{1}(1+|\zeta|)^{\gamma},$$

(1.5)

$$|k(\zeta,\zeta_{*})|\leq C_{1}|\zeta-\zeta_{*}|^{-1}(1+|\zeta|+|\zeta_{*}|)^{-(1-\gamma)}e^{-\frac{1-\delta}{4}\left(|\zeta-\zeta_{*}|^{2}+(\frac{|\zeta|^{2}-|\zeta_{*}|^{2}}{|\zeta-\zeta_{*}|})^{2}\right)},$$

(1.6)

$$|\nabla_{\zeta}k(\zeta,\zeta_{*})|\leq C_{2}\frac{1+|\zeta|}{|\zeta-\zeta_{*}|^{2}}(1+|\zeta|+|\zeta_{*}|)^{-(1-\gamma)}e^{-\frac{1-\delta}{4}\left(|\zeta-\zeta_{*}|^{2}+(\frac{|\zeta|^{2}-|\zeta_{*}|^{2}}{|\zeta-\zeta_{*}|})^{2}\right)}.$$

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Here, the constants $0 < \nu_0 < \nu_1$ may depend on the potential and $C_1$, and $C_2$ may depend on $\delta$ and the potential. Suppose $x$ is a point inside $\Omega$. We define $p(x,\zeta)$ to be the boundary point that the backward trajectory from $x$ with velocity $\zeta$ touches. The corresponding traveling time is denoted by $\tau_-(x,\zeta)$. We write

$$\zeta \cdot \nabla f(x,\zeta) + \nu(|\zeta|) f(x,\zeta) = K(f). \tag{1.7}$$

The corresponding integral equation is as follows:

(1.8)
$$f(x,\zeta) = f(p(x,\zeta),\zeta) e^{-\nu(|\zeta|)\tau_-(x,\zeta)} + \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} K(f)(x-\zeta s,\zeta)ds.$$

In this paper, we say $f$ is a solution to (1.1) if the integral equation above is satisfied almost everywhere.

We iterate the integral equation once more and obtain

(1.9)
$$\begin{aligned}
&f(x,\zeta) = f(p(x,\zeta),\zeta) e^{-\nu(|\zeta|)\tau_-(x,\zeta)} \\
&+ \int_0^{\tau_-(x,\zeta)} \int_{\mathbb{R}^3} e^{-\nu(|\zeta|)s} k(\zeta,\zeta') e^{-\nu(|\zeta'|)\tau_-(x-\zeta s,\zeta')} f\big(p(x-\zeta s,\zeta'),\zeta'\big) d\zeta' ds \\
&+ \int_0^{\tau_-(x,\zeta)} \int_{\mathbb{R}^3} \int_0^{\tau_-(x-\zeta s,\zeta')} e^{-\nu(|\zeta|)s} k(\zeta,\zeta') e^{-\nu(|\zeta'|)t} K(f)(x-\zeta s-\zeta' t,\zeta') dt d\zeta' ds \\
&=: I(x,\zeta) + II(x,\zeta) + F(x,\zeta).
\end{aligned}$$

The boundary of $\Omega$ is denoted by $\partial\Omega$, and the outer normal is denoted by $\overrightarrow{n}$. We define

$$\Gamma_- := \{(x,\zeta) | x \in \partial\Omega, \zeta \cdot \overrightarrow{n}(x) < 0\}. \tag{1.10}$$

We consider norms as follows:

$$\|g(\zeta)\|_{L^*_\zeta} := \left( \int_{\mathbb{R}^3} \nu(|\zeta|) |g(\zeta)|^2 d\zeta \right)^{\frac{1}{2}}, \tag{1.11}$$

$$\|f(x,\zeta)\|_{L^*_{x,\zeta}} := \left( \int_\Omega \int_{\mathbb{R}^3} \nu(|\zeta|) |f(x,\zeta)|^2 dx d\zeta \right)^{\frac{1}{2}}, \tag{1.12}$$

$$\|f(x,\zeta)\|_{L^\infty_x L^*_\zeta} := \sup_{x\in\Omega} \left( \int_{\mathbb{R}^3} \nu(|\zeta|) |f(x,\zeta)|^2 d\zeta \right)^{\frac{1}{2}}. \tag{1.13}$$

The indexes above denote the corresponding functional spaces.

The main conclusion in this paper is as follows.

