Growth properties for generalized Riesz potentials in central Herz-Morrey spaces

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Abstract

Riesz decomposition theorem says that a superharmonic function on the punctured unit ball $B_0$ is represented as the sum of a generalized potential and a harmonic function outside the origin. Our first aim in this note is to study growth properties near the origin for generalized Riesz potentials of functions in central Herz-Morrey spaces on $B_0$.

We know another Riesz decomposition theorem which says that a superharmonic function on the unit ball $B$ is represented as the sum of another generalized potential and a harmonic function on $B$. Our second aim in this note is to obtain growth properties near the boundary $\partial B$ for generalized Riesz potentials of functions in central Herz-Morrey spaces on $B$.

A continuous function $u$ on an open set $\Omega$ is called monotone in the sense of Lebesgue [18] if for every relatively compact open set $G \subset \Omega$,

$$\max_G u = \max_{\partial G} u \quad \text{and} \quad \min_G u = \min_{\partial G} u.$$ 

Harmonic functions on $\Omega$ are monotone in $\Omega$. More generally, solutions of elliptic partial differential equations of second order and weak solutions for variational problems may be monotone (see [15]). Our final aim in this note is concerned with growth properties for monotone Sobolev functions in central Herz-Morrey spaces.

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Part I
Isolated Singularities

1 Generalized Riesz potentials

Let us consider the Riesz kernel $I_{\alpha}(x) = |x|^{\alpha-n}$ of order $\alpha$ and a generalized kernel

$$I_{\alpha,\ell}(x, y) = I_{\alpha}(x-y) - \sum_{|\lambda| \leq \ell} \frac{(-y)^{\lambda}}{\lambda!} D^{\lambda} I_{\alpha}(x)$$

for an integer $\ell$; when $\ell \leq -1$, $I_{\alpha,\ell}(x, y) = I_{\alpha}(x-y)$. Note here that $I_{\alpha,\ell}(x, y)$ is Taylor’s remainder of $|x+z|^{\alpha-n}$ around $z = 0$.

We define the generalized Riesz potential of order $\alpha$ for a locally integrable function $f$ on the puncture unit ball $B_0$

$$I_{\alpha,\ell}f(x) = \int_{B_0} I_{\alpha,\ell}(x, y) f(y) \, dy.$$  

Here we prepare the estimates for generalized Riesz kernels.

**Lemma 1.1** (cf. [12, Lemma 3.2]). Let $\ell \geq 0$.

1. $|I_{\alpha,\ell}(x, y)| \leq C|x-y|^{\alpha-n}$ when $|x|/2 < |y| < 2|x|$.
2. $|I_{\alpha,\ell}(x, y)| \leq C|y|^{|\ell+1|}|x|^{\alpha-n-\ell-1}$ when $|y| < |x|/2$.
3. $|I_{\alpha,\ell}(x, y)| \leq C|y|^{|\ell|}|x|^{\alpha-n-\ell}$ when $2|x| < |y|$.

**Remark 1.2.** If $u$ is a superharmonic function on $B$, then $u$ is represented as the sum of a potential and a harmonic function (see e.g. [2], [3], [14], [23]).

Let $u$ be a superharmonic function on $B_0$. In view of Theorems 1.3 and 3.4 in [12], if $r^a S(|u|, r)$ is bounded in $(0, 1)$ for some $a > n - 2$, then

$$\sup_{0 < r < 1/2} r^{a+2-n} \mu(A(0, r)) < \infty \quad (A(0, r) = B(0, 2r) \setminus B(0, r))$$

and $u$ is represented as

$$u(x) = I_{2,\ell} \mu(x) + \text{ a harmonic function}$$

near the origin (except at the origin), where $(2-n+a)-1 < \ell \leq 2-n+a$ and $\mu = c(-\Delta)u$ is the Riesz measure; see also [8], [10], [11].
2 Central Herz-Morrey spaces

For \(1 \leq p < \infty\) and a real number \(\nu\), we consider the family \(M^{p,q,\nu}(B)\) of all measurable functions \(f\) on \(B\) satisfying

\[
\|f\|_{M^{p,q,\nu}(B)} = \left( \int_0^1 \left( r^\nu \|f\|_{L^p(A(0,r))} \right)^q \frac{dr}{r} \right)^{1/q} < \infty
\]

when \(0 < q < \infty\) and

\[
\|f\|_{M^{p,\infty,\nu}(B)} = \sup_{0<r<1} r^\nu \|f\|_{L^p(A(0,r))} < \infty;
\]

set \(f = 0\) outside \(B\) as before; see e.g. [4], [5], [16].

