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An a priori estimate for positive solutions of the Lane–Emden equation in a Lipschitz domain

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1 Introduction

This note presents an improvement of an a priori estimate for positive solutions of the Lane–Emden equation, given in many studies. Let us start with a simple introduction related to this. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ and let $\delta_\Omega(x)$ denote the usual distance from a point x to the boundary $\partial\Omega$ of Ω. The *Lane–Emden equation* is a nonlinear equation of the form

$$-\Delta u = |u|^{p-1} u, \tag{1.1}$$

where Δ is the Laplacian on $\mathbb{R}^n$ and $p > 1$. We consider the set $\mathscr{U}_p(\Omega)$ of all positive classical solutions of (1.1) in Ω. Let

$$p_S := \frac{n+2}{n-2}.$$

It is known that if $1 < p < p_S$, then there exists a positive constant C depending only on p and n such that

$$u(x) \leq C\delta_\Omega(x)^{-\frac{2}{p-1}} \tag{1.2}$$

holds for all $x \in \Omega$ and $u \in \mathscr{U}_p(\Omega)$. This estimate was utilized in many studies on (1.1). See Dancer [2] for the Dirichlet problem on exterior domains, Poláčik–Quittner–Souplet [6] for a Liouville type theorem: $\mathscr{U}_p(\mathbb{R}^n) = \emptyset$, and Serrin–Zou [7] for $-\Delta_q u = |u|^{p-1}u$ with Δ_q being the q-Laplacian on $\mathbb{R}^n$. Note that the exponent $-\frac{2}{p-1}$ comes from the scale invariant property of (1.1): if $u \in \mathscr{U}_p(\Omega)$ and $\lambda > 0$, then $\lambda^{\frac{2}{p-1}} u(\lambda x) \in \mathscr{U}_p(\frac{1}{\lambda}\Omega)$. Then there arises a natural question whether or not the growth rate in (1.2) is optimal?

To state our main result, we need to prepare some notations. We write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for $a, b \in \mathbb{R}$. Let $G(x, y)$ denote the (Dirichlet) Green function on Ω for Δ. For a fixed $x_0 \in \Omega$, we put

$$g(x) := G(x, x_0) \wedge 1.$$

Note that the boundary decay rate of g may vary at each boundary point when $\partial\Omega$ is non-smooth, whereas $g(x)$ is comparable to the distance function $\delta_\Omega(x)$ when $\partial\Omega$ is smooth.

We prove the following theorem.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 3)$ and let $1 < p < p_S$. Then there exists a positive constant C depending only on p, n and Ω such that every $u \in \mathscr{U}_p(\Omega)$ can be estimated by*

$$u(x) \leq \frac{C}{g(x)\delta_\Omega(x)^{n-2} \vee \delta_\Omega(x)^{\frac{2}{p-1}}} \tag{1.3}$$

for all $x \in \Omega$.

Here we give some remarks on the above estimate in a bounded Lipschitz domain Ω. For simplicity, we write

$$p_\alpha := \frac{n+\alpha}{n+\alpha-2}$$

for $\alpha \geq 0$.

- The inverse of $g(x)\delta_\Omega(x)^{n-2}$ is related to the boundary growth of positive harmonic functions on Ω. Indeed, on a nontangential region at $\xi \in \partial\Omega$, it is comparable to the Martin (Poisson) kernel at ξ. See Aikawa [1] and the author [3].
- Let $\xi \in \partial\Omega$. It is known that there are constants $\alpha_\xi > 0$ and $C > 1$ such that

$$g(x) \geq \frac{1}{C}\delta_\Omega(x)^{\alpha_\xi} \tag{1.4}$$

  on a nontangential region at ξ. Therefore, if $p < p_{\alpha_\xi}$, then

$$g(x)\delta_\Omega(x)^{n-2} > \delta_\Omega(x)^{\frac{2}{p-1}}$$

  on that set near ξ, which implies that (1.3) improves the earlier one.
- As we see from Theorem 3.1 below, the growth rate in (1.3) is optimal when $1 < p < p_{\alpha_\xi}$ (and also when $p_0 < p < p_S$ because it is known that there is a positive solution of (1.1) in Ω behaving like $\|x-\xi\|^{-\frac{2}{p-1}}$ near $\xi \in \partial\Omega$). For $p_{\alpha_\xi} \leq p \leq p_0$, we do not know whether or not (1.3) is optimal.