**Theorem 1.1.** *Let $\epsilon > 0$ and $\sigma_\epsilon = \frac{1}{3+\epsilon}$. Suppose $f \in L^\infty_x L^*_\zeta$ solves the stationary linearized Boltzmann equation* (1.1) *with cutoff hard potential, hard sphere, or cutoff Maxwellian gases.*

*Then, there exists a constant $C_0$ depending only on $\|f\|_{L^\infty_x L^*_\zeta}$, $\epsilon$, $\Omega$ and the potential such that, for any $x,\ y \in \Omega$ and $\zeta, \xi \in \mathbb{R}^3$,*

$$
\begin{aligned}
&|F(x,\zeta) - F(y,\xi)| \leq \\
&C_0(1+d_0^{-1})^3 \left(|\zeta-\xi|^2 + |x-y|^2\right)^{\frac{\sigma_\epsilon}{2}},
\end{aligned}
\tag{1.14}
$$

*where $d_0$ is the distance of $x,\ y$ to $\partial\Omega$.*

The key observation is that, in $F$, the regularity in velocity can first be increased by the integral part of the collision operator, $K$. Then, the regularity in velocity can be partially transferred to space through combination of transportation and collision. Readers familiar to the time evolutional Boltzmann equations may find an analogy to the Mixture Lemma studied by Liu and Yu, [8, 9], and later extend by Kuo, Liu, and Noh, [7], and Wu, [10]. This kind of idea can be traced back to the celebrated Velocity Averaging Lemma by Golse, Perthame, and Sentis, [5] discussing the regularity of moments. Although it is with the same spirit, in the context of stationary solution, a subtle interplay between velocity and space is needed to carry out the mixture effect, which will be discuss in detail in section 2.

This result is first used to study the regularity of stationary solution to the linearized Boltzmann equation. After that, the smoothing effect is further explored and resulting a better regularity of locally Hölder continuous up to order $\frac{1}{2}-$ in [3]. However, this original argument assumes less regularity of the solution $f$ and might be suitable to the discussion in other applications. Beside the major difference just pointed out, because of the evolutional relation between this paper and [3], there are some similarity in arguments and repeat calculations. In order to strike the balance between to be self-contained and concise, we keep some while refer others to [3].

## 2. Gaining regularity by collision and transportation

We will elaborate the smoothing effect due to the combination of collision and transportation in this section. The connection between velocity and space is a nature of transport equation. To use this to facilitate our analysis, we first change $\zeta'$ to the spherical coordinates so that

$$
\zeta' = (\rho\cos\theta, \rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi). \tag{2.1}
$$

Also, we change traveling time to the traveling distance:

$$
r = \rho t. \tag{2.2}
$$

Let $\hat{\zeta}' = \frac{\zeta'}{|\zeta'|}$. Then,

(2.3)

$$F(x,\zeta) = \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s}$$

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \int_0^{|\overline{xp(x-s\zeta,\zeta')}|} k(\zeta,\zeta') e^{-\frac{\nu(\rho)}{\rho}r} K(f)(x-\zeta s-\hat{\zeta}' r,\zeta')\rho \sin\theta dr d\phi d\theta d\rho ds$$

$$=: \int_0^{\tau_-(x,\zeta)} e^{-\nu(|\zeta|)s} G(x-\zeta s,\zeta) ds.$$

Notice that we can parametrize $\Omega$ by $\theta$, $\phi$, and $r$. Therefore, by regrouping the integrals, we can change the formulation to contain an integral over space. Let $x_0 = x - \zeta s$ and $y = x - \zeta s - \hat{\zeta}' r$. We have

(2.4)

$$G(x_0,\zeta) =$$

$$\int_0^\infty \int_\Omega k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|}) e^{-\nu(\rho)\frac{|x_0-y|}{\rho}} K(f)(y, \frac{(x_0-y)\rho}{|x_0-y|}) \frac{\rho}{|x_0-y|^2} dy d\rho.$$

Now, we will investigate the regularity of $K(f)$ in $\zeta$ first. Then, use the formula above to partially transfer the regularity to space. In [2], the locally Hölder continuity of $K(f)$ in velocity has been shown for $f \in L^2_\zeta$ and for hard sphere gases. Here, we use a similar technique to improve and extend the result.