If \(0 < q_1 < q_2 < \infty\), then

\[M^{p,q_1,\nu}(B) \subset M^{p,q_2,\nu}(B) \subset M^{p,\infty,\nu}(B)\]

Our space is somewhat a family of functions with finite weighted mixed norm

\[
\|f\|_{p,q,\omega} = \left( \int \left( \int |f(x,y)|^p \omega(x,y) dy \right)^{q/p} dx \right)^{1/q} < \infty.
\]

3 Sobolev’s inequality

Let \(p^t\) be the Sobolev exponent of \(p > 1\):

\[
1/p^t = 1/p - \alpha/n > 0.
\]

**Lemma 3.1** (Sobolev’s inequality (cf. [1], [23])). There is a constant \(C > 0\) such that

\[
\|I_\alpha f\|_{L^{p^t}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]

Sobolev’s inequality is extended to generalize Riesz potentials of functions in central Herz-Morrey spaces.

**Theorem 3.2.** Assume that \(\alpha - n/p < \nu < n - n/p\). Then

\[
\|I_\alpha f\|_{M^{p,q,\nu}(B)} \leq C \|f\|_{M^{p,q,\nu}(B)}.
\]

**Theorem 3.3** (Sobolev’s inequality for generalized Riesz potentials). Assume that \(\ell \geq 0\) and \(n - n/p + \ell < \nu < n - n/p + \ell + 1\). Then

\[
\|I_{\alpha,\ell} f\|_{M^{p,q,\nu}(B)} \leq C \|f\|_{M^{p,q,\nu}(B)}.
\]

To prove Sobolev’s inequality, for a real number \(\beta\) and \(0 < r < 1\), let us consider the Hardy type operators

\[
H_\beta^- f(r) = r^{-\beta} \int_{B(0,r)} |y|^{\beta-n} f(y) dy
\]

and

\[
H_\beta^+ f(r) = r^{-\beta} \int_{B \setminus B(0,r)} |y|^{\beta-n} f(y) dy
\]

for measurable functions \(f\) on \(B\).
**Lemma 3.4.** Let $\beta - \nu - n/p > \varepsilon > 0$. Then

$$H^{-}_\beta f(r) \leq C r^{\varepsilon - n/p - \nu} \left( \int_0^r \left( t^{\varepsilon + \nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

**Lemma 3.5.** Let $0 < \varepsilon < -\beta + \nu + n/p$. Then

$$H^+_\beta f(r) \leq C r^{\varepsilon - n/p - \nu} \left( \int_{r/2}^1 \left( t^{-\varepsilon + \nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}$$

for all $0 < r < 1$ and $f \in L^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof of Theorem 3.3.** Let $\|f\|_{M^{p,\nu}(B)} \leq 1$ and $f \geq 0$. For $x \in B$, set

$$I_{\alpha,\ell} f(x) = \int_{B(0,|x|/2)} I_{\alpha,\ell}(x, y)f(y)dy + \int_{B(0,2|x|) \setminus B(0,|x|/2)} I_{\alpha,\ell}(x, y)f(y)dy + \int_{B(0,1) \setminus B(0,2|x|)} I_{\alpha,\ell}(x, y)f(y)dy = u_1(x) + u_2(x) + u_3(x).$$