The plan of this note is as follows. In Section 2, we prove Theorem 1.1 using the global estimates of the Green function and the Martin kernel, a fundamental pointwise estimate of the Newton potential of a superharmonic density and some known results in potential theory. In Section 3, we prove the existence of a positive solution of (1.1) in Ω behaving like the Martin kernel in order to show that the growth rate in (1.3) is optimal. In the final section, we enumerate some properties one can get from Theorem 1.1.

2 Proof of Theorem 1.1

In what follows, we suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$ $(n \geq 3)$. By the symbol C, we denote an absolute positive constant whose value may vary at each occurrence. When C depends on some/all of the Lipschitz characters, $\operatorname{diam}\Omega$ and $\delta_\Omega(x_0)$, we say that C depends on Ω. If necessary, we use $C_1, C_2, \dots$ to specify them.

Let $\xi \in \partial\Omega$ and let $\beta > 0$. A *nontangential region* at ξ is defined by

$$\Gamma_\beta(\xi) := \{x \in \Omega : \|x - \xi\| \leq \beta\delta_\Omega(x)\}.$$

This set is nonempty and ξ is accessible from there whenever β is sufficiently large, say $\beta \geq \beta_\Omega$. Let us recall the global estimates for the Green function $G_\Omega(x, y)$ and the Martin kernel $M_\Omega(x,\xi)$ at $\xi \in \partial\Omega$ established in [4]. For $x, y \in \overline{\Omega}$ and $C_1 > 1$, we let

$$\mathscr{B}(x, y) := \left\{b \in \overline{\Omega} : \frac{1}{C_1}(\|x - b\| \vee \|b - y\|) \leq \|x - y\| \leq C_1\delta_\Omega(b)\right\}.$$

It is not difficult to see that $\mathscr{B}(x, y)$ is nonempty for any pair x, y whenever C_1 is sufficiently large. Then there exists a constant $C > 1$ depending only on n and Ω such that

$$\frac{1}{C}\frac{g(x)g(y)}{g(b_{xy})^2}\|x - y\|^{2-n} \leq G_\Omega(x, y) \leq C\frac{g(x)g(y)}{g(b_{xy})^2}\|x - y\|^{2-n} \tag{2.1}$$

for all $x, y \in \Omega$ and $b_{xy} \in \mathscr{B}(x, y)$;

$$\frac{1}{C}\frac{g(x)}{g(b_{x\xi})^2}\|x - \xi\|^{2-n} \leq M_\Omega(x,\xi) \leq C\frac{g(x)}{g(b_{x\xi})^2}\|x - \xi\|^{2-n} \tag{2.2}$$

for all $x \in \Omega$ and $b_{x\xi} \in \mathscr{B}(x,\xi)$. In particular, for each $\beta \geq \beta_\Omega$ there exists a constant $C > 1$ depending only on β, n and Ω such that

$$\frac{1}{C}\delta_\Omega(x)^{2-n} \leq g(x)M_\Omega(x,\xi) \leq C\delta_\Omega(x)^{2-n} \tag{2.3}$$

for all $x \in \Gamma_\beta(\xi)$ (see [3]). Also, we can see the following fact from the Harnack inequality and the Carleson estimate for positive harmonic functions:

- There exists a constant $C > 0$ depending only on n and Ω such that $g(x) \leq Cg(b)$ for any pair $x, y \in \overline{\Omega}$ and $b \in \mathscr{B}(x, y)$.

Note that this, together with (2.2) and (2.3), yields that

$$G_\Omega(x, y) \leq C\frac{g(y)}{g(x)}\|x - y\|^{2-n} \tag{2.4}$$

and

$$M_\Omega(x,\xi) \leq \frac{C}{g(x)\delta_\Omega(x)^{n-2}} \tag{2.5}$$

for all $x, y \in \Omega$. Also, we use the following elementary estimate.

Lemma 2.1. *Let u be a nonnegative superharmonic function on $B(x,r)$. Then*

$$\int_{B(x,r)} \frac{u(y)}{\|x-y\|^{n-2}}\,dy \le \frac{\sigma_n}{2} r^2 u(x),$$

where σ_n is the surface area of the unit sphere in $\mathbb{R}^n$.