**Lemma 2.1.** *If $f \in L^*_\zeta$ and $\epsilon > 0$, then*

$$|K(f)(\zeta) - K(f)(\zeta')| \le C_\epsilon |\zeta-\zeta'|^{\frac{1}{2+\epsilon}} \|f\|_{L^*_\zeta}. \tag{2.5}$$

*Proof.* Since $\|K(f)\|_{L^\infty_\zeta} \le C\|f\|_{L^2_\zeta}$, the inequality is trivial if $|\zeta-\zeta'| \ge 1$. We only need to check the case $|\zeta - \zeta'| < 1$.

$$K(f)(\zeta) - K(f)(\zeta') = \int_{\mathbb{R}^3} [k(\zeta,\zeta_*) - k(\zeta',\zeta_*)] f(\zeta_*) d\zeta_*. \tag{2.6}$$

Let $|\zeta-\zeta'| = l$ and $0 < b < 1$ to be determined later. We divide the domain of integration into two, $\mathbb{R}^3 \setminus B(\zeta, 2l^b)$ and $B(\zeta, 2l^b)$. The corresponding integrals are denoted by $H_1$ and $H_2$ respectively. We first deal with $H_1$. Let $\eta(z) = \zeta' + (\zeta-\zeta')z$. We have

$$\begin{aligned} |H_1| &= \left| \int_{\mathbb{R}^3\setminus B(\zeta,2l^b)} \int_0^1 \frac{d}{dz} k(\eta(z),\zeta_*) dz f(\zeta_*) d\zeta_* \right| \\ &\le \int_0^1 \int_{\mathbb{R}^3\setminus B(\zeta,2l^b)} |\nabla_\eta k(\eta,\zeta_*)\cdot(\zeta-\zeta')||f(\zeta_*|d\zeta_* du. \end{aligned} \tag{2.7}$$

Applying estimate (1.6) for $\nabla_\eta k(\eta,\zeta_*)$, we have
(2.8)

$$
\begin{aligned}
|H_1| &\le Cl\int_0^1\int_{\mathbb{R}^3\setminus B(\eta,l^b)}\frac{(1+|\eta|)^\gamma}{|\eta-\zeta_*|^2}e^{-\frac{1}{8}\left(|\eta-\zeta_*|^2+\frac{(|\eta|^2-|\zeta_*|^2)^2}{|\eta-\zeta_*|^2}\right)}|f(\zeta_*)|d\zeta_*du\\
&\le Cl^{1-b\frac{1+\epsilon}{2}}\int_0^1(1+|\eta|)^\gamma\left(\int_{\mathbb{R}^3\setminus B(\eta,l^b)}(\nu(\zeta_*)|f(\zeta_*)|^2d\zeta_*\right)^{\frac{1}{2}}\\
&\quad\cdot\left(\int_{\mathbb{R}^3\setminus B(\eta,l^b)}\frac{1}{|\eta-\zeta_*|^{3-\epsilon}}e^{-\frac{1}{4}\left(|\eta-\zeta_*|^2+\frac{(|\eta|^2-|\zeta_*|^2)^2}{|\eta-\zeta_*|^2}\right)}\frac{1}{(1+|\zeta_*|)^\gamma}|d\zeta_*\right)^{\frac{1}{2}}du\\
&\le C_\epsilon\left(\int_0^1(1+|\eta|)^{-\frac{1-\gamma}{2}}dz\right)l^{1-b\frac{1+\epsilon}{2}}\|f\|_{L^*_\zeta}\\
&\le C_\epsilon l^{1-b\frac{1+\epsilon}{2}}\|f\|_{L^*_\zeta}.
\end{aligned}
$$

Notice that in the inequalities above, we apply an estimate from [1]:

**Proposition 2.2.** *For any $\epsilon, a_1, a_2 > 0$,*

$$
\left|\int_{\mathbb{R}^3}\frac{1}{|\eta-\zeta_*|^{3-\epsilon}}e^{-a_1|\eta-\zeta_*|^2-a_2\frac{(|\eta|^2-|\zeta_*|^2)^2}{|\eta-\zeta_*|^2}}\right|\le C(1+|\eta|)^{-1}, \tag{2.9}
$$

*where $C$ may depend on $\epsilon, a_1,$ and $a_2$.*

On the other hand,

$$
\begin{aligned}
|H_2| &\le \Big(\int_{B(\zeta,2l^b)}|k(\zeta,\zeta_*)|^2+|k(\zeta',\zeta_*)|^2d\zeta_*\Big)^{\frac{1}{2}}\|f\|_{L_*}\\
&\le Cl^{\frac{b}{2}}\|f\|_{L^*_\zeta}.
\end{aligned} \tag{2.10}
$$

We find the optimal $b=\frac{2}{2+\epsilon}$ and conclude the lemma. □

We now return to the regularity in space, which is characterized by the following lemma.