Let $0 < r < 1$. By Lemma 1.1 we have

$$|u_2(x)| \leq C \int_{A(0,r/2) \cup A(0,r)} |x-y|^\alpha \|f(y)\|dy$$

for $x \in A(0, r)$, so that Lemma 3.4 gives

$$\|u_2\|_{L^p(A(0,r))} \leq C \|f\|_{L^p(A(0,r/2) \cup A(0,r))}.$$

Hence,

$$\int_0^1 \left( r^\varepsilon \|u_2\|_{L^p(A(0,r))} \right)^q \frac{dr}{r} \leq C \int_0^1 \left( r^\varepsilon \|f\|_{L^p(A(0,r/2) \cup A(0,r))} \right)^q \frac{dt}{t}.$$

By Lemma 1.1 we see that

$$|u_1(x)| \leq C |x|^\alpha \int_{B(0,|x|/2)} |y|^{\ell+1} f(y) dy$$

$$\leq C r^\alpha H^{-}_{n+\ell+1}(r)$$

for $x \in A(0, r)$. Hence, using Lemma 3.4, we find

$$\|u_1\|_{L^p(A(0,r))} \leq C r^{-\varepsilon - \nu} \left( \int_0^r \left( t^{\varepsilon + \nu} \|f\|_{L^p(A(0,t))} \right)^q \frac{dt}{t} \right)^{1/q}.$$
for $0 < \varepsilon < n + \ell + 1 - n/p - \nu$. Consequently,

$$
\int_{0}^{1} (r^{\nu} \| u_{\ell} \|_{L^{p}(A(0, r))})^{q} \frac{dr}{r} \leq C \int_{0}^{1} \left( r^{\varepsilon} \int_{0}^{r} \left( t^{\varepsilon + \nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \frac{dt}{t} \right) \frac{dr}{r}
$$

$$
\leq C \int_{0}^{1} \left( t^{\varepsilon + \nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \left( \int_{0}^{t} r^{\varepsilon q} \frac{dr}{r} \right) \frac{dt}{t}
$$

Similarly, by Lemma 1.1 we see that

$$
|u_{3}(x)| \leq C |x|^{\alpha - n - \ell} \int_{B \setminus B(0, 2|x|)} |y|^\ell f(y) \, dy
$$

$$
\leq Cr^{\alpha} H_{n+\ell}f(r)
$$

for $x \in A(0, r)$. Hence, using Lemma 3.5, we find

$$
\| u_{3} \|_{L^{p}(A(0, r))} \leq Cr^{\varepsilon - \nu} \left( \int_{r}^{1} \left( t^{-\varepsilon + \nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \frac{dt}{t} \right)^{1/q}
$$

for $0 < \varepsilon < -(n + \ell - n/p - \nu)$. Thus,

$$
\int_{0}^{1} (r^{\nu} \| u_{3} \|_{L^{p}(A(0, r))})^{q} \frac{dr}{r} \leq C \int_{0}^{1} \left( r^{\varepsilon} \int_{r}^{1} \left( t^{-\varepsilon + \nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \frac{dt}{t} \right) \frac{dr}{r}
$$

$$
\leq C \int_{0}^{1} \left( t^{-\varepsilon + \nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \left( \int_{0}^{t} r^{\varepsilon q} \frac{dr}{r} \right) \frac{dt}{t}
$$

$$
\leq C \int_{0}^{1} \left( t^{\nu} \| f \|_{L^{p}(A(0, t))} \right)^{q} \frac{dt}{t}
$$

4 Growth near the origin of spherical means

The $L^{q} (1 \leq q < \infty)$ means over the spherical surface $S(0, r)$ for a function $u$ is defined by

$$
S_{q}(u, r) = \left( \frac{1}{|S(0, r)|} \int_{S(0, r)} |u(x)|^{q} \, dS(x) \right)^{1/q}
$$

$$
= \left( \frac{1}{\omega_{n-1}} \int_{S(0, 1)} |u(r \sigma)|^{q} \, dS(\sigma) \right)^{1/q},
$$

where $S(0, r) = \partial B(0, r)$ and $|S(0, r)| = \omega_{n-1} r^{n-1}$ with $\omega_{n-1}$ denoting the area of the unit sphere.

Our aim is to find $d > 0$ such that

$$
\liminf_{r \to 0^{+}} r^{d} S_{q}(I_{\alpha, \ell} f, r) = 0
$$

for a function $f$ on $B$ satisfying Herz-Morrey type conditions.