Proof. By the polar coordinate representation and the spherical mean value inequality for superharmonic functions, we have

$$\begin{aligned}\int_{B(x,r)} \frac{u(y)}{\|x-y\|^{n-2}}\,dy &= \int_0^r \frac{1}{\rho^{n-2}} \int_{\partial B(x,\rho)} u(y)\,d\sigma(y)\,d\rho \\ &\le \sigma_n u(x) \int_0^r \rho\,d\rho = \frac{\sigma_n}{2} r^2 u(x).\end{aligned}$$

□

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u \in \mathscr{U}_p(\Omega)$. As stated in the introduction, we note that u satisfies (1.2). Therefore it satisfies the differential inequality

$$0 \le -\Delta u(x) \le C_2 \delta_\Omega(x)^{-2} u(x) \tag{2.6}$$

for all $x \in \Omega$, where C_2 is a positive constant depending only on p and n. To get (1.3), we have only to show that there exists a positive constant C depending only on p, n and Ω such that

$$u(x) \le \frac{C}{g(x)\delta_\Omega(x)^{n-2}} \tag{2.7}$$

holds for all $x \in \Omega$. By the Riesz decomposition theorem for nonnegative superharmonic functions, there exists a nonnegative harmonic function h on Ω such that

$$u(x) = h(x) + \int_\Omega G_\Omega(x,y)(-\Delta u(y))\,dy \tag{2.8}$$

for all $x \in \Omega$. Moreover, by substituting $x = x_0$ in (2.8), we have

$$\int_\Omega g(y)(-\Delta u(y))\,dy \le u(x_0). \tag{2.9}$$

Let $x \in \Omega$ and let $j \in \mathbb{N}$, which will be chosen later. We write $B_j := B(x, \delta_\Omega(x)/2^j)$ for simplicity. By (2.4), we have

$$G_\Omega(x,y) \le \frac{2^{j(n-2)}C}{g(x)\delta_\Omega(x)^{n-2}} g(y)$$

for all $y \in \Omega \setminus B_j$. Therefore, by (2.9),

$$\int_{\Omega\setminus B_j} G_\Omega(x,y)(-\Delta u(y))\,dy \le \frac{2^{j(n-2)}C}{g(x)\delta_\Omega(x)^{n-2}} u(x_0).$$

Since $G_\Omega(x,y) \le C\|x-y\|^{2-n}$ and u is superharmonic on Ω, it follows from (2.6) and Lemma 2.1 that

$$\int_{B_j} G_\Omega(x,y)(-\Delta u(y))\,dy \le \frac{C}{\delta_\Omega(x)^2}\int_{B_j}\frac{u(y)}{\|x-y\|^{n-2}}\,dy \le \frac{C_3}{2^{2j}}u(x),$$

where C_3 depends only on p, n and Ω. Moreover, by the Martin integral representation of h and (2.5), we get

$$h(x) \le \frac{C}{g(x)\delta_\Omega(x)^{n-2}}h(x_0) \le \frac{C}{g(x)\delta_\Omega(x)^{n-2}}u(x_0).$$

These estimates and (2.8) yield that

$$u(x) \le \frac{2^{j(n-2)}C}{g(x)\delta_\Omega(x)^{n-2}}u(x_0) + \frac{C_3}{2^{2j}}u(x).$$

Since $u(x_0) \le C$ by (1.2), we can obtain (2.7) by choosing j such that $C_3/2^{2j} \le 1/2$. □

3 Optimality of our estimate

The following theorem shows that the growth rate in (1.2) is optimal.

Theorem 3.1. *Let $1 < p < p_{\alpha_\xi}$, where α_ξ is the constant in (1.4). Then there exists a positive number λ_1 such that for any $\lambda \in (0, \lambda_1]$, there exists a positive classical solution u of (1.1) in Ω such that*

$$\frac{\lambda}{2}M_\Omega(x,\xi) \le u(x) \le \frac{3\lambda}{2}M_\Omega(x,\xi) \tag{3.1}$$

for all $x \in \Omega$.

To show this, we apply the Banach fixed point theorem to the following function class and operator. Let $\lambda > 0$. We consider the closed set

$$W_\lambda := \left\{ w \in C(\Omega) : \frac{\lambda}{2} \le w(x) \le \frac{3\lambda}{2} \text{ for } x \in \Omega \right\}$$

in the Banach space $(BC(\Omega), \|\cdot\|_\infty)$, the set of all bounded continuous functions on Ω equipped with the uniform norm, and the operator $\mathscr{T}_\lambda$ on W_λ defined by

$$\mathscr{T}_\lambda[w](x) := \lambda + \frac{1}{M_\Omega(x,\xi)}\int_\Omega G_\Omega(x,y)(w(y)M_\Omega(y,\xi))^p\,dy$$

for $x \in \Omega$. Using (2.1) and (2.2), we can show that if $1 < p < p_{\alpha_\xi}$, then

$$A := \sup_{x\in\Omega}\frac{1}{M_\Omega(x,\xi)}\int_\Omega G_\Omega(x,y)M_\Omega(y,\xi)^p\,dy$$

is finite. See [5] for details.