**Lemma 2.3.** *Suppose $f(x,\zeta)\in L^\infty_x L^*_\zeta$ is a solution to* (1.1) *and $x_0$, $x_1\in\Omega$. Then,*

$$
|G(x_0,\zeta)-G(x_1,\zeta)|\le C_\epsilon\|f\|_{L^\infty_x L^*_\zeta}|x_0-x_1|^{\sigma_\epsilon}, \tag{2.11}
$$

*where $\sigma_\epsilon=\frac{1}{3+\epsilon}$.*

For one space dimension, the Hölder continuity of $K(f)$ in space was proved in [2], thanks to the simple geometry, by the combination of transportation and collision, which is also the important ingredient in [4]. For the three space dimension problem we consider, we need Lemma 2.1, (2.4), together with the following observation from [3]:

**Proposition 2.4.** *Let $x_0$, $x_1$, and $y\in\mathbb{R}^3$ and $0<a<1$. We note $|x_0-x_1|=d$ and $\angle x_0yx_1=\theta$. If $|y-x_0|>2d^a$ and $d<1$, then $\theta<\frac{\pi}{4}d^{1-a}$.*

*Proof.* We first find circles passing both $x_0$ and $x_1$ with radius $d^a$. For any $y$ on the larger arcs of these circles, the angle $\theta$ is the same and

$$\sin(\theta)d^a = \frac{d}{2}. \tag{2.12}$$

Therefore, $\theta < \frac{\pi}{4}d^{1-a}$. We name the collection of these arcs $D$. Notice that $\theta$ for those points outside $D$ is smaller then those for points on $D$. By applying triangular inequality, we know that $|y - x_0| \leq 2d^a$ for any $y \in D$. Therefore, we conclude the proposition.

□

Now, we are ready to prove Lemma 2.3

*Proof of Lemma 2.3.* Let $\beta = \frac{1}{2+\epsilon}$. We add and subtract to obtain

(2.13)

$$\begin{aligned}
&|G(x_0,\zeta) - G(x_1,\zeta)| \leq \Big| \int_0^\infty \int_\Omega k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|}) \frac{\rho e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^2} \\
&\cdot \Big[K(f)(y, \frac{(x_0-y)\rho}{|x_0-y|}) - K(f)(y, \frac{(x_1-y)\rho}{|x_1-y|})\Big] dy d\rho \Big| \\
&+ \Big| \int_0^\infty \int_\Omega K(f)(y, \frac{(x_1-y)\rho}{|x_1-y|}) \\
&\cdot \Big[k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|}) \frac{\rho e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^2} - k(\zeta, \frac{(x_1-y)\rho}{|x_1-y|}) \frac{\rho e^{-\nu(\rho)\frac{|x_1-y|}{\rho}}}{|x_1-y|^2}\Big] dy d\rho \Big| \\
&=: G_K + G_O.
\end{aligned}$$

We first deal with $G_K$. We break the domain of integration into two, $\Omega_1 := \Omega \backslash B(x_0, 2d^a)$ and $\Omega_2 := \Omega \cap B(x_0, 2d^a)$ and name the corresponding integrals as $A_1$ and $A_2$ respectively. Applying Proposition 2.4, we have

(2.14)

$$\begin{aligned}
|A_1| &\leq C_\epsilon \|f\|_{L^\infty_x L^*_\zeta} \int_0^\infty \int_{\Omega_1} |k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|})| \frac{\rho e^{-\nu(\rho)\frac{|x_0-y|}{\rho}}}{|x_0-y|^2} |d^{(1-a)\beta} \rho^\beta| dy d\rho \\
&\leq C_\epsilon d^{(1-a)\beta} \|f\|_{L^\infty_x L^*_\zeta} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_{2d^a}^R |k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|})| \rho^{1+\beta} \sin\theta dr d\phi d\theta d\rho \\
&\leq C_\epsilon d^{(1-a)\beta} \|f\|_{L^\infty_x L^*_\zeta} \int_{2d^a}^R \int_{\mathbb{R}^3} |k(\zeta,\zeta')| \frac{1}{|\zeta'|^{1-\beta}} d\zeta' d\rho \\
&\leq C_\epsilon d^{(1-a)\beta} \|f\|_{L^\infty_x L^*_\zeta} R,
\end{aligned}$$

where $R$ is the diameter of $\Omega$.