Our result is a continuation of Gardiner's result ([13, 1988]):
REMARK 4.1. For a Green potential $G\mu$ on $B$,

(1) if $(n-1)/(n-2) \leq q < (n-1)/(n-3)$, then
$$\lim_{r \to 1} \inf_{1} (1-r)^{n-1-(n-1)/q}S_{q}(G\mu, r)=0;$$

(2) if $1 \leq q < (n-1)/(n-2)$, then
$$\lim_{r \to 1} (1-r)^{n-1-(n-1)/q}S_{q}(G\mu, r)=0.$$

THEOREM 4.2. Suppose $n-n/p + \ell < \nu < n-n/p + \ell + 1$. If $(n-\alpha p-1)/(p(n-1)) < 1/q \leq 1/p$, then
$$\lim_{r \to 0} \inf_{+} r^{(n-\alpha p + \nu p)/p}S_{q}(I_{\alpha,\ell}f, r)<\infty$$
for all $f \in M^{p,\nu}(B)$.

THEOREM 4.3. Suppose $n-n/p + \ell < \nu < n-n/p + \ell + 1$. If $(n-\alpha p-1)/(p(n-1)) < 1/q \leq 1/p$, then
$$\lim_{r \to 0} \inf_{+} r^{(n-\alpha p + \nu p)/p}S_{q}(I_{\alpha,\ell}f, r)=0$$
for all $f \in M^{p,\nu}_{0}(B)$.

Part II
Boundary growth properties

5 Superharmonic functions on $B$

The Riesz kernel is written as
$$|x-y|^{\alpha-n} = \sum_{\ell} (1-|y|)^{\ell} \phi_{\alpha,\ell}(x, \tilde{y}),$$
where $\tilde{y} = y/|y|$ and
$$\phi_{\alpha,\ell}(x, \tilde{y}) = \sum_{\ell/2 \leq k \leq \ell} a_{\alpha,\ell,k} |x-\tilde{y}|^{\alpha-n-2k}(x \cdot \tilde{y} - 1)^{2k-\ell}.$$

In fact, consider the Taylor expansion of
$$|x-y|^{\alpha-n} = |x-\tilde{y} + t\tilde{y}|^{\alpha-n}$$
and set $t = 1-|y|$.

Now define
$$K_{\alpha,m}(x, y) = \frac{1}{(n-\alpha)\sigma_n} \begin{cases} 
|x-y|^{\alpha-n} & (y \in B(0,1/2)); \\
|x-y|^{\alpha-n} - \sum_{\ell=0}^{m} (1-|y|)^{\ell} \phi_{\alpha,\ell}(x, \tilde{y}) & (y \in B \setminus B(0,1/2)).
\end{cases}$$

The following properties for $K_{2,m}$ are fundamental.
Lemma 5.1 (cf. [9, Lemma 2.2]). (1) $\Delta K_{2,m}(\cdot, y) = \delta_y$ when $n > 2$;

(2) $|K_{\alpha,m}(x, y)| \leq C|x - y|^{\alpha-n-m-1}(1 - |y|)^{m+1}$ when $1 - |y| \leq (\sqrt{2} - 1)|x - y|$.

We show Riesz decomposition for superharmonic functions on $B$.

Theorem 5.2. If $u$ is superharmonic in $B$ and

$$\liminf_{r \to 1} (1 - r)^{\alpha}S(u, r) > -\infty,$$

then

$$u(x) = \int_B K_{2,m}(x, y) \, d\mu(y) + h_0(x),$$

where $h_0$ is harmonic in $B$ and $m$ is an integer greater than $\alpha$.

Set

$$C(0, r) = B(0, r + (1-r)/2) \setminus B(0, r - (1-r)/2)$$

for $0 < r < 1$.

Denote by $\tilde{M}^{p, \nu}(B)$ the family of all functions $f \in L^1_{\text{loc}}(B)$ such that

$$\|f\|_{\tilde{M}^{p, \nu}(B)} = \sup_{0 < r < 1} (1 - r)^{\nu}\|f\|_{L^p(C(0, r)}) < \infty.$$

Now we give a continuation of the results by Gardiner [13].