Lemma 3.2. *$\mathcal{T}_\lambda(W_\lambda) \subset W_\lambda$ whenever λ is sufficiently small.*

Proof. Let $w \in W_\lambda$. Since $p > 1$, we get

$$\frac{\lambda}{2} \le \lambda - A\left(\frac{3\lambda}{2}\right)^p \le \mathcal{T}_\lambda[w](x) \le \lambda + A\left(\frac{3\lambda}{2}\right)^p \le \frac{3\lambda}{2}$$

for all $x \in \Omega$, whenever λ is sufficiently small. Since $(w(y)M_\Omega(y,\xi))^p$ is locally bounded on Ω, the classical result shows that the Green potential of that density is continuous on Ω, and so is $\mathcal{T}_\lambda[w]$. Hence $\mathcal{T}_\lambda[w] \in W_\lambda$. □

Lemma 3.3. *$\mathcal{T}_\lambda : W_\lambda \to W_\lambda$ is a contraction mapping whenever λ is sufficiently small.*

Proof. Let $w_1, w_2 \in W_\lambda$. For $x \in \Omega$, we get

$$\begin{aligned}|\mathcal{T}_\lambda[w_1](x) - \mathcal{T}_\lambda[w_2](x)| &\le \int_\Omega \frac{G_\Omega(x,y)M_\Omega(y,\xi)^p}{M_\Omega(x,\xi)} |w_1(y)^p - w_2(y)^p|\,dy \\ &\le A\|w_1^p - w_2^p\|_\infty.\end{aligned}$$

Since

$$\|w_1^p - w_2^p\|_\infty \le p\left(\frac{3\lambda}{2}\right)^{p-1} \|w_1 - w_2\|_\infty$$

by the mean value theorem, we can obtain

$$|\mathcal{T}_\lambda[w_1](x) - \mathcal{T}_\lambda[w_2](x)| \le \frac{1}{2}\|w_1 - w_2\|_\infty$$

for all $x \in \Omega$, whenever λ is small enough. Thus the lemma follows. □

Proof of Theorem 3.1. By the Banach fixed point theorem, there exists a unique $w_0 \in W_\lambda$ such that $\mathcal{T}_\lambda[w_0] = w_0$ on Ω. Letting $u(x) := w_0(x)M_\Omega(x,\xi)$, we have

$$u(x) = \lambda M_\Omega(x,\xi) + \int_\Omega G_\Omega(x,y)u(y)^p\,dy$$

for all $x \in \Omega$. Also, (3.1) holds. Since u is locally bounded on Ω, the classical regularity theorem shows that $u \in C^2(\Omega)$. Hence u is a positive classical solution of (1.1) in Ω. □

4 Remark

Theorem 1.1 has important and wide applications. Indeed, with the help of some results in potential theory, one can get

- a strong Harnack inequality: for each small $0 < \kappa \ll 1$, there exists a constant $c(\kappa)$ depending only on κ, p and n such that $u(x) \le c(\kappa)u(y)$ for all $u \in \mathcal{U}_p(\Omega)$ and any pair $x, y \in \Omega$ satisfying $\|x - y\| \le \kappa\delta_\Omega(x)$. Moreover, $c(\kappa)$ enjoys $c(\kappa) \ge 1$ and $c(\kappa) \to 1$ as $\kappa \to 0+$;

- the existence of nontangential limits of $u \in \mathscr{U}_p(\Omega)$ and the ratio $u/M_\Omega(\cdot,\xi)$;
- a Harnack convergence theorem: any sequence in $\mathscr{U}_p(\Omega)$ has a subsequence which converges uniformly to a function in $\mathscr{U}_p(\Omega) \cup \{0\}$ on each compact subset of Ω.
- the existence and nonexistence of a positive classical solution of

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = \lambda \delta_\xi & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

  where $\lambda > 0$ and δ_ξ is the Dirac measure concentrated at $\xi \in \partial\Omega$. Indeed, we can find a critical number λ^* such that if $\lambda \le \lambda^*$, then (4.1) has a positive solution, but if $\lambda > \lambda^*$, then (4.1) has no positive solution.

These results and their proofs can be found in [5].

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