On the other hand,

(2.15)

$$|A_2| \leq C\|f\|_{L^\infty_x L^2_\zeta} \int_0^\infty \int_{B(x_0,2d^a)} |k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|})| e^{-\nu(\rho)\frac{|x_0-y|}{\rho}} \frac{\rho}{|x_0-y|^2} dy d\rho$$

$$\leq C\|f\|_{L^\infty_x L^2_\zeta} \int_0^\infty \int_0^\pi \int_0^{2\pi} \int_0^{2d^a} |k(\zeta, \frac{(x_0-y)\rho}{|x_0-y|})| e^{-\nu(\rho)\frac{|x_0-y|}{\rho}} \rho \sin\theta dr d\phi d\theta d\rho$$

$$\leq C\|f\|_{L^\infty_x L^2_\zeta} \int_0^{2d^a} \int_{\mathbb{R}^3} |k(\zeta,\zeta')| \frac{1}{|\zeta'|} d\zeta' d\rho \leq C\|f\|_{L^\infty_x L^2_\zeta} d^a.$$

To optimize the estimate, we choose $a = \frac{\beta}{1+\beta} = \frac{1}{3+\epsilon}$, which gives the desired estimate.

Now, we proceed to estimate $G_O$. We dived the domain of integration into two, $\Omega_3 := \Omega \setminus B(x_1, 2d)$ and $\Omega_4 := \Omega \cap B(x_1, 2d)$, and name the corresponding integrals as $A_3$ and $A_4$ respectively.

We first deal with $A_3$. Let $X(u) = x_1 + (x_0 - x_1)u$. We will use (1.5), (1.6) together with

$$|K(f)| \leq C\|f\|_{L^*_\zeta}(1+|\zeta|)^{-\frac{3-\gamma}{2}} \tag{2.16}$$

in Proposition 4.1 in the following estimate:

(2.17)

$|A_3| =$

$$|\int_0^\infty \int_{\Omega_3} \int_0^1 \frac{d}{du}[k(\zeta, \frac{(X(u)-y)\rho}{|X(u)-y|}) \frac{\rho e^{-\nu(\rho)\frac{|X(u)-y|}{\rho}}}{|X(u)-y|^2}] du K(f)(y, \frac{(x_1-y)\rho}{|x_1-y|}) dy d\rho|$$

$$\leq C|x_1 - x_0| \|f\|_{L^\infty_x L^*_\zeta}$$

$$\cdot \left| \int_0^1 \int_0^\infty \int_{\Omega_3} e^{-\frac{1}{8}|\zeta - \frac{(X(u)-y)\rho}{|X(u)-y|}|^2} \frac{(1+\rho)^{-\frac{3}{2}(1-\gamma)}}{|\zeta - \frac{(X(u)-y)\rho}{|X(u)-y|}|^2} \frac{\rho^2}{|X(u)-y|^3} dy d\rho du \right|$$

$$\leq Cd\|f\|_{L^\infty_x L^*_\zeta}$$

$$\cdot \left| \int_0^1 \int_{2d}^R \int_0^\pi \int_0^{2\pi} \int_0^\infty \frac{e^{-\frac{1}{8}|\zeta - \frac{(X(u)-y)\rho}{|X(u)-y|}|^2}}{|\zeta - \frac{(X(u)-y)\rho}{|X(u)-y|}|^2} \frac{\rho^2}{r} \sin\theta d\rho d\phi d\theta dr du \right|$$

$$\leq Cd\|f\|_{L^\infty_x L^*_\zeta} | \int_0^1 \int_{2d}^R \int_{\mathbb{R}^3} e^{-\frac{1}{8}|\zeta-\zeta'|^2} \frac{1}{|\zeta-\zeta'|^2} d\zeta' \frac{1}{r} dr du$$

$$\leq Cd(1 + |\ln d|)\|f\|_{L^\infty_x L^*_\zeta}.$$

Notice that both $x_0$ and $x_1$ belongs to $\Omega_4$. We can obtain

$$|A_4| \leq Cd\|f\|_{L^\infty_x L^*_\zeta}. \tag{2.18}$$

Combining all the estimates above, we conclude the lemma.