Theorem 5.3. Let $1 \leq q < \infty$ and suppose $\|F\|_{\tilde{M}^{p, \nu}(B)} \leq 1$ with $F(y) = (1 - |y|)f(y)$. Then there exists a constant $C > 0$ such that

(1) if $n + m - 1 - \alpha p < (n - 1)/q \leq n + m - \alpha p$, then

$$\liminf_{r \to 1} (1 - r)^{(n-\alpha p+\nu)/p-(n-1)/q}S_q(K_{\alpha,m}f, r) \leq C;$$

(2) if $n - \alpha p + m < (n - 1)/q < n + m + 1 - \alpha p$, then

$$\sup_{1/2 < r < 1} (1 - r)^{(n-\alpha p+\nu)/p-(n-1)/q}S_q(K_{\alpha,m}f, r) \leq C.$$

6 Isolated singularities for monotone functions in the sense of Lebesgue [18, 1907]

A continuous function $u$ on a domain $D$ is said to be monotone in the sense of Lebesgue [18] if for every subdomain $G$, $\bar{G} \subset D$,

$$\max_{\bar{G}} u = \max_{\partial \bar{G}} u$$

and

$$\min_{\bar{G}} u = \min_{\partial \bar{G}} u.$$

see Heinonen-Kilpeläinen-Martio [15], Koskela-Manfredi-Villamor [17], Manfredi-Villamor [20, 21], the author [22, 23], the author-Shimomura [24, 25], Villamor-Li [29], Vuorinen [30, 31].
**Theorem 6.1.** Suppose \( n - 1 < p < \nu + n \). Let \( u \) be a function on \( B \setminus \{0\} \) which is monotone in the sense of Lebesgue and satisfies

\[
\sup_{0<r<1} r^{\nu} \int_{A(0,r)} |\nabla u(x)|^p \, dx \leq 1 \quad (p > n - 1).
\]

Then

\[
\sup_{x \in B} r^{(n-p+\nu)/p} |u(x)| \leq C < \infty.
\]

For monotone functions in the sense of Lebesgue, the following is a crucial tool.

**Lemma 6.2.** (cf. [20], [21], [23]). If \( u \) is monotone in \( B(x_0, 2r) \) in the sense of Lebesgue and \( p_1 > n - 1 \), then \( \forall x, y \in B(x_0, r) \)

\[
|u(x) - u(y)|^{p_1} \leq C r^{p_1-n} \int_{B(x_0,2r)} |\nabla u(z)|^{p_1} \, dz.
\]

(6.1)

**Example 6.3.** For \( \beta > 0 \), consider \( u(x) = |x|^{-\beta} \). Then

- \( u \) is monotone in \( B \setminus \{0\} \) in the sense of Lebesgue;
- \( |\nabla u(x)| \leq C |x|^{-\beta-1} \).

If \( -(\beta+1)p + \nu + n \geq 0 \),

\[
\sup_{0<r<1} r^{\nu} \int_{A(0,r)} |\nabla u(x)|^p \, dx < \infty.
\]

Hence, letting \( \beta = (n-p+\nu)/p \geq 0 \), we find

\[
\lim_{x \to 0} |x|^{(n-p+\nu)/p} u(x) = 1.
\]

Finally we show boundary growth for monotone functions on \( B \) in the sense of Lebesgue.

**Theorem 6.4.** Suppose \( n - 1 < p < \nu + n \), \( p < q < \infty \). Let \( u \) be a function on \( B \) which is monotone in the sense of Lebesgue and satisfies

\[
\sup_{0<r<1} (1-r)^{\nu} \int_{C(0,r)} |\nabla u(x)|^p \, dx \leq 1.
\]

If \( (n-1)/q < (n-p+\nu)/p \), then

\[
\sup_{0<r<1} (1-r)^{(n-p-\nu)/(p-(n-1)/q)} S_q(u, r) \leq C < \infty.
\]
References


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