∎

In the end of this section, we would like to discuss the regularity of $G$ in velocity. Notice that

$$(2.19) \qquad |\int_0^{\tau_-(x_0,\zeta')} e^{-\nu t} K(f)(x_0 - \zeta' t, \zeta') dt| < \|f\|_{L^\infty_x L^*_\zeta}.$$

Use an approach similar to (2.7) and (2.8) in the proof of Lemma 2.1, we can conclude

**Proposition 2.5.**

$$(2.20) \qquad |G(x_0, \zeta_1) - G(x_0, \zeta_2)| \leq \|f\|_{L^\infty_x L^*_\zeta} |\zeta_1 - \zeta_2|.$$

## 3. Influence from the domain

In this section, we will introduce some properties of a convex domain, which are discussed in [3], and then apply to the proof of regularity of $F$. These properties are all based on the same observation, that is, a line joint an interior point and a boundary point can not be too tangent in a convex domain.

**Proposition 3.1.** *Let $\Omega$ be a bounded $C^1$ convex domain in $\mathbb{R}^3$ and $x$, $y \in \Omega$. Suppose the distance of $x$ and $y$ to $\partial\Omega$ is $d_0 > 0$. Then,*

$$(3.1) \qquad |p(x,\zeta) - p(y,\zeta)| \leq C(1 + d_0^{-1})|x - y|,$$

$$(3.2) \qquad |\tau_-(x,\zeta) - \tau_-(y,\zeta)| \leq C(1 + d_0^{-1}) \frac{|x-y|}{|\zeta|},$$

*for all nonzero $\zeta \in \mathbb{R}^3$.*

Similarly, for two trajectories passing the same point, we have the following proposition.

**Proposition 3.2.** *Suppose $\Omega$ is a bounded $C^1$ convex domain and $\zeta_1, \zeta_2 \in \mathbb{R}^3$ and $x \in \Omega$. Let $P_1 = p(x,\zeta_1)$ and $P_2 = p(x,\zeta_2)$. Let $d_0$ be the distance between $x$ and $\partial\Omega$ and $\theta$ be the angle between $\zeta_1$ and $\zeta_2$. Then,*

$$(3.3) \qquad |P_1 - P_2| \leq C(1 + d_0^{-1})\theta,$$

$$(3.4) \qquad \big||\overline{xP_1}| - |\overline{xP_2}|\big| \leq C(1 + d_0^{-1})\theta.$$

For the detail of the proof, please see [3]. Now we are ready to prove our main theorem.

*Proof of Theorem 1.1.* Since $F$ is bounded the cases for $|x - y| > 1$ or $|\zeta_1 - \zeta_2| > 1$ are trivial. We only consider the case $|x - y| \leq 1$ and $|\zeta_1 - \zeta_2| \leq 1$. Let $\bar{\zeta_1} = \frac{|\zeta_2|}{|\zeta_1|}\zeta_1$ and $\zeta \in \mathbb{R}^3$. We break the estimate into three parts:

$$(3.5) \qquad |F(x,\zeta) - F(y,\zeta)| \leq C(1 + d_0^{-1})^2 \|f\|_{L^\infty_x L^*_\zeta} |x_0 - x_1|^{\sigma_\epsilon},$$

$$(3.6) \qquad |F(x,\bar{\zeta_1}) - F(x,\zeta_2)| \leq C(1 + d_0^{-1})^3 \|f\|_{L^\infty_x L^*_\zeta} |\zeta_1 - \zeta_2|^{\sigma_\epsilon},$$

$$(3.7) \qquad |F(x,\bar{\zeta_1}) - F(x,\zeta_1)| \leq C \|f\|_{L^\infty_x L^*_\zeta} |\zeta_1 - \zeta_2|^{\sigma_\epsilon}.$$

We start with (3.5). We will assume $\tau_-(y,\zeta) \geq \tau_-(x,\zeta)$ and use the notations in the proof of Proposition 3.1.

(3.8)

$$
\begin{aligned}
&|F(x,\zeta) - F(y,\zeta)| \leq \\
&\left|\int_0^{\frac{|x-X_0|}{|\zeta|}} e^{-\nu s}\big[G(x-s\zeta,\zeta) - G(y-s\zeta,\zeta)\big]ds\right| + \left|\int_{\tau_-(x,\zeta)}^{\tau_-(y,\zeta)} e^{-\nu s} G(y-s\zeta,\zeta)ds\right| \\
&\leq C\|f\|_{L^\infty_x L^*_\zeta}|x-y|^{\sigma_\epsilon} + C(1+d_0^{-1})\frac{|x-y|}{|\zeta|}e^{-\nu_0\frac{d_0}{|\zeta|}}\|f\|_{L^\infty_x L^*_\zeta} \\
&\leq C(1+d_0^{-1})^2\|f\|_{L^\infty_x L^*_\zeta}|x_0 - x_1|^{\sigma_\epsilon}.
\end{aligned}
$$

Notice that we used $|x-y| \leq 1$ in the last inequality above.

As for (3.6), we may assume $\tau_-(x,\bar{\zeta_1}) \geq \tau_-(x,\zeta_2)$ without lost of generality. Let $P_2 = p(x,\zeta_2)$. Then,

(3.9)

$$
\begin{aligned}
&|F(x,\bar{\zeta_1}) - F(x,\zeta_2)| \leq \\
&\left|\int_0^{\frac{|\overline{P_2 x}|}{|\zeta_2|}} e^{-\nu s}\big[G(x-s\bar{\zeta_1},\bar{\zeta_1}) - G(x-s\zeta_2,\zeta_2)\big]ds\right| + \left|\int_{\tau_-(x,\zeta_2)}^{\tau_-(x,\bar{\zeta_1})} e^{-\nu s} G(x-s\bar{\zeta_1},\zeta)ds\right| \\
&=: D_d + R_m.
\end{aligned}
$$

The $R_m$ can be estimated by adopting 3.2 and similar argument in the corresponding part in (3.8) above. For $D_d$, we will subtract and add one term and then apply Lemma 2.3 and Proposition 2.5:

$$
\begin{aligned}
D_d \leq &\left|\int_0^{\frac{|\overline{P_2 x}|}{|\zeta_2|}} e^{-\nu s}\big(G(x-s\bar{\zeta_1},\bar{\zeta_1}) - G(x-s\zeta_2,\bar{\zeta_1})\big)ds\right| \\
&+\left|\int_0^{\frac{|\overline{P_2 x}|}{|\zeta_2|}} e^{-\nu s}\big(G(x-s\zeta_2,\bar{\zeta_1}) - G(x-s\zeta_2,\zeta_2)\big)ds\right| \\
\leq &C\|f\|_{L^\infty_x L^*_\zeta}\left|\int_0^{\frac{|\overline{P_2 x}|}{|\zeta_2|}} e^{-\nu s}\big((|\bar{\zeta_1}-\zeta_2|s)^{\sigma_\epsilon} + |\bar{\zeta_1}-\zeta_2|\big)ds\right| \\
\leq &C\|f\|_{L^\infty_x L^*_\zeta}|\zeta_1 - \zeta_2|^{\sigma_\epsilon}.
\end{aligned}
\tag{3.10}
$$

To prove (3.7), we first change the variable from traveling time to traveling distance and let $\hat{\zeta_1} = \frac{\zeta_1}{|\zeta_1|}$, then

$$
\begin{aligned}
&|F(x,\bar{\zeta_1}) - F(x,\zeta_1)| \leq \\
&\left| \int_0^{|\overline{xP_1}|} e^{-\frac{\nu(|\zeta_2|)}{|\zeta_2|}r} G(x - r\hat{\zeta_1}, \bar{\zeta_1}) - e^{-\frac{\nu(|\zeta_1|)}{|\zeta_1|}r} G(x - r\hat{\zeta_1}, \zeta_1) ds \right| \\
&\leq \left| \int_0^{|\overline{xP_1}|} e^{-\frac{\nu(|\zeta_2|)}{|\zeta_2|}r} \left[ G(x - r\hat{\zeta_1}, \bar{\zeta_1}) - G(x - r\hat{\zeta_1}, \zeta_1) \right] ds \right| \\
&\quad + \left| \int_0^{|\overline{xP_1}|} \left[ e^{-\frac{\nu(|\zeta_2|)}{|\zeta_2|}r} - e^{-\frac{\nu(|\zeta_1|)}{|\zeta_1|}r} \right] G(x - r\hat{\zeta_1}, \zeta_1) ds \right| \\
&=: D_L + D_P.
\end{aligned} \tag{3.11}
$$

We can control $D_L$ by applying Proposition 2.5. For $D_P$, we first observe

$$
\begin{aligned}
| \nabla \nu_\gamma(|\zeta|)| &= \left| \beta_0 \int_{\mathbb{R}^3} e^{-|\eta|^2} |\eta - \zeta|^{\gamma-1} \frac{\eta - \zeta}{|\eta - \zeta|} d\eta \right| \\
&\leq \nu_{\gamma-1}(|\zeta|),
\end{aligned} \tag{3.12}
$$

where we use the lower index to indicate different potential. Therefore, we have

$$
\left|\frac{d}{d\rho}\frac{\nu(\rho)}{\rho}\right| \leq C \begin{cases} \frac{1}{\rho^2}, & 0 < \rho < 1, \\ (1+\rho)^{\gamma-2}, & \rho \geq 1. \end{cases} \tag{3.13}
$$

By the Mean Value Theorem, there exist $z$ between $|\zeta_1|$ and $|\zeta_2|$ such that

$$
\begin{aligned}
D_P \leq& C\|f\|_{L^\infty_x L^*_\zeta} \big||\zeta_1| - |\zeta_2|\big| \int_0^{|\overline{xP_1}|} (1 + \frac{1}{z^2}) e^{-\frac{\nu(z)}{z}r} dr \\
\leq& C\|f\|_{L^\infty_x L^*_\zeta} \big||\zeta_1 - \zeta_2|.
\end{aligned} \tag{3.14}
$$

Therefore, we prove the regularity of the $F$. □

## 4. APPENDIX

The following estimate also appears in [3]. For the completeness of this paper, we include its proof in this appendix.

**Proposition 4.1.** *For any* $0 \leq \gamma \leq 1$,

$$
|K(f)| \leq C\|f\|_{L^*_\zeta} (1 + |\zeta|)^{-\frac{3-\gamma}{2}}. \tag{4.1}
$$

*The constant* $C$ *above may depend on* $\gamma$.

*Proof.*

(4.2)

$$
\begin{aligned}
|k(f)(\zeta)| =& \left| \int_{\mathbb{R}^3} k(\zeta,\zeta_*) f(\zeta_*) d\zeta_* \right| \\
=& \left( \int_{\mathbb{R}^3} |k(\zeta,\zeta_*)|^2 \frac{1}{|\nu(\zeta_*)|} d\zeta_* \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nu(\zeta_*)| |f(\zeta_*)|^2 d\zeta_* \right)^{\frac{1}{2}} \\
\leq& \|f\|_{L^*_\zeta} \left( \int_{\mathbb{R}^3} \frac{e^{-\frac{1}{4}\left(|\zeta-\zeta_*|^2 + (\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}}{|\zeta-\zeta_*|^2(1+|\zeta|+|\zeta_*|)^{2(1-\gamma)}(1+|\zeta_*|)^\gamma} d\zeta_* \right)^{\frac{1}{2}} \\
\leq& C\|f\|_{L^*_\zeta} \left( \int_{|\zeta-\zeta_*|\leq\frac{|\zeta|}{2}} \frac{e^{-\frac{1}{4}\left(|\zeta-\zeta_*|^2 + (\frac{|\zeta|^2-|\zeta_*|^2}{|\zeta-\zeta_*|})^2\right)}}{|\zeta-\zeta_*|^2(1+|\zeta|)^{2(1-\gamma)}(1+\frac{|\zeta|}{2})^\gamma} d\zeta_* \right. \\
&+ \left. \int_{|\zeta-\zeta_*>\frac{|\zeta|}{2}} \frac{e^{-\frac{1}{4}|\zeta-\zeta_*|^2}}{|\zeta-\zeta_*|^2(1+|\zeta|)^{2(1-\gamma)}} d\zeta_* \right)^{\frac{1}{2}} \\
\leq& C\|f\|_{L^*_\zeta} \left[ ((1+|\zeta|)^{-(3-\gamma)} + |\zeta|^{-2}(1+|\zeta|)^{-(2-2\gamma)} \right]^{\frac{1}{2}}.
\end{aligned}
$$

Notice that $K(f)$ is also bounded. Therefore, we conclude. □

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